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THE GEOMETRY OF INTERACTION

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1 — THE UNDERSIDE OF THE WORLD

- **Nothing like a *password* file kept by Tarski and his pupils.**
- **Explicitation as a *process :* never finished.**

  Cognitive onion : alternating Q/A.
- **Find the *geometric* space where processes belong :**

  Process algebras : mediocre, reflect our prejudices. *Avoid !*
  Operator algebras : sophisticated, but *how to use them ?*
- **Processes should not *collapse* to Q/A diagrams :**

  Kripke models : paragon of what should be *impossible.*
  Abductive (inductive) methods should be *impossible.*
  η-expansion : should be trivially *wrong.*
- ***Wrong :* impossible for *geometric* reasons.**

  Not out of a *religious* interdiction, e.g., « predicativism. »

I — QUANTUM COHERENT SPACES

2 — Coherent spaces

- On *carrier* $|X|$, define *duality* between subsets :

$$a \mathrel{\underset{\sim}{\downarrow}} b \quad \Leftrightarrow \quad \sharp(a \cap b) \leq 1$$

- *Coherent space* of *carrier* (a.k.a. web) $|X|$:

$$X \subset \wp(|X|) \quad \text{s.t.} \quad X = {\sim\sim} X$$

- *Adjunction :* if $f \subset |X| \times |Y|, a \subset |X|, b \subset |Y|$

$$\sharp([f]a \cap b) = \sharp(f \cap a \times b)$$

$$\Delta_X := \{(x, x); x \in |X|\} \quad \text{then} \quad [I_X]a = a$$

- Yields *linear implication* $\quad \Delta_X \in X \multimap X$

$$X \multimap Y \quad := \quad \{f \subset |X| \times |Y| \; ; \; \forall a \in X \; [f]a \in Y\}$$

- The true origin of *linear logic.*
- T.V.S. compatible : *coherent Banach spaces, QCS.*

3 — Quantum coherent spaces

- *Carrier* $|E|$: finite dimensional *Hilbert space.*
- *Duality* between *hermitians* $u = u^*$ of $\mathcal{B}(|E|)$:

$$u \overset{\perp}{\sim} v \quad \Leftrightarrow \quad 0 \leq \mathrm{tr}(uv) \leq 1$$

- *QCS* of *carrier* $|E|$: $E \subset \mathcal{H}(|E|)$ equal to its *bipolar.*
- *Adjunction :* if $f \in \mathcal{H}(|E| \otimes |F|)$, $a \in \mathcal{H}(|E|)$, $b \in \mathcal{H}(|F|)$

$$\mathrm{tr}([f]a \cdot b) \quad = \quad \mathrm{tr}(f \cdot a \otimes b)$$

- *Identity* becomes the *flip :*

$$\sigma_E(x \otimes y) \quad := \quad y \otimes x$$

$$\mathrm{tr}(\sigma_E \cdot a \otimes b) \quad = \quad \mathrm{tr}(a \cdot b)$$

- Still *linear implication* $\sigma_E \in E \multimap E$

$$E \multimap F \quad := \quad \{f \in \mathcal{H}(|E| \otimes |F|)\ ;\ \forall a \in E\ [f]a \in F\}$$

4 — Non commutativity : revisiting Kant

- $|X| \rightsquigarrow \mathbb{C}^{|X|}$ $a \rightsquigarrow \mathbb{C}^a$ *projection,* idempotent hermitian.

$$\mathrm{tr}(\mathbb{C}^a\mathbb{C}^b) = \mathrm{tr}(\mathbb{C}^{a\cap b}) = \sharp(a \cap b)$$

- η*-expansion* fails : $\sigma_{\mathbb{C}^{|X|}} \neq \mathbb{C}^{\Delta_X}$.
- *Syntax/semantics* paradigm (Frege & al.) :

  Ready made dichotomy object/subject.
- *Non commutativity :* the *subject* as the choice of a *basis.*

  Diagonal matrices : subjective, functions $i \mapsto a_{ii}$.
  Commutativity : of diagonal matrices.
- A commutative *operator algebra* is a function space :

  C^*-algebras : $C(X)$ *continuous* functions on *compact* X.
  von Neumann : $\mathcal{L}^\infty(X, \mu)$, bounded *measurable* functions.
- *Constitution of subject :* operators become *sets,* functions.

5 — The unfinished

- **QCS do not live in *infinite* dimension ($\neq$ CS).**

  Type I : *semi-finite* trace. Diverges, but for *trace class.*
  Type II_1 : finite trace. But *flip* gets a null trace.

- ***Fock space* (exterior algebra) : $\Lambda E := \bigoplus \Lambda^n E$**

$$\langle x_1 \wedge \ldots \wedge x_n \mid y_1 \wedge \ldots \wedge y_n \rangle := \det((\langle x_i \mid y_j \rangle))$$

- **Finite dimension : $\mathrm{tr}(u) = \det(1 + \Lambda u)$.**

- **Replace trace with determinant, $u \rightsquigarrow \Lambda i u$:**

$$u \underset{\sim}{\downarrow} v \quad \Leftrightarrow \quad \det(1 - uv) \neq 1$$

- **Λuv : all possible *partial travels* in uv.**

- **Infinite dimension : Fock space « diverges » (type I) :**

  Travel space : diverges. The *demise* of Kripke models !
  Determinant : still makes sense in II_1 algebras.

II — DESIGNS AND CONDUCTS

6 — The GoI adjunction

- W.r.t. *direct sum decomposition*, provided $\varrho(FA) < 1$

$$\mathrm{ldet}(1 - (F \cdot A \oplus B)) = \mathrm{ldet}(1 - FA) + \mathrm{ldet}(1 - ([F]A \cdot B))$$

with : $\mathrm{ldet}(1 - u) := \mathrm{tr}(u + u^2/2 + u^3/3 + \ldots)$

If $F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$ then $[F]A := (F_{21}A(F_{11}A)^{-1}F_{12})$

- Analogous to *QCS* : $\mathrm{tr} \rightsquigarrow \mathrm{ldet}, \quad \otimes \rightsquigarrow \oplus$

  Heating : $\mathrm{ldet}(1 - FA) \quad (= \mathrm{ldet}(1 - F_{11}A))$
  Logic : $F_{11}A$ *nilpotent*, thence no heating.
  Truth value : neglected by « actualisation. »

- Constitution of dichotomy *object/subject* from adjunction.

  No ready made semantics/syntax or object/morphism.
  Morphological question : find *adequate* restrictions.

7 — von Neumann algebras

- *$*$-subalgebra* $\mathcal{A} \subset \mathcal{B}(\mathbb{H})$ equal to its *bicommutant.*
- *Dual* $*$-algebra s.t. $\|uu^*\| = \|u\|^2$

  Predual $\mathcal{A}_\sharp$ **:** made of *ultraweakly* continuous forms.
  Means : weakly continuous *on unit ball.*
- *Commutative case :* measure spaces $\mathcal{L}^\infty(X, \mu)$.

  Predual : $\mathcal{L}^\infty(X, \mu)_\sharp = \mathcal{L}^1(X, \mu)$.
- *Factors :* connected algebras, *never* commutative !

  Trace : unique (up to a scalar) on factors.
  Exists for types $\neq III$.
  Type II **:** $\{\mathrm{tr}(p)\,;\, p \ \texttt{proj.}\} = [0, \lambda], \lambda \in]0, \infty]$
  Splits into : II_1 (λ finite) and II_∞.
  Trace ultraweakly continuous (in type II_1 case).
- Only one *hyperfinite* factor of type II_1 (resp. II_∞).

8 — Designs

- $\mathcal{R}$ « the » hyperfinite factor of type II_∞, tr « its » trace.
- ***Design*** $\mathfrak{a} = (p, a, \mathcal{A}, \alpha, A)$ notation $a \cdot\!+\!\cdot \alpha + A$

  Carrier : *finite* projection $p \in \mathcal{R}$: $\mathrm{tr}(p) < \infty$.

  Wager : $a \in \mathbb{R} \cup \{\infty\}$.

  Dialect : algebra $\mathcal{A}$ embeddable in hyperfinite II_1 factor.

  Diatrace : form $\alpha \in \mathcal{A}_\sharp$ s.t. $\alpha(uv) = \alpha(vu), \alpha(1_\mathcal{A}) \neq 0$.

  Plot : $A \in p\mathcal{R}p \otimes \mathcal{A}$, with $\|A\| \leq 1$ and $A = A^*$.
- If $\mathfrak{b} = (p, b, \mathcal{B}, \beta, B)$ (same carrier), i.e., $b \cdot\!+\!\cdot \beta + B$

  Isos : $(\cdot)^\dagger, (\cdot)^\ddagger \ : \ \mathcal{R} \otimes \mathcal{A}, \mathcal{R} \otimes \mathcal{B} \ \mapsto \ \mathcal{R} \otimes \mathcal{A} \otimes \mathcal{B}$.

  $\ll \mathfrak{a} \,|\, \mathfrak{b} \gg := a\beta(1_\mathcal{B}) + b\alpha(1_\mathcal{A}) + \mathrm{ldet}_{\mathrm{tr}\otimes\alpha\otimes\beta}(1 - A^\ddagger B^\dagger)$

  Duality : $\mathfrak{a} \underset{\sim}{\downarrow} \mathfrak{b} \quad \Leftrightarrow \quad \ll \mathfrak{a} \,|\, \mathfrak{b} \gg \neq 0$

  Correctness criterion : subsumes $\varrho(A^\ddagger B^\dagger) < 1$.

9 — Conducts

- *Conduct* $\mathbb{A}$ of carrier p : *set* of designs equal to its bipolar.
- $\mathbb{A}, \mathbb{B}$ of disjoint carriers p, q. Define conducts of carrier $p + q$

$$\mathbb{A} \otimes \mathbb{B} := \sim\sim \{\mathfrak{a} \otimes \mathfrak{b} ; \mathfrak{a} \in \mathbb{A}, \mathfrak{b} \in \mathbb{B}\}$$

$$\mathbb{A} \multimap \mathbb{B} := \{\mathfrak{f} \; ; \; \forall \mathfrak{a} \in \mathbb{A} \; [\mathfrak{f}]\mathfrak{a} \in \mathbb{B}\}$$

- The two *multiplicatives* are *adjoint.*

$$\ll \mathfrak{f} \,|\, \mathfrak{a} \otimes \mathfrak{b} \gg = \ll [\mathfrak{f}]\mathfrak{a} \,|\, \mathfrak{b} \gg$$

- *Universal quantification* defined as an arbitrary *intersection.*
- *Second order :* $\forall X \, A[X] \quad = \quad \bigcap_X A[X]$
  X varies over all conducts of given carrier $r \neq 0$.
- Cannot substitute $X \multimap X / X$: carrier of *too big* trace $2\mathrm{tr}(r)$.
- A thunderbolt. . . *Geometrical* restriction on second order.Nothing to do with *predicativist* self-floggings !

III — SUMMARY OF RESULTS

10 — POLARISATION

- Stronger *morphology* handling additives and exponentials :

  Positive : contains all *daimons* $a \cdot\!+\!\cdot 1 + 0,\ a \neq 0$

  but not $0 \cdot\!+\!\cdot 1 + 0$.

  Negative : all wagers *boolean* $a + a = a$

  at least one design $0 \cdot\!+\!\cdot \alpha + A$.

- *Additives* as expected :

$\&$	<0	>0
<0	<0	$-$
>0	$-$	$-$

$\oplus$	<0	>0
<0	$-$	$-$
>0	$-$	>0

- *Multiplicative* polarity table unexpected :

$\otimes$	<0	>0
<0	<0	>0
>0	>0	$-$

$\multimap$	<0	>0
<0	<0	>0
>0	$-$	<0

11 — EXPONENTIALS

- *Weakening* free of charge on *positive* conducts.
- *Contraction* holds for *coperennial* conducts :

  No duplication of dialect possible.
  Perenniality : *dialect-free* designs $a \cdot\!+\!\cdot 1 + A$.
- *Perennialisation :* kill the dialect by means of

  $\Omega \;:\; \mathcal{R} \otimes \mathcal{H} \mapsto \mathcal{R}$ ($\mathcal{H}$: the hyperfinite factor of type II_1).

  $$!_\Omega \mathbb{A} \;:=\; \sim\sim\{a \cdot\!+\!\cdot 1 + \Omega(A); a \cdot\!+\!\cdot \mathrm{tr} + A \in \mathbb{A}\}$$
- « Bang » sends negative to negative (and perennial)

  $$!_\Omega(\mathbb{A} \& \mathbb{B}) \;=\; !_\Omega\mathbb{A} \otimes !_\Omega\mathbb{B}$$
- *Standard bang* $!A$: *amenable* group containing free monoid.

  Contextual promotion : « from $\Gamma \vdash A$ get $!\Gamma \vdash !A$ » holds.
  ELL : *elementary* linear logic.

12 — LATERALISATION

- Need for *changes of polarity.*
- $¥p$: positive conduct, contains the $\quad 0 \cdot + \cdot 1 + q \ ; \ q \subset p$

  Right behaviour of *base* $b \subset p$: positive $\mathbb{A}$ s.t. $\mathbb{A}_b \subset ¥b.$

  Left behaviour : defined dually.
- Behaviours preserved by *all* connectives but « bang. »
- Behaviours disantangle *Gustave* paradox :

  Intrication of $⅋, \oplus.$
  Witness theorem : $\quad pq = 0 \quad \Rightarrow \quad ¥p ⅋ ¥q = ¥p \oplus ¥q$
- If $\mathbb{A}, \mathbb{B}$ left behaviours, then $!\mathbb{A} \multimap \mathbb{B}$ left behaviour.
- *Secularisation* $\quad !\mathbb{A} \subset \S\mathbb{A}.$

  Contextual promotion : « from $\Gamma \vdash A$ get $!\Gamma \vdash \S A$ » holds.
  LLL : *light* linear logic.

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