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THREE LIGHTINGS OF LOGIC

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1 — An entangled relation

- Relation *question/answer:* abuse of *presuppositions*.
- What is an *answer?*

  Meaning should be expelled.
  Basic opposition: implicit/explicit.
  Explicitation a.k.a. *normalisation.*
- What is a *question?*

  Chosen to ensure *explicitability.*
  Basic opposition: formatted/informal.
  Rules set through deontic dialogue.
- What conveys *certainty?*

  Schizophrenia production/utlisation vs. rights/duties.
  Identity axiom: duties *do* ensure rights.
  Cut rule: rights *should* match duties. Purely conjectural.

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I — First light: Answers

2 — EXPLICIT VS. IMPLICIT

- **Reliable inefficiency vs. unreliable efficiency.**
- ***Explicit* answers.**

  Mathematics: numerical equations $2 + 2 = 4$.
  Computers: ***constative*** keyboard: ↲.
  Economy: barter.
- ***Implicit* answers.**

  Mathematics: theorems yield ***corollaries.***
  Computers: ***performative*** keyboard launches ***programs:*** ↲.
  Economy: cheques can be ***cashed.***
- **Gentzen (1934) distinguishes between:**

  Implication: $\Rightarrow$ explicit.
  Entailment: $\vdash$ implicit.
  Relative: explicit = no meaning, no further use, ***analytic.***

3 — TRANSPARENCY

- ***Reasoning:*** **about what we don't have, don't fully control.**
- ***Transparency:*** **totalitarian reduction implicit $\rightsquigarrow$ explicit.**

  Politics: Big Brother, Jeremy Bentham, NSA.
  Economics: replace money with barter.
  Semantics: replace consequence with *subsequence.*
- ***Failsafe*** **ideology at war with intelligence. Stumbles on:**

  Too many data: no way of processing them.
  Explicit bank should milk cows!
  Incompleteness: consequence $\neq$ subsequence.
- ***Gödel*** **1931, *Turing* 1936:**

  Reduction implicit/explicit: hazardous.
  Unanswerable question, even by cheating.
  Complexity: refutation of concrete transparency.

4 — Stars & Constellations

- ***Analytic*** **space of answers: meaningless.**

  Explicitation only matters: no relation with questions.
  Unification (matching) as a universal paradigm.

- **Equations $t = u$ between functional terms.**

  Literal solution by unification: $t\theta \equiv u\theta$.
  Most general unifier: mother of all unifiers (*Herbrand* 1930).

- ***Star*** $[\![t_1, \ldots, t_n]\!]$ **with** $n \neq 0$ ***rays*****.**

  Rays with the same variables and disjoint (not *matchable*).

- ***Constellation:*** **finite set of stars.**

  Rays pairwise disjoint.

- ***Coloured*** **stars, constellations, e.g., $[\![t, u, v]\!]$.**

  Distinct colours considered disjoint.

5 — NORMALISATION

- **Explicit/implicit as *monochrome/coloured.***

  Normalisation as elimination of colours.
  Complementary colours: $\mathtt{red}/\mathtt{cyan}, \mathtt{magenta}/\mathtt{green}$.

- ***Diagrams* of a constellation: trees built from its stars.**

  Edges: formal vertices $\mathtt{t} = \mathtt{u}$.
  Correct iff formal vertices can be *matched*.

- ***Strong normalisation:* finitely many correct diagrams.**

  Open: always a free ray.
  Acyclic: free rays of complementary colours don't match.
  Normal form: all correct (reduced) monochrome diagrams.

- ***Church-Rosser:* in presence of two pairs of colours.**

  Either: normalise $\mathtt{red}/\mathtt{cyan}$, **then** $\mathtt{magenta}/\mathtt{green}$.
  Or: identify $\mathtt{red} = \mathtt{magenta}, \mathtt{cyan} = \mathtt{green}$ **then normalise.**

II — SECOND LIGHT: QUESTIONS

6 — FORMATTED VS. INFORMAL

- The *format* at work in all activities.

  Art: musical forms (symphony, sonata, etc.).
  Politics: systems (democracy, tyranny, etc., laws, the family.
  Computation: programming language, typing systems.
  Logic: formal systems.
  Categories: *morphisms* preserve the format.

- Shell of the tortoise: useful, but what a burden!

  Protective: education as formation.
  Repressive: education as formatting.

- *Richard* 1905: smallest integer not definable in ≤ 20 words.

  Unformatted definability yields antinomy.
  Formatted definability DEFINABLE too restrictive.
  Richard with DEFINABLE: definition, but not DEFINITION.

7 — QUALUNQUISM

- **Down with formats! Direct access to « reality. »**

  Populists, libertarians: no politicians, no state!
  Analytical philosophy: don't think, ask logic.

- ***Hidden* formats: « some were more equal than others. »**

  Duce interprets free will of *l'uomo qualunque.*
  God: how to say it logically? ∞? Is God denumerable?

- **The *treasure hunt* format, e.g., *whodunits*, selects clues.**

  Shady-looking = evil: true *conjecture* for Sherlock Holmes.
  Abduction: qualunquist mistake, e.g., expensive $\Rightarrow$ better.

- **Logical formatting ensures *strong normalisation.***

  Use is limited: not all combinations allowed (typing).
  Meaning created by format through use.

8 — THE DEONTIC DIALOGUE

- ***Prussian formalism:*** **don't discuss, obey! Natural deduction.**

  $\Rightarrow, \wedge, \forall$ **: natural w.r.t. tree-like format.**

  $\vee, \exists$ **: awkward: reformating by *commutative conversions*.**

- **Linear logic 1986: too many positive, e.g.,** $\otimes$**.**

  Proof-nets: proofs with *implicit* format.

  Correctness criterion uses *switches.*

- ***Herbrand* 1930: prenex form** $Q\vec{x}\vec{y}A[\vec{x},\vec{y}]$**.**

  Unformatted solution $\vec{Y}[\vec{x}]$ **s.t.** $A[\vec{x},\vec{Y}[\vec{x}]]$**.**

  A posteriori: $x \rightsquigarrow f(y)$ **ensures** y **does not depend on** x**.**

- **Logical dialogue: either follow rule or questions it.**

  Alethic: (= truth) the winner is right. Negation *refutes.*

  Deontic: (duties), no winner. Negation *recuses.*« Objection, Your Honor! »

9 — VEHICLES AND GABARITS

- ***Vehicle*** $\mathcal{V}$**: the upper part of the proof-net.**

  Define: $p_A(x) := p_{A\otimes B}(\mathtt{l}\cdot x), p_B(x) := p_{A\otimes B}(\mathtt{r}\cdot x)$.
  Identity links: **binary stars** $[\![\, p_A(x), p_{\sim A}(x) \,]\!]$.

- ***Ordeal*** $\mathcal{O}$***:*** $q_A(x) := p_A(\mathtt{g}\cdot x)$; **the** q_A **disjoint.**

  Literals: $[\![\, p_A(x), q_A(x) \,]\!]$.
  $\otimes$: $[\![\, q_{A\otimes B}(x), q_A(x), q_B(x) \,]\!]$.
  $\parr_L$: $[\![\, q_{A\parr B}(x), q_A(x) \,]\!]$ **and** $[\![\, q_B(x) \,]\!]$.
  $\parr_R$: $[\![\, q_{A\parr B}(x), q_B(x) \,]\!]$ **and** $[\![\, q_A(x) \,]\!]$.
  Conclusion: $[\![\, q_A(x), p_A(x) \,]\!]$.

- **Correctness (= completeness): paint** $\mathcal{V}$ **in green:** $\mathcal{V}$.

  Strong normalisation of $\mathcal{V}\cup\mathcal{O}$ **with**
  Normal form: $[\![\, A_1(x), \ldots, A_n(x) \,]\!]$.

- ***Gabarit:*** **finite set of ordeals induced by switchings.**

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III — THIRD LIGHT: CERTAINTY

10 — A MISFIRE: CONSISTENCY

- **XIXth century antinomies:** ***Burali-Forti*** **1897,** ***Russell*** **1902.**

  Formalism misses nothing; can only ***overprove.***
  Hilbert 1925: fix doubts by reducing certainty to:
  Consistency: $A, \neg A$ **not both provable.**

- ***Incompleteness (I):*** **system « misses » Gödel sentence.**

  Status of limitations: consistency $\not\Rightarrow$ ***confidence.***
  Paraconsistency: complete loss of confidence.

- ***Incompleteness (II):*** **even consistency is out of reach.**

  XXth century doubts: after 1931 proofs no longer prove.

- ***Transcendentalism*****: conditions of possibility of** ***prediction.***

  Doubts: failure of ***apodictic*****, irrefragable certainty.**
  Sufficient conditions: not necessary ($\neq$ **Kant).**
  Epidictic: reasonable, limited, certainty.

11 — THE BHK APORIA

- **General problem with deontic dialogues:**

  Infinite: too many ordeals: infinite gabarits?
  Dissension: which test is dismissed? ***Logical complexity***.
  Explains logical law, but cannot ***enforce*** it.

- **Similar problem with *Brouwer-Heyting-Kolmogoroff* 1930.**

  Proofs as « functions »: typically, $n \rightsquigarrow \text{proof of A[n]}$.
  Known in advance: $n \rightsquigarrow \text{verification of A[n].}$
  Certainty: how come that the trivial function is a proof?

- **Failure of *realism* (= fetichism of reality).**

  Subsequence cannot explain consequence.

  Incompleteness $\neq$ non-euclidian geometries.

- **Can be ascribed to *dissociation* object/subject. *Derealism.***

  Épure: object + views of the object, vehicle + gabarit.

12 — RIGHTS AND DUTIES

- **Finiteness of gabarits fails for *second order* (i.e., $\mathbb{N}$).**

  Gabarits finite as *virtual* checking: schizophrenia right/duty.
  Right: production, construction, analytic, meaningless.
  Duty: utilisation, destruction, synthetic, meaningful.

- **Cut-free proofs: analytic, duty-free.**

  $A \vdash A$: utilisation stronger than production.
  Dinaturality: hexagonal diagrams.
  3-valued logic: not false (side wheels).

- **Cut rule: consume our production. Problem of prediction.**

  Hexagons don't compose.
  Balanced? Gabarits for $A, \sim A$ need not match.

- **Everyting analytic, meaningless, but for:**

  Transcendental hypothesis: right and duties do match.

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THE END