LUDICS I : INTRODUCTION

Jean-Yves Girard

Le jeu de Bonneteau

$\boxed{\heartsuit}$ $\boxed{\spadesuit}$ $\boxed{\diamondsuit}$

Question : where is the meaning ?

Le jeu de Bonneteau

♡ ♠ ♢

Syntaxe Sémantique

Le jeu de Bonneteau

$\boxed{\heartsuit}$	$\boxed{\spadesuit}$	$\boxed{\diamondsuit}$
Syntaxe	Sémantique	Méta
$\tau\acute{\alpha}\xi\iota\varsigma$	$\sigma\tilde{\eta}\mu\alpha$	$\mu\acute{\epsilon}\tau\alpha$

The meaning of meaning

- Meaning is not to be found in « *semantics* ».

  Gesticulation : interprete the *Broccoli* axiom $(A \diamondsuit B) \Rightarrow ((A \diamondsuit A) \diamondsuit B)$ by semantics $a \heartsuit b \leqslant (a \heartsuit a) \heartsuit b$... construct the *free Broccolo*.

  Treason : devious interpretations (model theory)... not the point.

  Tarskism : interprete conjunction by meta-conjunction etc.[a].

- Meaning should rather be internal.

  External completeness If A is a closed Π^1 formula and A is *true*, then A is provable[b].

  Internal completeness If A is closed and Π^1 then proofs of A enjoy *subformula property*.

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[a] Tarskian truth vs. *vérité de La Palisse*.

[b] Limits *a priori* full completeness results.

Down with Tarskism !

- No schizophrenia syntax/semantics ; just plain logic, *monism*.
- Dualism replaced with homogeneous duality[a]. *Proofs vs. counterproofs*.
- Interaction corresponds to
  - Cut-elimination between $\vdash A$ and $A \vdash$.
  - Interactive proof-search.
- *Para*proofs : proofs with mistakes of logic *(invisible)*
  - Enough of them to separate points.
  - Not too many of them ; should enjoy cut-elimination.
  - Find out what visibility means.

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[a] Dialectics (???) or better *ludics*.

Analysis : time in logic

- Two *polarities*, i.e. relative parities, positive = same, negative=diverse.

  Negative : invertible logical connectives[a]. $(\wedge, \Rightarrow), \&, \parr, \top, \bot, \forall$.

  Positive : you commit youlself, but can perform *clusters* *(Andreoli)*. $(\vee), \oplus, \otimes, 0, 1, \exists$. Association of connectives of same polarity[b].

- Clock incremented by alternation of polarities[c].

- Paradigm extended to atoms : α is positive, no identity axiom, rather infinite non-well founded η-expansion. ! not a connective, but two steps $!N = \uparrow \sharp N$ ($\sharp N$ real exponential, negative, $\uparrow$ changes polarity).

- Restriction to sequents $\vdash \Gamma$ with at most one negative formula. Rewrite as $\vdash \Gamma$ (positive sequent) or $A \vdash \Gamma$ (negative sequent).

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[a] Going « downwards » in natural deduction.

[b] Graphical styles mnemonise association, e.g. distribution $\otimes / \oplus$.

[c] $\Phi(A, B, C) = A \& (B \oplus C)$ is not a connective.

Analysis : cut-elimination

- Replace connectives with clusters, and combine with negation. See $((L \oplus M) \otimes N)$ as $\Phi(L^\perp, M^\perp, N^\perp)$.

- Write cluster rules

$$\frac{\vdash \Lambda, P, R \quad \vdash \Lambda, Q, R}{\Phi(P^\perp, Q^\perp, R^\perp) \vdash \Lambda}\ (\Phi \vdash)$$

$$\frac{P \vdash \Gamma \quad R \vdash \Delta}{\vdash \Gamma, \Delta, \Phi(P^\perp, Q^\perp, R^\perp)}\ (l \vdash \Phi)$$

$$\frac{Q \vdash \Gamma \quad R \vdash \Delta}{\vdash \Gamma, \Delta, \Phi(P^\perp, Q^\perp, R^\perp)}\ (r \vdash \Phi)$$

  - *The* rule $\Phi \vdash$ has *2* premises, one for each rule $\vdash \Phi$
  - The right premise of $\Phi \vdash$ matches the right rule $\vdash \Phi$.

- Cut between $\Phi(P^\perp, Q^\perp, R^\perp) \vdash \Lambda$ and $\vdash \Gamma, \Delta, \Phi(P^\perp, Q^\perp, R^\perp)$ replaced with two cuts between $\vdash \Lambda, Q, R$, $\quad Q \vdash \Gamma$, $\quad R \vdash \Delta$[a].

[a] Order of cuts irrelevant : use cut-*links*, and not cut-rules !

Analysis : space in logic

- Forget formulas, remember only *locations* : locus solus.
  - Locii are *addresses* of subformulas $\xi = \langle i_1, \ldots, i_n \rangle$
  - $i_k \in \mathbb{N}$ is called a *bias*. P, Q, R correspond to $3, 4, 7$
  - the *parity* of ξ is the parity of n.
  - Locii are either disjoint *(space)* or comparable *(time)*.
  - Premises of negative rules distinguished by their *ramifications* : $\{3, 7\}$ and $\{4, 7\}$.
- Sequents become *pitchforks* $\Xi \vdash \Lambda$
  - Ξ, Λ finite sets of addresses, pairwise disjoint *(handle*, *tines)*.
  - $\sharp(\Xi) \leq 1$; $\vdash \Lambda$ is a *brush —positive—* ; $\xi \vdash \Lambda$ is *negative*.
  - The addresses in Λ have the same parity (the *parity* of the pitchfork), opposite to the parity of Ξ. $\vdash$ has both parities.

Synthesis :

- Analysis easy, synthesis difficult : *Pierre Ménard autor del Quijote*.
- Principle : *AVOID TARSKISM*[a]. This induces considerable constraints, but also considerable gains.
  - Although the paradigm is interactive, *no rule of game*, everything is permitted. Like in real life, the « rule » is a result of free interaction.
    - Possibility of subtyping.
    - Logical operations strictly associative and commutative (records).
  - Objects should be describable through their interactions ($\mathcal{T}_0$ topology). Need of additional proofs (paraproofs).
    - Real exponential cannot be uniform.
    - First-order quantification is not a connective : only makes sense as a complex.

[a] Tarskism := keeping on changing the « rule of the game »

Synthesis (cont[d])

- No quotients, no observational equivalence, just plain objects.
  - Incarnation $|\mathfrak{D}| \subset \mathfrak{D}$.
  - *Miracle of incarnation :* $\mathbf{C} \,\&\, \mathbf{D} := \mathbf{C} \cap \mathbf{D}$, but $|\mathbf{C} \,\&\, \mathbf{D}| \simeq |\mathbf{C}| \times |\mathbf{D}|$.
- No logical relations for visibility, typically no *candidats de visibilité*[a].
  - Winning
  - Parsimony
  - Uniformity

[a]e.g. totality.

Pierre Ménard

El método inicial que imaginó era relativamente sencillo. Conocer bien el español, recuperar la fe católica, guerrear contra los moros o contra el turco, olvidar la historia de Europa entre los años de 1602 y de 1918, ser *Miguel de Cervantes. Pierre Ménard estudió ese procedimiento (sé que logró un manejo bastante fiel del español del siglo diecisiete), pero lo descartó por fácil. [. . .]*

Mi complaciente precursor no rehusó la colaboración del azar : iba componiendo la obra inmortal un poco à la diable*, llevado por inercias del lenguaje y de la invención. Yo he contraído el misterioso deber de reconstruir literalmente su obra espontánea. Mi solitario juego está gobernado por dos leyes polares. La primera me permite ensayar variantes de tipo formal o psicológico ; la segunda me obliga a sacrificarlas al texto « original » y a razonar de un modo irrefutable esa aniquilación.*

J. L. Borges *Pierre Ménard autor del Quijote*, 1939.

LUDICS II : DESIGNS

Jean-Yves Girard

Ludics : Designs

- Infinite, non recursive, non well-founded proof using 3 or 4 rules[a].
  - *Cut* defined by coincidence *handle/tine* between two designs. Deterministic cut-elimination : *associativity* theorem.
  - When $\mathfrak{D}, \mathfrak{E}$ of bases $\vdash \xi$, $\xi \vdash$, form $\langle \mathfrak{D}, \mathfrak{E} \rangle$
    - $*$ either $\langle \mathfrak{D}, \mathfrak{E} \rangle = \mho$ *(converges*, orthogonality : $\mathfrak{D} \perp \mathfrak{E}$).
    - $*$ or $\langle \mathfrak{D}, \mathfrak{E} \rangle = \Omega$[b] *(diverges)*.
  - Topology generated by sets $\mathbf{C}^{\perp\perp}$ is $\mathcal{T}_0$: *separation* theorem. $\mathfrak{D}$ is a *subdesign* of $\mathfrak{E}$ iff $\mathfrak{E} \in \mathfrak{D}^{\perp\perp}$, notation $\mathfrak{D} \leq \mathfrak{E}$.
  - *Closure* principle : reduction to closed systems *(Associativity + Separation)*. Yields basic *adjunction* $\langle \mathfrak{F}(\mathfrak{D}), \mathfrak{E} \rangle = \langle \mathfrak{F}, \mathfrak{D} \otimes \mathfrak{E} \rangle$.
  - Normalisation increasing, commutes to compatible intersections[c].

[a] To be seen as symmetrisation of pure λ-calculus, $\ll$ non-leaking $\gg$ π-calculus.
[b] Notation reminiscent of λ-calculus.
[c] Infinite pullbacks, stability etc.

Light cavalry

Bias : an integer $i \in \mathbb{N}$ (index of « immediate » subformula).

Address : sequence $\xi = \langle i_1, \ldots, i_n \rangle$ of biases (location of subformula).

- The *parity* of ξ is the parity of n.
- Locii are either disjoint *(space)* or comparable *(time)*.

Ramification : finite set $I \in \wp_f(\mathbb{N})$ of biases ; $\xi * I$ is short for $\{\xi * i; i \in I\}$.

Pitchfork : expression $\Xi \vdash \Lambda$; Ξ *handle*, Λ *tines*.

- Ξ, Λ finite sets of addresses, pairwise disjoint ; $\sharp(\Xi) \leq 1$[a].
  - $\vdash \Lambda$ is a *brush —positive—*, i.e. active.
  - $\xi \vdash \Lambda$ is *negative*, passive.
  - The addresses in Λ have the same parity (the *parity* of the pitchfork), opposite to the parity of Ξ. $\vdash$ has both parities.

[a] Sort of mirror intutionistic sequents : IL is based on negative operations ($\Rightarrow, \wedge, \forall$).

Designs *(dessins*, 3 rules)

Give up : too late !

$$\frac{}{\vdash \Lambda}\,\maltese$$

Positive rule : [a] I is a ramification, for $i \in I$ the Λ_i are pairwise disjoint and included in Λ : one can apply the rule (finite, one premise for each $i \in I$)

$$\frac{\ldots, \xi * i \vdash \Lambda_i, \ldots}{\vdash \Lambda, \xi}\,(\xi, I)$$

Negative rule : [b] $\mathcal{N}$ is a set of ramifications, for all $I \in \mathcal{N}$ $\Lambda_I \subset \Lambda$: one can apply the rule : perhaps infinite, one premise for each $I \in \mathcal{N}$)

$$\frac{\ldots, \vdash \Lambda_I, \xi * I, \ldots}{\xi \vdash \Lambda}\,(\xi, \mathcal{N})$$

- No assumption of finiteness, well-foundedness, recursivity etc.

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[a] Abstract form of $\oplus \otimes (.)^{\perp}$.

[b] Abstract form of $\& \parr (.)^{\perp}$.

Example : the Fax

- If ξ and ξ' are disjoint and of opposite parities, then one defines $\mathfrak{Fax}_{\xi,\xi'}$, a design of basis $\xi \vdash \xi'$.

$$
\dfrac{\ldots \qquad \dfrac{\dfrac{\vdots\, \mathfrak{Fax}_{\xi' * i, \xi * i}}{\ldots \xi' * i \vdash \xi * i \ldots}}{\vdash \xi', \xi * I}\,(\xi', I) \qquad \ldots}{\xi \vdash \xi'}\,(\xi, \wp_f(\mathbb{N}))
$$

- In fact $2^{\aleph_0}$ faxes, one for each isomorphism from ξ to ξ'.
- Corresponds to identity axioms $A \vdash A$ (spiritual notation) or $\Phi(A) \vdash \Psi(A)$ (locative notation, more accurate).

The Jesuit

- Copies *perinde ac cadaver*, in the immediate future ; $\mathfrak{Jes}^-_\xi$ is

$$\dfrac{\dfrac{\dfrac{\vdots\ \mathfrak{Jes}^-_{\xi*0*0}}{\xi*0*0\vdash}}{\vdash\xi*0}(\xi,\{0\})}{\xi\vdash}(\xi,\{\{0\}\})$$

- Positive Jesuit exists as well.
- $2^{\aleph_0}$ Jesuits (one for each iso of ξ into some $\xi*i$).
- Jesuit interprets a form of *tertium non datur*, i.e. $A\oplus\uparrow A^\perp$, (here $\oplus$ is strongly locative).

Designs (*dessins*, 4 rules)

- Variant useful for theoretical purposes, an additional rule is added, and the negative rule is restricted to the *full* case.

Failure : please wait !

$$\frac{}{\vdash \Lambda}\,\Omega$$

Negative rule : $\mathcal{N} = \wp_f(\mathbb{N})$

- Failure corresponds to absent premises in negative rules.
- A design cannot end with failure.
- Failure corresponds indeed to infinite loops.
- « Failure » is *partial* (undefined, inexistent), « Give up » is *total*. *Knowing that not* (giveup, Ctrl-C) $\neq$ *Not knowing* (failure, infinite loop)[a].

[a] Founding mistake of junk « logics ».

Designs *(desseins)*

- To any finite branch in a *dessin* associate a sequence of adresses and ramifications. In $\mathfrak{Fax}_{\xi,\xi'}$ this yields all sequences *(chronicles)*

  $\langle(\xi, I_1), (\xi', I_1), (\xi' * i_1, I_2), (\xi * i_1, I_2), (\xi * i_1 * i_2, I_3), (\xi' * i_1 * i_2, I_3),$
  $(\xi' * i_1 * i_2 * i_3, I_4), \ldots\rangle$ with $i_1 \in I_1, i_2 \in I_2, i_3 \in I_3 \ldots$.

  The (ξ, I) are *proper actions*, they are positive or negative, polarity alternates. There is an *unproper action*, $\mho$. The *dessein* corresponding to $\mathfrak{D}$ is the set of its chronicles.

- *Desseins* can be defined independently as certain sets of chronicles. They are the real objects, *dessins* being nothing but a convenient representation : the splitting of contexts is not indicated in *desseins*. *Location* post mortem *of contexts.*

- Although not intrinsic, *dessins* can in practice be confused with *desseins* : use the ambiguous expression *designs*.

Chronicles

Actions : fix a basis $\Xi \vdash \Lambda$. A *proper action* is a pair $\kappa = (\xi, I)$ of an address (the *focus*) and a ramification. The polarity of (ξ, I) relates the parity of ξ and the parity of the basis. The *unproper action* $\mho$ is positive.

Chronicles : a *chronicle* $\mathfrak{c}$ of basis $\Xi \vdash \Lambda$, and *duration* $\natural\mathfrak{c} = n$ is a sequence of actions $\langle \kappa_1, \ldots, \kappa_n \rangle$ such that :

Alternation : polarities alternate and start with the polarity of basis.

Give up : all actions are proper except perhaps the last one ; chronicle $\mathfrak{c}$ is *proper* when it does not use $\mho$.

Negative actions : a negative focus ξ_k is chosen in Ξ (if $k = 0$ and the basis is negative), or in $\xi_{k-1} * I_{k-1}$.

Positive actions : a positive focus ξ_k is chosen either in Λ, or in one $\xi_m * I_m$, where (ξ_m, I_m) is a previous negative action.

Destruction of focuses : focuses can only be used once in $\mathfrak{c}$[a].

[a]No longer true in *multiple theory*, dealing with exponenials (not considered here).

Coherence

Coherence : chronicles $\mathfrak{c}, \mathfrak{d}$ are *coherent* when

- ***Comparability :*** either one extends the other, or they first differ on negative actions.
- ***Separation :*** if $\mathfrak{c}, \mathfrak{d}$ first differ on negative actions with distinct focuses, all further focuses are distinct[a].

Designs : a design $\mathfrak{D}$ of basis $\Xi \vdash \Lambda$ is a set of chronicles such that :

- ***Tree :*** $\mathfrak{D}$ closed under restriction and non-empty.
- ***Coherence :*** elements of $\mathfrak{D}$ are pairwise coherent.
- ***Positivity :*** if $\mathfrak{d} \in \mathfrak{D}$ is maximal, either $\mathfrak{d}$ is empty and the basis is negative, or the last action of $\mathfrak{d}$ is positive.

Parity, polarity : $\mathfrak{D}$ has the parity, polarity of its basis.

[a] Implicit splitting of contexts : focuses above distinct premises of $\otimes$ are distinct.

LUDICS III : BEHAVIOURS

Jean-Yves Girard

Nets

- A *cut* is a coincidence *handle/tine* between two bases of designs. More generally define *nets* : incidence graph *connected/acyclic*. A net has a base, the non-shared locii. Closed net when basis is $\vdash$. Main design is

  Positive basis : the only positive design of the net.

  Negative basis $\xi \vdash \Lambda$: the design with a basis $\xi \vdash \Delta$.

- Normalisation is deterministic :

  - Done starting with the conclusion[a], « stream-style ».
  - Always converges when basis negative.
  - When basis positive may diverge (or yield the non-design Ω). Divergence due to absence of premise in negative rule, or to infinite sequence of conversions ; 4-rules presentation symmetrical in $\mho/\Omega$, but for infinite loops which yield Ω.

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[a] See *Proof theory and logical complexity*.

Normalisation

Positive case : conclusion brush, main design $\mathfrak{D}$ with last rule R :

Give up : if $R = \mho$, normalise the net into $\mho$.

Main case : $R = (\xi, I)$ and ξ is « cut » with $\xi \vdash \Pi$, basis of $\mathfrak{E}$ with last rule $S = (\xi, \mathcal{N})$. Two subcases :

Conversion : if $I \in \mathcal{S}$, replace $\mathfrak{D}$ by designs above (ξ, I) and $\mathfrak{E}$ by design above premise of index I of S, yielding cuts on the $\xi * i$, for each $i \in I$.

Immediate failure : if $I \notin \mathcal{N}$, normalisation fails.

Positive commutation : $R = (\xi, I)$ and ξ is not cut ; choose $i \in I$ and $\vdash \Lambda, \xi$ by premise $\xi * i \vdash \Lambda$ and apply procedure, (which converges) and yields $\mathfrak{E}_i$; apply rule (ξ, I) to the $\mathfrak{E}_i$.

Negative case : principal pitchfork $\xi \vdash \Lambda$ obtained by $(\xi, \mathcal{N})$. For $I \in \mathcal{N}$ replace $\xi \vdash \Lambda$ by $\vdash \Lambda, \xi * I$ and apply procedure which converges for $I \in \mathcal{M} \subset \mathcal{N}$ yielding $\mathfrak{E}_I$) : apply rule $(\xi, \mathcal{M})$ to the $\mathfrak{E}_I$.

Examples

- Let $\mathfrak{D}$ be a design of basis $\vdash \xi$. Then the net $\{\mathfrak{D}, \mathfrak{Fax}_{\xi,\xi'}\}$ normalises into $\mathfrak{D}' = (\mathfrak{D})$, where Φ is the isomorphism *(delocation)* $\Phi(\xi * \sigma) = \xi' * \sigma$.

- Let $\mho^+, \mho^-$ be the designs *(positive, negative suicides)*

$$\frac{}{\vdash \Lambda}\ \mho$$

$$\frac{\dots \quad \dfrac{}{\vdash \Lambda_I, \xi * I}\ \mho \quad , \dots , \quad \dfrac{}{\vdash \Lambda_J, \xi * J}\ \mho \quad \dots}{\xi \vdash \Lambda}\ (\xi, \wp_f(\mathbb{N}))$$

Every cut with a suicide normalizes into a suicide of the right polarity[a].

[a]The suicide is the paradigm of an *invisible design* ; the suicide is a total object, the non-design Ω being partial. The suicide is maximum w.r.t. $\leq$.

Orthogonality

- Designs $\mathfrak{D}, \mathfrak{E}$ of bases $\vdash \xi$ and $\xi \vdash$ are *orthogonal* when normalisation of the closed net $\{\mathfrak{D}, \mathfrak{E}\}$ converges, notation $\langle \mathfrak{D}, \mathfrak{E} \rangle = \mho$, or $\mathfrak{D} \perp \mathfrak{E}$.
- Topology on designs of a given basis generated by closed sets $\mathfrak{E}^{\perp}$.
- Weak separation : topology is $\mathcal{T}_0$[a].
- Equivalently $\mathfrak{D} \leq \mathfrak{E}$ defined as $\mathfrak{E} \in \mathfrak{D}^{\perp\perp}$ is an order relation.
- $\mathfrak{D} \leq \mathfrak{E}$ corresponds to an ordering of positive rules.

$$\Omega < (\xi, I) < \mho$$

- Normalisation increasing w.r.t. $\leq$ and commutes to *compatible* intersections, i.e. $\mathfrak{D} \cap \mathfrak{D}'$ when $\mathfrak{D} \cup \mathfrak{D}'$ is a design.

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[a]Means that $\mathfrak{D} \neq \mathfrak{D}' \Rightarrow \exists \mathfrak{E} \quad \langle \mathfrak{D}, \mathfrak{E} \rangle \neq \langle \mathfrak{D}', \mathfrak{E} \rangle$.

Associativity

- Normalisation is *associative* : if net $\mathfrak{R} = \{\mathfrak{D}_o, \ldots, \mathfrak{D}_n\}$ normalises into $\mathfrak{D}$ and $\{\mathfrak{E}_o, \ldots, \mathfrak{E}_m\}$ normalises into $\mathfrak{D}_0$, then $\{\mathfrak{E}_o, \ldots, \mathfrak{E}_m, \mathfrak{D}_1, \ldots, \mathfrak{D}_m\}$ normalises into $\mathfrak{D}$.

- Associativity still works when $\mathfrak{D}_0$ (resp. $\mathfrak{D}$) is the non-design Ω.

- Principle of *closure* : enough to consider closed nets. Typically, the normal form $\mathfrak{F}$ of a cut between $\mathfrak{D}$ (basis $\vdash \xi$) and $\mathfrak{E}$ (basis $\xi \vdash \lambda$) is determined by the result of all cuts between $\mathfrak{F}$ and some $\mathfrak{E}'$ (basis $\lambda \vdash$) (*separation*). By *associativity* this result is nothing but the normal form of the closed net $\{\mathfrak{D}, \mathfrak{E}, \mathfrak{E}'\}$.

- The principle of closure is behind the essential adjonction of logic

$$\langle \mathfrak{F}(\mathfrak{D}), \mathfrak{E} \rangle = \langle \mathfrak{F}, \mathfrak{D} \otimes \mathfrak{E} \rangle$$

.

Ludics : behaviours

- Set of designs equal to its biorthogonal. Corresponds to the idea of a formula, a proposition, a type etc.
  - Closed under superdesigns and compatible intersections.
  - Always non-empty, contains suicide, $\mho^+$ or $\mho^-$ depending on polarity.
  - Existence of *incarnation*. $|\mathfrak{D}|_{\mathbf{C}} = \bigcap\{\mathfrak{E}; \mathfrak{E} \subset \mathfrak{D}, \mathfrak{E} \in \mathbf{C}\}$.
  - Completeness is $\mathbf{C} = \mathbf{C}^{\perp\perp}$!!! Purely internal.
  - Existential types badly incomplete... the mysterious invisible integers.

Behaviours

Behaviours : a behaviour is a set $\mathbf{C}$ of designs (of a given basis) equal to its biorthogonal. A behaviour has the polarity and parity of its basis. Immediate properties of behaviours :

Non-emptiness : a behaviour is never empty (contains the suicide).

Monotonicity : if $\mathfrak{D} \in \mathbf{C}$ and $\mathfrak{D} \leq \mathfrak{E}$, then $\mathfrak{E} \in \mathbf{C}$.

Stability : if $\mathfrak{D}_k \in \mathbf{C}$ and $\bigcup_k \mathfrak{D}_k$ is a design, then $\bigcap_k \mathfrak{D}_k \in \mathbf{C}$.

Incarnation : if $\mathfrak{D} \in \mathbf{C}$ there is a smallest $\mathfrak{E} \subset \mathfrak{D}$ (w.r.t. inclusion) such that and $\mathfrak{E} \in \mathbf{C}$, notation $\mathfrak{E} = |\mathfrak{D}|_{\mathbf{C}}$. A *useful part* of $\mathbf{C}$ is a subset $\mathbf{D} \subset \mathbf{C}$ such that $\{|\mathfrak{D}|; \mathfrak{D} \in \mathbf{D}\} = \{|\mathfrak{D}|; \mathfrak{D} \in \mathbf{C}\}$, typically the *incarnation* of $\mathbf{C}$ i.e. the set $|\mathbf{C}| = \{|\mathfrak{D}|; \mathfrak{D} \in \mathbf{C}\}$.

Behaviours vs. Games

- Games are a *useful intuition* not a valuable mathematic approach[a].
- Only solution : *no rule* (more precisely, a lax general rule for everybody). Like in real life, rules are themselves part of the interaction.
- Ludics cannot be seen as a sort of game : the first player (positive side) can always play $(\xi, \emptyset)$, and game over.
- However behaviours are *sort of games* : the rule of $\mathbf{C}$ is given by $\mathbf{C}^{\perp}$ and vice-versa, $\mathbf{C}$ is the rule for $\mathbf{C}^{\perp}$. No referree, no junk games.
- When in a behaviour, a design is a *sort of strategy*. Indeed the strategies induced by $\mathfrak{D}, \mathfrak{E}$ are equal iff $\mathfrak{D}, \mathfrak{E}$ have the same incarnation in $\mathbf{C}$[b] : incarnation is the passage to games. But it would be a serious want of taste to replace a design by its incarnation, e.g. no subtyping !

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[a] Why not : « You propose a formula, I say YES or NO. » ?

[b] The existence of the incarnation is the strongest result of ludics.

LUDICS IV : ADDITIVE CONNECTIVES

Jean-Yves Girard

Logical connectives

- Build new behaviours from extant ones, *the noble art of socialisation*.

Delocation : image of behaviour under isomorphism.

Shift : Pure change of polarity $\uparrow\uparrow \mathbf{C} \neq \mathbf{C}$.

Intersection : Yields strictly associative/commutative version of $\&$.
Every negative behaviour is the $\ll$ with $\gg$ of its connected components. Intersection also defined on positive behaviours, yields new connectives.

Multiplicatives : Strictly commutative/associative version of $\otimes$.
Dependent type generalisation, corresponding to prime decomposition.

Exponentials : Need *multiple* and *uniform* features.

Locative phenomenons

- Usual logic is *spiritual* : formulas refer to an external reality. Due to destruction of the paradigm syntax/semantics, spirituality fails. Locative phenomenons existed beforehand but not recognised as meaningful[a].
- We can refuse it and systematically delocate, sort of general α-conversion. But the phenomenon exists anyway !
- Or we accept it ; as we shall see *logical interference* is violent : A, B may be true but $A \otimes B$ false, or A, B may be false but $A \otimes B$ true.
- Like it or nor, second-order quantification is locative (intersection of behaviours sharing a location), hence spiritual logic (any kind of model : Krpike, topological, phase...) mistreats them. Typically, ludics establishes that every second-order proposition has a prenex form[b].

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[a] For instance a name of variable is a location : in $\sigma \wedge \sigma \Rightarrow \tau$, we shall use two distinct variables of type σ, but the two σ have distinct locations.

[b] Indeed not only equivalent to its prenex form, but simply... equal.

Delocation

Delocation : injective function Φ from sublocii of ξ to sublocii of ξ' s.t. $\Phi(\xi) = \xi'$ and for all σ there is a function Φ_σ s.t. $\Phi(\sigma * i) = \Phi(\sigma) * \Phi_\sigma(j)$.

A delocation is positive or negative, depending on ξ, ξ'[a].Easy to define delocation of chronicles, designs, behaviours. Observe that $\Phi(\mathbf{C})$ is the biorthogonal of $\mathbf{D} = \{\Phi(\mathfrak{D}); \mathfrak{D} \in \mathbf{C}\}$; $\mathbf{D}$ is a useful part of $\Phi(\mathbf{C})$.

Propositional variables : α stands for the unknown behaviour

Polarity : we only consider positive behaviours. If you want to quantify over negative behaviours, just use $\alpha^\perp$.

Location : locate α at basis $\vdash \langle\rangle$ and handle occurences by delocations : $\alpha \vdash \alpha$ (spiritual style) is interpreted by $\Phi(\mathbf{C}) \vdash \Psi(\mathbf{C})$, where $\mathbf{C}$ is based on $\vdash \langle\rangle$ and Φ, Ψ are appropriate delocations.

« ***New*** » **:** use delocation to « create » fresh addresses.

[a]The delocation at work in the Fax or the Jesuit is negative.

The shift

Shift : artifacts of basis $\vdash \xi * i$ (resp. $\xi * i \vdash$) are shifted into artifacts of basis $\xi \vdash$ (resp. $\vdash \xi$). The shift swaps both parity and polarity.

Chronicles : $\uparrow \mathfrak{c} := (\xi, \{i\}) * \mathfrak{c}$.

Designs : $\uparrow \mathfrak{D} := \{\uparrow \mathfrak{c}; \mathfrak{c} \in \mathfrak{D}\}$.

Behaviours : $\uparrow \mathbf{C} := \{\uparrow \mathfrak{D}; \mathfrak{D} \in \mathbf{C}\}^{\perp\perp}$. A behaviour of the form $\uparrow \mathbf{C}$ is *prime*.

Results : unary connective that does nothing but a change of polarity ; non-involutive[a]

Negative case : $\uparrow \mathbf{C} = \{\uparrow \mathfrak{D}^{+}; \mathfrak{D} \in \mathbf{C}\} \cup \{\mho\}$.

Positive case : $\{\uparrow \mathfrak{D}; \mathfrak{D} \in \mathbf{C}\}$ is a useful part of $\mathbf{C}$.

Negation : $(\uparrow \mathbf{C})^{\perp} = \uparrow (\mathbf{C}^{\perp})$.

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[a] Anderssen opening a2-a3 in Chess.

The additive conjunction

- The intersection *With* $\&_p \mathbf{C}_p$ of any family of negative behaviours (basis $\xi \vdash$) is still a negative behaviour.
- *Strictly* associative, commutative with neutral element $\top$ (the set of all designs) and absorbing element $\{\mho^-\}$.
- Dual operation *Plus* $\oplus_p \mathbf{C}_p$ on positive behaviours : strictly associative, commutative, with neutral $0 = \{\mho^+\}$ and absorber $\{\mho^-\}^\perp$, the set of all positive designs[a].
- « With » corresponds to three operations

  Additive linear conjunction : modulo delocation.

  Second-order quantification : $\forall \alpha \mathbf{C}[\alpha]$; opportunist connective.

  Intersection types : usually excluded from logic, since non-spiritual.

[a]Greatest positive design, also equal to $\emptyset^\perp$.

Linear additives

- Linear logic is spiritual (refers to phase semantics).
- Usual (spiritual) *with* defined as $\Phi(\mathbf{C}) \,\&\, \Psi(\mathbf{D})$, with $\Phi(\xi * i * \sigma) = \xi * 2i * \sigma$, $\Psi(\xi * i * \sigma) = \xi * 2i + 1 * \sigma$.
- Usual *plus* defined as $\Phi(\mathbf{C}) \oplus \Psi(\mathbf{D})$.
- Associativity, commutativity neutrality etc. only up to canonical isomorphisms. A real novelty : *absorbers* : $\mathbf{C} \,\&\, \forall\alpha.\alpha^{\perp} \simeq \forall\alpha.\alpha^{\perp}$.
- Additive spirituality amounts to *disjunction* :
  - $\mathbf{C}, \mathbf{D}$ positive are *disjoint* when $\mathbf{C} \cap \mathbf{D} = 0$.
  - $\mathbf{C}, \mathbf{D}$ negative are *disjoint* when $\mathbf{C}^{\perp}, \mathbf{D}^{\perp}$ are disjoint.

The miracle of *encarnación*

Subtyping : if $\mathbf{C} \subset \mathbf{D}$ and $\mathfrak{D} \in \mathbf{C}$, then $|\mathfrak{D}|_{\mathbf{C}} \supset |\mathfrak{D}|_{\mathbf{D}}$

Negative case : ▸ $\mathbf{C} \,\&\, \mathbf{D} \subset \mathbf{C}$.

- $|\mathfrak{D}|_{\mathbf{C}\&\mathbf{D}} = |\mathfrak{D}|_{\mathbf{C}} \cup |\mathfrak{D}|_{\mathbf{D}}$.
- If $\mathbf{C}, \mathbf{D}$ are disjoint, the union is disjoint[a]

$$|\mathbf{C} \,\&\, \mathbf{D}| \simeq |\mathbf{C}| \times |\mathbf{D}|$$

- Spiritual conjunction is both an intersection and a product !

Positive case : ▸ $\mathbf{C} \subset \mathbf{C} \oplus \mathbf{D}$.

- $|\mathfrak{D}|_{\mathbf{C}\oplus\mathbf{D}} \subset |\mathfrak{D}|_{\mathbf{C}}$.
- If $\mathbf{C}, \mathbf{D}$ disjoint, $|\mathfrak{D}|_{\mathbf{C}\oplus\mathbf{D}} = |\mathfrak{D}|_{\mathbf{C}}$; moreover $\mathbf{C} \oplus \mathbf{D} = \mathbf{C} \cup \mathbf{D}$.

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[a] *Morally* disjoint : the empty chronicle belongs to both.

Completeness

Limitations : completeness is limited to *spiritual operations*, typically delocalised $\oplus$ and $\&$ ($\mathbf{C}, \mathbf{D}$ disjoint).

General form : give a *presentation* of behaviour $\mathbf{C}$ as $\mathbf{C}'^{\perp\perp}$; show that if $\mathfrak{D} \in \mathbf{C}$ is incarnated and visible, then $\mathfrak{D} \in \mathbf{C}'$.

Conjunction : if $\mathfrak{D} \in \mathbf{C} \,\&\, \mathbf{D}$, then $\mathfrak{D}$ can be written as $\mathfrak{D}_1 \cup \mathfrak{D}_2$. Unicity in case $\mathfrak{D}$ is incarnated[a].

Disjunction : present $\mathbf{C} \oplus \mathbf{D}$ by means of $\mathbf{C} \cup \mathbf{D}$; the behaviour equal to its presentation : $\mathbf{C} \oplus \mathbf{D} = \mathbf{C} \cup \mathbf{D}$. If $\mathfrak{D} \in \mathbf{C} \oplus \mathbf{D}$ is visible, then it cannot belong to both.

Final proof : we can produce the last rule ($\&, \oplus_l, \oplus_r$) and then proceed with premises.

---

[a]Nothing important in terms of presentation, since *with* is negative.

Additive decomposition

Positive case : assume that $\mathbf{C}$ is positive

- $\mathbf{C}$ is *connected* if not a non-trivial disjoint union (delocated plus).
- Define $[\mathbf{C}]_I \subset \mathbf{C}$ by : $\mathfrak{D} \in [\mathbf{C}]_I$ iff $\mathfrak{D} = \mho^+$ or $\mathfrak{D}$ « begins with » (ξ, I).
- $\mathbf{C}$ is the « plus » (union) of its *connected components* (unique decomposition) : $\mathbf{C} = \cup\{[\mathbf{C}]_I; \mathbf{C}_I \neq 0\}$

Negative case : assume that $\mathbf{C}$ is negative

- Define $[\mathbf{C}]_I = ([\mathbf{C}^\perp]_I)^\perp$:
- **C** is the « with » its *connected components* (unique decomposition) : $\mathbf{C} = \&\{[\mathbf{C}]_I; \mathbf{C}_I \neq \top\}$
- $|\mathbf{C}_I|$ corresponds to *slices* of index I in additive box.The set $\{I; \mathbf{C}_I \neq \top\}$ is the *arity* of the box.

Locative[a] aspects

- Neutral element $\top$ incarnated by sole design

$$\frac{}{\xi \vdash}\ (\xi, \emptyset)$$

- Difference $\mathbf{C} \,\&\, \mathbf{C} = \mathbf{C}$ vs. $\mathbf{C}' \,\&\, \mathbf{C}''$ which is sort of Cartesian product.
- In general « *intersection types* ». Completeness not expected.
- Projections available : split set $\wp_p(\mathbb{N})$ in two « halves ». Projection can either be the identity (again subtyping) or be *reifyed* (coertion), by means of incarnation. $\pi_{\mathcal{N}}\pi_{\mathcal{M}} = \pi_{\mathcal{N}\cap\mathcal{M}}$. Coertions are not functions, unless we delocalise source/target, in which case they are *partial faxes*.
- *Cointersection types*, not equal to unions (again completeness is irrealistic).

[a] Here : object-oriented.

Inter and Cointer

- The intersection of positive behaviours yields new connective *inter*, dual *cointer*, $\bigcap$, $\biguplus$.
- Associative, commutative, neutral $\mho^{-\perp}$[a], absorber 0.
- $\uparrow \&_p \mathbf{C}_p = \bigcap_p \uparrow \mathbf{C}_p$, etc.
- Universal quantifier opportunist : $\forall\alpha\mathbf{C}[\alpha]$ also defined when $\mathbf{C}[\alpha]$ is positive (major change w.r.t. extant theory).
- Imagine other quantifiers (on subtypes of a given type etc.)... only limit is good taste !

[a] Greatest positive behaviour, made of all designs. Can be seen as $\exists\alpha\alpha$ or $\emptyset^{\perp}$.

LUDICS V : MULTIPLICATIVE CONNECTIVES

Jean-Yves Girard

The multiplicative conjunction

- If the $\mathfrak{D}_p$ are designs of basis $\vdash \xi$, one defines the design $\mathfrak{E} = \bigotimes_p \mathfrak{D}_p$:
  - If any of the $\mathfrak{D}_p$ equals $\mho^+$, $\mathfrak{E} = \mho^+$.
  - Otherwise let (ξ, I_p) be the respective first actions of the $\mathfrak{D}_p$. Then
    * If the I_p are not pairwise disjoint, or if union infinite, then $\mathfrak{E} = \mho^-$.
    * Otherwise $\mathfrak{E}$ starts with action $(\xi, I) = (\xi, \bigcup_p I_p)$; its non-empty chronicles are the $(\xi, I) * \mathfrak{c}$ such that for some (unique) p $(\xi, I_p) * \mathfrak{c} \in \mathbf{C}_p$.
- Let $\mathbf{C}_p$ be positive behaviours. Define the set $\bigodot_p \mathbf{C}_p$ of all tensors $\bigotimes_p \mathfrak{D}_p$, where $\mathfrak{D}_p \in \mathbf{C}_p$ for all p.
- The *tensor product* $\bigotimes_p \mathbf{C}_p$ of the $\mathbf{C}_p$ is the biorthogonal of $\bigodot_p \mathbf{C}_p$; empty tensor product is noted $\mathbf{1}$.
- The connective *Par* is defined by duality as well as the constant $\perp$.

Basic multiplicative theory

- $\otimes$ is *strictly* associative and commutative, with neutral 1 and absorber 0[a].
- $\otimes$ *strictly* distributes over $\oplus$.
- Dual results for $⅋, \perp, \&$.
- $[\bigotimes_p \mathbf{C}_p]_I = \bigoplus_{I=\sum I_p} \bigotimes [\mathbf{C}_p]_{I_p}$
- Observe that $1 \oplus 1 = 1$; cannot be used for booleans, because 1 cannot be delocated (empty ramification). Use the non-isomorphic $!\top$ instead[b].

[a] The tensor of all behaviours !

[b] The facts that $1 \oplus 1 = 1$ and $!\top \neq 1$ are the only « mistakes » detected by ludics in linear logic

Linear multiplicatives

- Usual (spiritual) *tensor* defined as $\Phi(\mathbf{C}) \otimes \Psi(\mathbf{D})$.
- Usual *par* defined as $\Phi(\mathbf{C}) \parr \Psi(\mathbf{D})$.
- Associativity, commutativity neutrality etc. only up to canonical isomorphisms.
- Multiplicative spirituality amounts to *relative primality* :
  - the $\mathbf{C}_p$ (positive) are *relatively prime* when for all ramification I there is at most one decomposition $I = \bigcup_p I_p$ with $[\mathbf{C}]_{I_p} \neq 0$.
  - the $\mathbf{C}_p$ (negative) are *relatively prime* when their negations are r.p.

The main adjunction

Application : let $\mathfrak{C}, \mathfrak{F}$ of respective bases $\vdash \xi$, $\xi \vdash$. Define design $\mathfrak{F}(\mathfrak{C})$ of basis $\xi \vdash$ as the unique solution of adjunction $\langle \mathfrak{F}(\mathfrak{C}), \mathfrak{D} \rangle = \langle \mathfrak{F}, \mathfrak{C} \otimes \mathfrak{D} \rangle$ (for all $\mathfrak{D}$ positive of basis $\vdash \xi$).

Par : if $\mathbf{C}, \mathbf{D}$ of basis ξ are negative and *relatively prime*[a], then

- $\mathfrak{F} \in \mathbf{C} \parr \mathbf{D}$ iff for all $\mathfrak{D} \in \mathbf{C}^\perp$ $\mathfrak{F}(\mathfrak{D}) \in \mathbf{D}$.
- $\mathfrak{F} \in \mathbf{C} \parr \mathbf{D}$ iff for all $\mathfrak{D} \in \mathbf{D}^\perp$ $\mathfrak{F}(\mathfrak{D}) \in \mathbf{C}$[b].

Tensor : if $\mathbf{C}, \mathbf{D}$ of basis ξ are positive and *relatively prime* then $\mathbf{C} \otimes \mathbf{D} = \mathbf{C} \odot \mathbf{D}$. The paralogism of *weakening* is essential here.

Completeness : essentially contained in previous results ; in practice need to define *sequents of behaviours* based on arbitrary pitchforks.

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[a] Spiritual hypothesis of standard LL.

[b] Observe locative miracle : no distinction between left and right application, i.e. adjunction is the identity ! $F(a)(b) = F(b)(a) = F(a \otimes b) = F(b \otimes a)$.

Multiplicative decomposition

- Split set $\mathbb{N}$ of biases into disjoint $\mathbb{X}$, $\mathbb{Y}$. In $\mathbf{C}$ write any $\mathfrak{D} \in \mathbf{C}$ as tensor product of rel. prime $\mathfrak{D}_\mathbb{X}$, $\mathfrak{D}_\mathbb{Y}$ (uniquely determined if $\mathfrak{D} \neq \mho^+$).
- Define $\mathbf{C}_\mathbb{X} = \{\mathfrak{D}_\mathbb{X}, \mathfrak{D} \in \mathbf{C}\}$, $\mathbf{C}_\mathbb{Y} = \{\mathfrak{D}_\mathbb{Y}, \mathfrak{D} \in \mathbf{C}\}$. Obviously $\mathbf{C} \subset \mathbf{C}_\mathbb{X} \otimes \mathbf{C}_\mathbb{Y}$, but *equality fails*, e.g. $\mathbf{C} = \exists\alpha(\Phi(\alpha) \otimes \Psi(\uparrow \alpha^\perp))$.
- For each $\mathfrak{D} \in \mathbf{C}_\mathbb{X}$ let $\mathbf{C}_{\mathbb{Y},\mathfrak{D}} = \{\mathfrak{E} \in \mathbf{C}_\mathbb{Y}; \mathfrak{D} \otimes \mathfrak{E} \in \mathbf{C}\}$. Then $\mathbf{C}_{\mathbb{Y},\mathfrak{D}}$ is a behaviour, and we can write *dependent sum :*

  - $$\mathbf{C} = \bigotimes_{\mathfrak{D} \in \mathbf{C}_\mathbb{X}} \mathbf{C}_{\mathbb{Y},\mathfrak{D}}$$

- For negative behaviours *dependent products :*

$$\mathbf{C}^\perp = \bigwedge_{\mathfrak{D} \in \mathbf{C}_\mathbb{X}} \mathbf{C}_{\mathbb{Y},\mathfrak{D}}{}^\perp$$

Dependent types

- Symmetry

$$\mathbf{C} = \bigotimes_{\mathfrak{D} \in \mathbf{C}_{\mathbb{Y}}} \mathbf{C}_{\mathbb{X},\mathfrak{D}}$$

$$\mathbf{C}^{\perp} = \bigwedge_{\mathfrak{D} \in \mathbf{C}_{\mathbb{Y}}} \mathbf{C}_{\mathbb{X},\mathfrak{D}}{}^{\perp}$$

- Observe that $\mathbf{C}_{\mathbb{Y}} = \mathbf{C}_{\mathbb{Y},\mho^{+}}$.

- Open question : find conditions for a parametric family $\mathbf{C}_{\mathbb{Y},\mathfrak{D}}; \mathfrak{D} \in \mathbf{C}_{\mathbb{X}}$ to correspond to the multiplicative decomposition[a]. Completeness of *Martin-Løf* syntax ?

- Multiplicative decomposition reduces connected behaviours to prime behaviours, i.e. to shifts.

[a]Positive parametrism is contravariant, negative parametrism is covariant.

Prenex forms

- If $\mathbf{C}, \mathbf{D}[\alpha]$ are *relatively prime* for all α, then

$$\mathbf{C} \otimes \forall \alpha \mathbf{D}[\alpha] = \forall \alpha (\mathbf{C} \otimes \mathbf{D}[\alpha])$$

- Similar results for $\oplus$, provided $\mathbf{C}, \mathbf{D}[\alpha]$ are *disjoint* for all α

$$\mathbf{C} \oplus \forall \alpha \mathbf{D}[\alpha] = \forall \alpha (\mathbf{C} \oplus \mathbf{D}[\alpha])$$

- Every proposition is *equal* to its prenex form.
- Prenex forms *impossible* with Tarskian straightjacket.
- New form of incompleteness, non-Gödelian, i.e. *non-enumerative*.
- *Challenge :* add prenex forms to type theory.

LUDICS VI : VISIBILITY

Jean-Yves Girard

Invisible designs

- *Visibility*, i.e. first class citizenship, should be *absolute*, i.e. independent of « typing ». Exclude any sort of logical relation (hereditary notion).
- Invisible designs are essential (play the role of *counter-models*).
- Three conditions

  Winning : no give up.

  Parsimony : no weakening, sort of axiom links.

  Uniformity : express that

  Exponentials : all copies in $\sharp A$ isomorphic.

  Second order quantification : distinct ramifications are related.

  Non uniform *(invisible)* integers (type $\forall \alpha . !(\alpha \multimap \alpha) \multimap (\alpha \multimap \alpha)$) yield different values according to ramification. Can they parametrise complexity ?

Winning

- No give up.
- Obviously closed under normalisation (no way to create give up).
- But not enough to win all plays : *morality. . .* i.e. sort of additional *coherence conditions* on designs. These additional conditions *(parsimony, uniformity)* cannot be systematically required, since it is important to violate them. Typically *weakening* (non parsimonious waste of adresses) is essential in multiplicative completeness.

Parsimony

- A design $\mathfrak{D}$ is *strictly parsimonious* when its negative pitchforks are of the form $\xi \vdash$ or $\xi \vdash \lambda$ justified by[a] :

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \lambda * j \vdash \Xi_j\end{array}}{\ldots \vdash \lambda, \xi * I \ldots}\,(\lambda, J)}{\xi \vdash \lambda}\,(\xi, \mathcal{N})$$

- A positive rule is *strict* when $\Lambda = \bigcup_{i \in I} \Lambda_i$.

- A negative rule is *strict* when $\Lambda_I = \Lambda$ for all $I \in \mathcal{N}$.

- A design $\mathfrak{D}$ is *parsimonious* when all rules are strict and every branch is eventually strictly parsimonious.

[a]Sort of generalised Fax or Jesuit.

Parsimony under cut

- Parsimony is a form of well-foundedness : ordinal *height* $\hbar\mathfrak{D}$ of design.
- Parsimony implies the existence of ordinal *degree* $d\mathfrak{D}$, relative to a *slice*.
- Show that parsimony is closed under cut : $\{\mathfrak{D}, \mathfrak{E}\}$ parsimonious and *normalises* into $\mathfrak{F}$ implies $\mathfrak{F}$ parsimonious.
- Induction on $(d\mathfrak{D}, \hbar\mathfrak{D})$; in simple case, height and degree stay finite etc.

Visibility and completeness

- Soundness *always* interesting (and non-trivial : difficult to find a *Broccolo*, besides the free one).
- Completeness is a test for the *right* notions ; but completeness is death.
- More interesting to investigate *incomplete* situations, e.g. intersection types, invisible integers, interference.
- *Question :* to which extent can we interpret classical logic in a « single thread » ($\Vdash \xi \quad \xi * 0 \Vdash \quad \Vdash \xi * 0 * 0 \ldots$), by using plain boolean operations and no delocation... The Jesuit would do for $A \vee \neg A$.

Logical interference

- $\mathbf{C}$ *true* iff visibly inhabited, *false* iff $\mathbf{C}^{\perp}$ visibly inhabited.
- Spiritual (e.g. linear) logic does not follow classical truth tables, but does not contradict them either.
- In locative situation, possible to find
  - $\mathbf{C}, \mathbf{D}$ *true*, but $\mathbf{C} \otimes \mathbf{D}$ *false* (example 1).
  - $\mathbf{C}, \mathbf{D}$ *false*, but $\mathbf{C} \otimes \mathbf{D}$ *true* (example 2).
- Interference = logical status of *side effects* (capture of variables).

Example 1

- For $i = 0, 1, 2$ define positive designs $\mathfrak{C}_i$ with basis $\vdash \xi$:

$$
\cfrac{\cfrac{}{\xi i \vdash}\;(\xi i, \emptyset)}{\vdash \xi}\;(\xi, \{i\})
$$

$\mathfrak{C}_0$ is *visible*, hence $\mathbf{C_0} = \mathfrak{C_0}^{\perp\perp}$ is *true*, but $\mathbf{C}_0 \otimes \mathbf{C}_0 = 0$ is *false*.

Example 2

- For $i = 0, 1, 2$ consider the positive designs $\mathfrak{D}_i$ of basis $\vdash \xi$:

$$\dfrac{\dfrac{\dfrac{}{\vdash \xi i i}\,\mho}{\xi i \vdash}\,(\xi i, \{\{i\}\})}{\vdash \xi}\,(\xi, \{i\})$$

- Then $\mathbf{D} = \{\mathfrak{D}_0, \mathfrak{D}_1, \mathfrak{D}_2\}^{\perp\perp}$ is *false*, as well as $\mathbf{E} = \{\mathfrak{C}_0 \otimes \mathfrak{D}_1, \mathfrak{C}_1 \otimes \mathfrak{D}_2, \mathfrak{C}_2 \otimes \mathfrak{D}_0\}^{\perp\perp}$; typically $\mathfrak{C}_0 \otimes \mathfrak{D}_1$ equals

$$\dfrac{\dfrac{}{\xi 0 \vdash}\,(\xi 0, \emptyset) \qquad \dfrac{\dfrac{}{\vdash \xi 1 1}\,\mho}{\xi 1 \vdash}\,(\xi 1, \{\{1\}\})}{\vdash \xi}\,(\xi, \{0, 1\})$$

Example 2 (cont^{d})

$$
\dfrac{
\dfrac{\dfrac{}{\xi 11 \vdash \xi 0}\,(\xi 11, \emptyset)}{\vdash \xi 0, \xi 1}\,(\xi 1, \{1\})
\qquad
\dfrac{\dfrac{}{\xi 22 \vdash \xi 1}\,(\xi 22, \emptyset)}{\vdash \xi 1, \xi 2}\,(\xi 2, \{2\})
\qquad
\dfrac{\dfrac{}{\xi 00 \vdash \xi 2}\,(\xi 00, \emptyset)}{\vdash \xi 2, \xi 0}\,(\xi 0, \{0\})
}{\xi \vdash}\,(\xi, I)
$$

with $I = \{\{0,1\},\{1,2\},\{2,0\}\}$, is a *visible* design in $\mathbf{E}^{\perp}$.

- $\mathbf{D} \otimes \mathbf{E}$ is *true* : this behaviour is the biorthogonal of three designs $\{\mathfrak{C}_0 \otimes \mathfrak{D}_1 \otimes \mathfrak{D}_2, \mathfrak{C}_1 \otimes \mathfrak{D}_2 \otimes \mathfrak{D}_0, \mathfrak{C}_2 \otimes \mathfrak{D}_0 \otimes \mathfrak{D}_1\}$.
  Since their union $\mathfrak{D}_0 \otimes \mathfrak{D}_1 \otimes \mathfrak{D}_2$ *is a design, their intersection* :

$$
\dfrac{
\dfrac{}{\xi 0 \vdash}\,(\xi 0, \emptyset)
\qquad
\dfrac{}{\xi 1 \vdash}\,(\xi 1, \emptyset)
\qquad
\dfrac{}{\xi 2 \vdash}\,(\xi 2, \emptyset)
}{\vdash \xi}\,(\xi, \{0,1,2\})
$$

  is in $\mathbf{C} \otimes \mathbf{D}$; this design is *visible*, so $\mathbf{D} \otimes \mathbf{E}$ is *true*.

Claude Debussy/Charles Péguy

Oui, c'est imbécile ce que je dis ! Seulement je ne sais pas comment concilier tout ça. Il est sûr que je ne me sens libre que parce que j'ai fait mes classes et que je ne sors de la fugue que parce que je la sais.

Claude-Achille Debussy *Entretiens avec Ernest Guiraud*, ~ 1890.

Il faut toujours dire ce que l'on voit. Surtout il faut toujours, ce qui est plus difficile, voir ce que l'on voit.

Charles Péguy, *Notre jeunesse*, 1910.