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THIRTY YEARS LATER

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1 — The four horsemen of cognition

	Analytic	Synthetic
Explicit	Constat	Usine
Implicit	Performance	Usage

2 — L'USINE, A.K.A. PROOF-NETS

- Main novelty of linear logic: *factory* tests.

  Cut-free: cut seen as special conclusion $[A \otimes \sim A]$.

  Testing anticipated by Herbrand: $\forall = f(\exists)$.

  Non compositional: dinaturals. Opposite to BHK (*usage*).

- Upper *answer*, solution vs. lower *question*, problem.

  Independent: e.g., η-expansion.

  Upper: analytic, type-free, meaningless objects.

  Analytic normalisation by plugging identity links.

  Lower: deals with logic, type, meaning.

  Synthetic normalisation replaces cuts with simpler ones.

- *Sequentialisation* not needed: logical *soundness* suffices.

  Non sequential $¶ := \{\{1,2\},\{3,4\}\} + \{\{2,3\},\{4,1\}\}$.

  Adequation usine/usage (deductive use).

  Factory tests preserved by synthetic normalisation.

3 — ANALYTICS: DETERMINISTIC CASE

- **Meaningless, untyped, self-contained:** ***beyond discussion*.**

  Constat **finite result (normal):** ***incremental*.**

  Performance **program, self-executed:** ***destructive*.**

  Colours: **constats black, performance = colour-elimination.**

- **Stars** $[\![t_1, \ldots, t_n]\!]$**;** ***rays*** **with same variables (needed for** η**).**

  Constellation: **finite set of stars.**

  Determinism: **rays of constellation not matchable.**

- ***Dendrites*** **(= analytic cut): plugging rays of comp. colours.**

  Strong normalisation: **no dendrite of size** $> N$**.**

  Normal form: **dendrites with black rays, seen as stars.**

- **Correctness: uses links like** $[\![p_{A\otimes B}(x), p_A(x), p_B(x)]\!]$**.**

  Normal form **should be** $[\![p_\Gamma(x)]\!]$**.**

  Multiplicatives: **preserved by synthetic normalisation.**

4 — ANALYTICS: NON DETERMINISTIC CASE

- *Additives:* **boxes and weights excluded, not self-contained.**

  Matchable rays: Alzheimer non-determinism.

  Coherence between rays; ensures non-matchability.

- **Not enough for correctness; *Simulacron 3*, *The matrix*.**

  Choice $A \,\&\, B$ possibly biased (slicing).

  Coh. star: $[\![\, p_{A\&B}(x), p_A(x), p_B(x) \,]\!]$, with $p_A(x) \smile p_B(x)$.

- *Canvas* **(= analytic cut): like a dendrite, but not tree-like.**

  Select anticlique for each star; result should be a dendrite.

  Dendrites of a canvas: parallel executions, slices.

- *Correctness:* **normal form with dendrites $[\![\, p_\Gamma(x) \,]\!]$.**

  Preserved by synthetic normalisation.

  Sequentialisation: impossible with usual sequents.

  Suggests series/parallel $⅋/\oplus$ sequents with canvas-cut.

5 — Exponentials

- **Basic idea of $?A$: normal form $[\![\, p_\Gamma(x) \setminus p_{?\Gamma}(x) \,]\!]$.**

  Problem: no way to recover, since location lost.

  Independence proof/test not preserved.

  Proof-nets for exponentials (and $1, \perp$) thus impossible.

- **But $A \otimes B = !A \otimes B$ and $A \ltimes B = ?A ⅋ B$ tractable.**

  One-sided version of disjunction (Mogbil) can be used.

- ***Problem:* quid of synthetic normalisation?**

  Duplicated switchings should be independent.

  Solution unique switching (with obvious coherence)

  $$[\![\, p_{A⅋B}(x), p_A(x), p_B(x) \,]\!] + p_A(x) + p_B(x).$$

- ***Intuitionistic* disjunction $!A \oplus !B$ does not survive.**

  Untractable linearity; comm. conversions not analytic.

  However $\vee, !, ?, 1, \perp$ possible at second order.

6 — DEREALISM A.K.A. SECOND ORDER

- ***Second order*** **quantifications: over** ***propositions*****.** $\dfrac{A}{\forall X\, A}$ $\dfrac{A[T/X]}{\exists X\, A}$

- **Can be handled by** ***usine*** **(proof-nets).**

  Universal qfr. **switch** $X := \cdot / \otimes / \parr$**, hence** $\sim X := \cdot / \parr / \otimes$**.**

  Existential qfr. **witness** T **provides its** ***own*** **switchings.**

- **However,** T **is part of the** ***derealist*** **answer.**

  Object/Subject **no longer valid: answer partly** ***subjective*****.**

  Épure: **combination vehicle +** ***mould*****, e.g.,** $T + \sim T$**.**

  Balance: **how do we know that** $T + \sim T$ **actually** ***match*** **?**

  Volkswagen **effect: cop can turn mafioso.**

  Answer **combines** ***analytic*** **and** ***synthetic*** **features.**

  Animism: **when analytic and synthetic cannot be split.**

  Consistency: **no type is empty, admits animist inhabitants.**

7 — Predicate calculus is wrong

- System $\mathbb{F}$: propositions are (roughly) enough.

  Forgetful functor: keeps computational (analytic) contents.
  Realisability: awkward reduction predicate $\rightsquigarrow$ proposition.

- *Predicate calculus:* XIX$^{\mathrm{th}}$ century legacy.

  Axiomatics: cannot avoid « Barbari » $\forall x A \vdash \exists x A$.
  Semantics: models non-empty; example of *selfie*.

- Dubious principle: besides *eigen* variables, used for $\vdash \forall$

  Junk variables: dedicated to the sole *Barbari.*

- Intrusion of reality through *external* domain.

  Variables, functions: proceed from the Sky.
  Equality: mistreated through axiomatics.

- *Replace* ethereal individuals with concrete *connectives*.

  Equality: becomes equivalence.

8 — INDIVIDUALS AS MULTIPLICATIVES

- ***Individual = proposition*** **forbidden by realistic prejudice.**

  Classical: $t \equiv u \vee u \equiv v \vee v \equiv t$. **Only two individuals.**

  Intuitionistic: $\neg\neg(t \equiv u \vee u \equiv v \vee v \equiv t)$. **Not more than** 2.

  Linear: **with** $(t \multimap u) \mathbin{\&} (u \multimap t)$ **as equivalence, OK.**

- n**-ary multiplicative: set of partitions of** $\{1, \ldots, n\}$.

  Duality: $\mathcal{C} \perp \mathcal{D}$ **iff their incidence graph is a tree** $(n \neq 0)$.

  Multiplicative: **non-trivial set of partitions equal to bidual.**

  Example: $\otimes := \{\{1, 2\}\}$ **vs.** $⅋ := \{\{1\}, \{2\}\}$.

  Series/parallel: $\P := \{\{1, 2\}, \{3, 4\}\} + \{\{2, 3\}, \{4, 1\}\}$.

  Not sequential: $\P$ **admits proof-nets, no sequent calculus.**

- ***Linear*** **implication between multiplicatives:**

  Same n **:** **typically,** $\cdot \otimes (\cdot ⅋ \cdot) \multimap (\cdot \otimes \cdot) ⅋ \cdot$ **with** $n = 3$.

  $\sharp$ **partitions:** **decreases; equal in case of equivalence.**

  Equality: **equivalence yields two *isomorphisms,* not related.**

9 — FUNCTIONS AND PREDICATES

- Functional *terms* come from same multiplicative matrix:

  Positive multiplicatives with possible repetitions.

  Example: $x \parr (x \otimes y)$. No constant, no *Barbari,* no regrets.

  Pairing: ensured by $(x \parr y) \otimes (x \parr x \parr y)$.

- *Predicate* variables $P, Q, \ldots$ as variable *connectives.*

  Pt handled by unknown binary connective K.

  Usage: all possible uses $Kt\tilde{t}$ of individual t and negation $\tilde{t}$.

  Usine: enough to test with $K = \otimes$ and $K = \parr$.

  Equality principle: $t = u \Rightarrow (Pt \multimap Pu)$ OK'ed by l'usine.

  Refused: $t = u \Rightarrow (Pt \multimap Qu)$ and $t = u \multimap (Pt \multimap Pu)$.

- *Equality* handled by: $(\tilde{t} \parr u) \,\&\, (t \parr \tilde{u})$.

- First-order quantification: restriction of « full » case.

  Existential witnesses: among multiplicative connectives.

10 — THE ARITHMETIC CHALLENGE

- ***Desaxiomatisation:*** **integers are logical.**
- **Dedekind:**

  $t \in \mathrm{nat} := \forall X(\forall x(Xx \multimap X(Sx)) \Rightarrow (X0 \multimap Xt))$
- **Third and fourth Peano axioms should become *theorems*.**

  3 $Sx \neq 0$

  4 $Sx = Sy \Rightarrow x = y$.

  Anti-classicism: propositions [3] and [4] classically false.
- **Search for *classically false* linear theorems.**