Geometry of Interaction IV : the Feedback Equation

Jean-Yves Girard
Institut de Mathématiques de Luminy, UPR 9016 – **CNRS**
163, Avenue de Luminy, Case 930, F-13288 Marseille Cedex 09

girard@iml.univ-mrs.fr

December 9, 2005

Abstract

The first three papers on Geometry of Interaction [9, 10, 11] did establish the universality of the *feedback equation* as an explanation of logic ; this equation corresponds to the fundamental operation of logic, namely cut-elimination, i.e., *logical consequence* ; this is also the oldest approach to logic, *syllogistics* ! But the equation was essentially studied for those Hilbert space operators coming from *actual* logical proofs.

In this paper, we take the opposite viewpoint, on the arguable basis that operator algebra is more primitive than logic : we study the *general* feedback equation of Geometry of Interaction, $\mathsf{h}(x \oplus y) = x' \oplus \sigma(y)$, where h, σ are hermitian, $\|\mathsf{h}\| \leq 1$, and σ is a partial symmetry, $\sigma^3 = \sigma$. We show that the *normal form* which yields the solution $\sigma[\![\mathsf{h}]\!](x) = x'$ in the *invertible* case can be extended in a unique way to the general case, by various techniques, basically *order-continuity* and *associativity*.

From this we expect a definite break with *essentialism* à la Tarski : an interpretation of logic which does not presuppose logic !

1 Introduction

We are essentially concerned with the technical contents of this precise paper in the series. For the general significance of Geometry of Interaction, see appendix A.

1.1 Non-commutativity in Logic

First of all, we are not using "non-commutative" in the sense of the non-commutative logic of Ruet and Abrusci [1], but rather in the sense of the non-commutative geometry of Connes [4]. The basic idea is to overcome the limitations of set theory[1] so as to give a sort of "quantum" interpretation of logic[2]. Set theory explains everything with *atoms*, a very useful *reduction*, even if criticised by various mathematicians. Category theory would rather focus on *morphisms*, but this approach does not quite expel the atoms, the *points* : it rather pretends not to see them. The non-commutative approach is more radical : the set-theoretic atoms appear as *eigenvectors*, they are related to interaction, observation. As far as the observer remains the same, "his" eigenvectors are used, and everything looks commutative, "set-theoretic" ; technically speaking, we deal with diagonal matrices, i.e., stay in a *commutative* operator algebra. Non-commutativity is nothing but the *relativisation* of the subject, i.e., the oblivion of the distinguished "basis", or commutative algebra, associated with the observer.

We are concerned with the foundational part of logic, *proof-theory*. What follows is a short dictionary of the "non-commutative" analogues of familiar logical artifacts.

Proofs, Functions, Programs : Hermitians of norm at most 1. These hermitians need not be positive : the most basic *identity axiom* $A \vdash A$ is interpreted by the flip —a.k.a. extension cord— of $\mathcal{H}\oplus\mathcal{H}$: $\mathsf{h}(x\oplus y) := y\oplus x$.

Deduction, Composition : Our hermitians usually come together with a *feedback* σ which is a partial symmetry, i.e., a hermitian of spectrum within $\{-1, 0, +1\}$. The feedback corresponds to a logical deduction, in which the same formula occurs twice, both as a result (lemma) and as a hypothesis for the theorem : the feedback swaps the two copies. The basic artifact, corresponding to a proof with cuts, to a program before execution, is therefore a *cut-system* $(\mathcal{H}, \mathsf{h}, \sigma)$.

Execution : There is a dynamics, in the sense of a *performance*. Performing the cut, executing a program, amounts at actually "plugging" the feedback σ with h, i.e., at solving the *feedback equation* : $\mathsf{h}(x\oplus y) = x'\oplus\sigma(y)$: x is the input, x' is the output and y is the computation.

---

[1] And category theory as well.

[2] Of course this is radically different from the project of the unfortunate quantum "logic", who tried to "tame" the quantum world.

1.2 Solving the Feedback Equation

Section 4 is devoted to the most down-to-earth case, namely when the equation has always a solution, the *terminating case*. This solution is then shown to be unique as to x', y being unique up to a *deadlock*, i.e., something which is completely inaccessible, does not interfere with the system, and therefore can be ignored. A notion of "computational size", basically the norm of the operator yielding $x \oplus y$ as a function of x is introduced, and is shown to enjoy a remarkable *associativity* inequality.

The most important (and natural) terminating case is *invertibility* (of $\mathrm{I} - \sigma\mathsf{h}$), in which case the feedback equation can be solved by inversion. The typical invertible cases arise from logic : if h interprets a logical proof, $\sigma\mathsf{h}$ is nilpotent, see [9], and the inverse is given by a finite power series. Logical rules[3] can be read as gimmicks enabling one to reduce complex feedback equations to simpler ones, basically by iterated substitutions, this is why the power series is finite.

Most of the paper is devoted to the study of the extension of the *normal form* $\sigma[\![\mathsf{h}]\!](x) = x'$ obtained in the invertible case to general systems.

1.3 Continuity

As usual, the first intuition is continuity. However, since the invertible case makes use of inversion, we cannot expect any reasonable topological continuity. Here there is a conflict between the mathematical tradition and some (recent) logical tradition : in the late sixties, following previous work by Kleene, Kreisel, Gandy... Dana Scott introduced his non-Hausdorff *domains*, which are indeed ordered structures. The question whether or not this is topology is controversial, but an essential role is played anyway by a partial ordering of objects. The same phenomenon is observed here : there is a partial order (the usual pointwise ordering of hermitians) and the first important phenomenon is that :

The normal form is monotonous (increasing).

The suggestion coming from Scott domains is to try *order-continuity*, i.e., commutation to directed sups *and* infs. Encouraging point : the normal form is order-continuous in the invertible case. This is why we introduce *semi-invertibility* : lower-semi-invertibles (l.s.i.) are l.u.b. of invertible systems, upper-semi-invertibles (u.s.i.) are g.l.b. of invertibles, and the miracle is that invertibility is the same as the two semi-invertibilities. We can extend by means of l.u.b. (resp. g.l.b.) the normal form to l.s.i. (resp. u.s.i.) systems, and this consistently.

[3] Indeed, *cut-free rules*, i.e., the part of logic which doesn't deal with logical consequence.

To come back to Scott domains and the logical "tradition" : it was possible —using weird topologies— to style order-continuity "topological", only because the sole commutation was commutation to suprema. Here we have two conflicting commutations, one to suprema, one to infima, no global topology —weird or not—, would account for order-continuity on both sides. Take the simplest example : on $\mathbb{R}$, upwards continuity corresponds to the open sets $]x, +\infty[$, whereas downwards continuity corresponds to the open sets $]-\infty, y[$; continuous functions from X to $\mathbb{R}$ equipped with the "upwards" (resp. "downwards") topology are indeed l.s.c. (resp. u.s.c.) functions. These two $\mathcal{T}_0$-topologies have a supremum, the usual topology (since $]x, y[\, = \,]x, +\infty[\, \cap \,]-\infty, y[$), in other terms, full order-continuity does not correspond to continuity in any topological sense, natural or not.

But, even if order-continuity does not quite make sense topologically speaking, it can be used in conjunction with standard topologies (usually the weak and the strong topologies, both weaker than the norm topology). This is what makes our semi-invertible extension possible : order-continuity works like an *upgrader*, from weak to strong in the case of operators, from pointwise to uniform in Dini's theorem, see *infra*.

But order-continuity does not work beyond the semi-invertible case.

1.4 Lebesgue Integration

The comparison with a classic, the Lebesgue integral, is illuminating. The problem at stake is the extension of the Riemann integral. We describe the main steps, pointing out the analogies/differences.

(i) The Riemann integral is a continuous linear map from the Banach space $\mathbb{R}([0, 1])$ of continuous real-valued functions on $[0, 1]$ into $\mathbb{R}$. Our case is similar : we start with a norm-continuous function $\sigma[\![\mathsf{h}]\!]$ (the normal form) defined on certain operators h (invertible case) of norm ≤ 1. We can even assume the output space of dimension 1, see proposition 9, hence belongs to $\mathbb{R}$. An essential difference : the normal form is not linear. However the output remains bounded : $\|\sigma[\![\cdot]\!]\| \leq 1$.

(ii) Coming back to integration, the next step is to remark that the Riemann integral is monotonous. The idea is to extend it to lower-semi-continuous functions, which are suprema of continuous functions. In the same way, we extend our normal form to l.s.i. systems by means of suprema.

(iii) Of course, only a monotonous function can be extended in that way. Both the Riemann integral and the normal form are monotonous. Moreover, the extension should be consistent with the starting point. In the case of integration, one uses Dini's theorem : if f_n is an increasing sequence

of continuous functions with a continuous supremum f, then the convergence $f_n \to f$ is uniform, i.e., a norm-convergence. The normal form is sup-continuous too : this relies on the strong convergence of bounded increasing nets, see proposition 18, appendix C.4, and the strong continuity of composition on balls.

(iv) This extension is, in both cases, consistent with the symmetric extension by infima. Because continuous = l.s.c. $\cap$ u.s.c. (resp. invertible = l.s.i. $\cap$ u.s.i.) ; moreover, the full semi-continuous (semi-invertible) case remains monotonous.

(v) Another step must be performed : after suprema, infima. This is the end of the story : w.r.t. Lebesgue integration, every measurable function is equivalent to an infimum $\inf_n f_n$ of l.s.c. functions. In our case, every system is the infimum of l.s.i. system.

(vi) But the answers are different : Lebesgue integration can perform this second step, basically because the extension to l.s.c. functions by suprema commutes to infima. Here we say goodbye to our model : the normal form, extended by suprema to l.s.i. systems, does not commute to infima, see section 6.6. Something else must be found.

No doubt that this ultimate divergence is due to the non-commutativity of the normal form.

1.5 Associativity

To go beyond the semi-invertible case, one should introduce another idea, alien to the idea of approximation —topological, or order-theoretic. Here, we use one of the milestones of logic, the *Church-Rosser Property*, that we interpreted as *associativity* in ludics [13]. The question is the following : is an iterated normal form the same as a single normal form, in other terms, if the feedback splits as a direct sum $\sigma + \tau$, can we solve the equation in two steps, first with the sole feedback σ, then with the feedback τ applied to the solution ?

The answer is positive in the invertible case ; in the semi-invertible case, there is no obvious order-continuity argument, for we may have to relate semi-invertible systems which are on "different sides", l.s.i., versus u.s.i. ; the situation seems desperate.

In fact this problem is the key to the full solution : every feedback is the difference of two projections, $\sigma = \sigma^+ - \sigma^-$. In case of a *positive* feedback, all systems are l.s.i., and the normal form is upwards continuous ; in case of a *negative* feedback, all systems are u.s.i. and the normal form is downwards continuous. In order to get a full normal form, it is enough to prove associativity in case of *lopsided* feedbacks of opposite signs. This is not achieved by order-continuity, but by providing a sort of explicit formula. By the way, we heavily rely on the main property of positive hermitians : the existence of a square root.

1.6 Stability

I was surely one of the first persons to express strong doubts as to the topological nature of Scott domains. These doubts prompted me to introduce (rather to rediscover after Berry [3]) a competitor to the Scott ordering, the *stable* ordering, roughly corresponding to inclusion ; this was the origin of *coherent spaces* and further developments such as *linear logic*.

The "inclusion" between hermitians is defined by $\mathsf{kh} = \mathsf{h}^2$, together with $\mathsf{k}\sigma\mathsf{h} = \mathsf{h}\sigma\mathsf{h}$, in case of a feedback. Contrarily to the standard ordering, inclusion has the structure of a (downwards) conditional lattice ; moreover the solution $\sigma[\![\mathsf{h}]\!]$ of the feedback equation is monotonous and preserves conditional g.l.b., i.e, pull-backs. From this, it follows that the cut-system $(\mathcal{H}, \mathsf{h}, \sigma)$ has a unique *incarnation* $(\mathcal{H}, \mathsf{k}, \sigma)$, where $\mathsf{k} \sqsubset \mathsf{h}$ is the part of h "actually used" in the computation of $\sigma[\![\mathsf{h}]\!]$.

In a commutative —set-theoretic— setting, typically in ludics [13], inclusion is a refinement of the pointwise order. In a non-commutative world, the two orders are independent : typically $\mathsf{h} \sqsubset \mathsf{k} \Rightarrow -\mathsf{h} \sqsubset -\mathsf{k}$, whereas $\mathsf{h} \leq \mathsf{k} \Rightarrow -\mathsf{k} \leq -\mathsf{h}$.

1.7 Winning

Ludics [13] mainly rests upon the notion of *polarity*, which roughly corresponds to the natural notion of *signature*, the most basic invariant of logic. Logic is then interpreted by sort of *bipartite graphs*, with positive/negative nodes (answers/questions, etc).

What we try to mimic here by means of *bipartite hermitians* : assuming that our Hilbert space splits into a direct sum $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, we require that $\mathcal{H}^+\mathsf{h}\mathcal{H}^+ \geq \mathsf{0}$ and $\mathcal{H}^-\mathsf{h}\mathcal{H}^- \leq \mathsf{0}$. We must also restrict to feedbacks enjoying $\mathcal{H}^+\sigma\mathcal{H}^+ = \mathsf{0}$, $\mathcal{H}^-\sigma\mathcal{H}^- = \mathsf{0}$. Under such hypothesis, the surprising fact is that the solution of the feedback equation remains bipartite.

Indeed, the interpretation of logical proofs, λ-expressions, is not only bipartite, it enjoys $\mathcal{H}^+\mathsf{h}\mathcal{H}^+ = \mathsf{0}$, $\mathcal{H}^-\mathsf{h}\mathcal{H}^- = \mathsf{0}$. If we call this "winning", it is immediate that winning is preserved by composition, i.e., by normal forms. The importance of the preservation of winning lies in its relation to logical *consistency*, see [13].

1.8 Immediate Questions

A few immediate questions that I had not the time to investigate :

(i) Assume that $(\mathcal{H}, \mathsf{h}, \sigma)$ is terminating. Does the solution correspond to the normal form ?

(ii) More generally, if a normal form is achieved by various means, does this output corresponds to our normal form ? The main example is *weak nilpotency*, $\langle (\mathsf{h}\sigma)^{\mathsf{n}}(x) \mid x \rangle \to 0$, which yields an *unbounded* execution, but a perfectly bounded output, see [10].

1.9 Further Work

Further work should involve the definition of a notion of *polarity*[4] between hermitians, in such a way that, of two hermitians, at most one of them can be *winning*, and enjoying a *separation* property, see [13]. Then, last but not least, the remake of GoI, but not necessarily in a type I von Neumann algebra like $B(\mathcal{H})$: maybe a type II or a type III algebra[5] is more appropriate, especially in view of a subtler approach to logic.

2 Cut-Systems

2.1 The Feedback Equation

Definition 1 (Cut-Systems)
A cut-system *is a 3-tuple* $(\mathcal{H}, \mathsf{h}, \sigma)$ *such that :*

★ $\mathcal{H}$ *is a complex Hilbert space.*

★ h *is a hermitian —i.e., self-adjoint— operator on* $\mathcal{H}$ *of norm at most* 1 *:* $\mathsf{h}^* = \mathsf{h}, \|\mathsf{h}\| \leq 1$.

★ σ *(the* cut, *the* loop, *the* feedback*) is a* partial symmetry, *i.e.,* $\sigma = \sigma^* = \sigma^3$, *see appendix C.6.*

Since $\sigma^3 = \sigma$, σ^2 is the orthogonal projection of a closed subspace $\mathcal{S} \subset \mathcal{H}$; we can therefore write $\mathcal{H} = \mathcal{R} \oplus \mathcal{S}$, with $\mathcal{R} := \mathcal{S}^\perp$. Accordingly to the standard abuse of notations, $\mathcal{S} = \sigma^2, \mathcal{R} = \mathrm{I} - \sigma^2$.

[4] Formerly called orthogonality.

[5] Algebras in which projections can be "halved", i.e., in which there is no minimal subspace, see [18].

The cut-system $(\mathcal{H}, \mathsf{h}, \sigma)$ induces a *feedback equation* : given $x \in \mathcal{R}$ find $x' \in \mathcal{R}$, $y \in \mathcal{S}$, such that :

$$\mathsf{h}(x \oplus y) = x' \oplus \sigma(y) \tag{1}$$

Indeed, we are mostly interested in the "visible" part of the equation, i.e., the output, the *result* x' ; the component y is rather perceived as "internal", and quite corresponds to the *computation*.

To tell the truth, it is not quite true that this equation is always solvable, *stricto sensu*. But it is anyway our starting point.

Remark 1
This definition covers everything done under the name "GoI" in [9, 10, 11], but for a small exception, namely the interpretation of *weakening* in [11], which introduces a non-hermitian operator, in contrast to what was previously done in [9] ; this fancy variant was supposed to achieved effects of "connectedness", but this never worked. So let us get back to the original, hermitian, definition given in [9].

In fact, it turns out that the h constructed in [9, 10, 11] are not only hermitian, but also partial symmetries. But we cannot limit ourselves to partial symmetries, for the simple reason that, if we don't make any heavy additional hypothesis, the solution to the feedback equation need not be given by a partial symmetry.

Remark 2
One may wonder why the loop is a partial symmetry, and not a projection. Obviously $(\mathcal{H}, \mathsf{h}, \sigma)$ behaves like $(\mathcal{H}, \mathsf{h}(\mathcal{R} + \sigma), \mathcal{S})$; but there is a deep conceptual difference, h is hermitian, $\mathsf{h}(\mathcal{R} + \sigma)$ is a product of hermitians, i.e., a nothing !

Remark 3
The original equation is not between a hermitian and a feedback, it is between two hermitians, which are put in duality by the equation. The feedback equation is basically the remark that, w.l.o.g., we can assume that one of the two hermitians, the feedback, is of a very simple form. See appendix B.2 for a discussion.

2.2 Bipartism and Winning

Definition 2 (Bipartism)
A bipartite *cut-system is a cut-system $(\mathcal{H}, \mathsf{h}, \sigma)$, together with a decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, such that :*

(i) $\mathcal{H}^+ \mathsf{h} \mathcal{H}^+ \geq 0$, $\mathcal{H}^- \mathsf{h} \mathcal{H}^- \leq 0$.

(ii) $\mathcal{H}^+\sigma\mathcal{H}^+ = \mathsf{0}$, $\mathcal{H}^-\sigma\mathcal{H}^- = \mathsf{0}$.

Definition 3 (Winning)
A bipartite cut-system is winning *when :* $\mathcal{H}^+\mathsf{h}\mathcal{H}^+ = \mathsf{0}$, $\mathcal{H}^-\mathsf{h}\mathcal{H}^- = \mathsf{0}$.

Proposition 1
h *is winning iff* $-\mathsf{h}$ *is bipartite.*

Proof : Trivial. Equivalently, h is winning if it remains bipartite when we swap $\mathcal{H}^+$ and $\mathcal{H}^-$. □

Remark 4
All cut-systems constructed in [9, 10, 11] are bipartite. Indeed, the notion of *signature* (positive, negative occurrences) induces a natural splitting of the Hilbert spaces at work.

Moreover, all these systems are winning. This should not surprise us, since logical rules are supposed to construct sort of "winning strategies", and GoI interprets logical rules. But, if we leave room for "losing" devices such as the *daimon* of ludics [13], then we may encounter bipartite cut-systems with $\mathcal{H}^+\mathsf{h}\mathcal{H}^+ \neq \mathsf{0}$, $\mathcal{H}^-\mathsf{h}\mathcal{H}^- \neq \mathsf{0}$.

Among the properties to be checked later, let us mention the fact that the solution of the feedback equation of a bipartite system is still bipartite ; similarly, winning will be preserved.

2.3 Orderings

Definition 4 (Pointwise Order)
We order cut-systems with the same underlying space and feedback in the obvious way :

$$(\mathcal{H}, \mathsf{h}, \sigma) \leq (\mathcal{H}, \mathsf{k}, \sigma) \quad \Leftrightarrow \quad \mathsf{k} - \mathsf{h} \geq \mathsf{0} \tag{2}$$

The pointwise ordering[6] admits directed l.u.b. and g.l.b. ; but it is not a lattice.

Definition 5 (Order-continuity)
A monotonous (increasing) map Φ *from a subset of* $Her(\mathcal{H})$ *to* $Her(\mathcal{R})$ *is* order-continuous *when it preserves (directed) l.u.b. and g.l.b.*

We shall apply this terminology to the *normal form* which associates to cut-systems $(\mathcal{H}, \mathsf{h}, \sigma)$ a hermitian of $\sigma[\![\mathsf{h}]\!] \in Her(\mathcal{R})$, to mean order-continuity w.r.t. the sole parameter h, consistently with definition 4.

[6] Logicians would rather speak of "extensional" order ; "pointwise", like in "pointwise convergence", seems however more standard.

Definition 6 (Stable order)
The stable ordering is defined as follows : $(\mathcal{H}, \mathsf{h}, \sigma) \sqsubset (\mathcal{H}, \mathsf{k}, \sigma)$ *iff*

(i) $\mathsf{kh} = \mathsf{h}^2$ *and*

(ii) $\mathsf{k}\sigma\mathsf{h} = \mathsf{h}\sigma\mathsf{h}$.

Proposition 2
$(\mathcal{H}, \mathsf{h}, \sigma) \sqsubset (\mathcal{H}, \mathsf{k}, \sigma)$ *iff there exists a closed subspace* $\mathcal{E}$ *such that :*

(i) $\mathcal{E}\sigma = \sigma\mathcal{E}$ *and*

(ii) $\mathsf{h} = \mathsf{k}\mathcal{E}$.

Proof : Assume that $(\mathcal{H}, \mathsf{h}, \sigma) \sqsubset (\mathcal{H}, \mathsf{k}, \sigma)$; then k and h coincide on the spaces $\mathrm{rg}(\mathsf{h})$ and $\mathrm{rg}(\sigma\mathsf{h})$, hence on the closure $\mathcal{E}$ of $\mathrm{rg}(\mathsf{h})+\mathrm{rg}(\sigma\mathsf{h})$. Obviously $\mathcal{E}\sigma = \sigma\mathcal{E}$ and $\mathsf{h}\mathcal{E} = \mathsf{k}\mathcal{E}$; moreover, $\mathcal{E}\mathsf{h} = \mathsf{h}$ hence, by taking adjoints, $\mathsf{h}\mathcal{E} = \mathsf{h}$ and $\mathsf{k}\mathcal{E} = \mathsf{h}$.

Conversely, if the condition of the proposition holds, then $\mathsf{h} = \mathcal{E}\mathsf{k} = \mathcal{E}\mathsf{h}$ and $\mathsf{kh} = \mathsf{k}\mathcal{E}\mathsf{h} = \mathsf{h}^2$; also, $\mathsf{k}\sigma\mathsf{h} = \mathsf{k}\sigma\mathcal{E}\mathsf{h} = \mathsf{k}\mathcal{E}\sigma\mathsf{h} = \mathsf{h}\sigma\mathsf{h}$. □

Corollary 2.1
$\sqsubset$ *is an order relation. Moreover, any (non-empty) bounded family* $(\mathcal{H}, \mathsf{h}_i, \sigma)$ *admits a g.l.b. with respect to* $\sqsubset$.

Proof : Transitivity, antisymmetry are obvious. If $\mathcal{E}_i\sigma = \sigma\mathcal{E}_i$ and $\mathsf{h_i} = \mathsf{k}\mathcal{E}_\mathsf{i}$, then $\bigsqcap_i \mathsf{h}_i := \mathsf{k}\bigcap_i \mathcal{E}_i$ is the desired greatest lower bound. □

Pointwise directed l.u.b. are like direct limits, whereas bounded stable g.l.b. are like pull-backs ; some distributivity is therefore expected. We state it in the case of a binary pull-back, just for readability.

Proposition 3
Assume that $(\mathcal{H}, \mathsf{g}_i, \sigma)$, $(\mathcal{H}, \mathsf{h}_i, \sigma)$, *and* $(\mathcal{H}, \mathsf{k}_i, \sigma)$ *are directed increasing nets w.r.t. the pointwise order, with respective l.u.b.* $(\mathcal{H}, \mathsf{g}, \sigma)$, $(\mathcal{H}, \mathsf{h}, \sigma)$, *and* $(\mathcal{H}, \mathsf{k}, \sigma)$ *; assume that* $(\mathcal{H}, \mathsf{g}_i, \sigma) \sqsubset (\mathcal{H}, \mathsf{k}_i, \sigma)$ *and* $(\mathcal{H}, \mathsf{h}_i, \sigma) \sqsubset (\mathcal{H}, \mathsf{k}_i, \sigma)$ *and let* $\mathsf{f}_i = \mathsf{g}_i \sqcap \mathsf{h}_i$. *Then* $(\mathcal{H}, \mathsf{g}, \sigma) \sqsubset (\mathcal{H}, \mathsf{k}, \sigma)$, *and* $(\mathcal{H}, \mathsf{h}, \sigma) \sqsubset (\mathcal{H}, \mathsf{k}, \sigma)$ *; moreover* $(\mathcal{H}, \mathsf{h}_i, \sigma)$ *is an increasing directed net whose pointwise l.u.b.* $(\mathcal{H}, \mathsf{f}, \sigma)$ *is equal to* $(\mathcal{H}, \mathsf{g}, \sigma) \sqcap (\mathcal{H}, \mathsf{h}, \sigma)$.

Proof : For instance, from $\mathsf{g}_i\mathsf{k}_i = \mathsf{g}_i^2$, we can, using a strong continuity argument (see proposition 18, appendix C.4) and the strong continuity of composition on balls, get $\mathsf{gk} = \mathsf{g}^2$. □

Remark 5
Since $(\mathcal{H}, \mathsf{h}, \sigma) \sqsubset (\mathcal{H}, \mathsf{k}, \sigma) \Rightarrow (\mathcal{H}, -\mathsf{h}, \sigma) \sqsubset (\mathcal{H}, -\mathsf{k}, \sigma)$, proposition 3 works also for *decreasing* nets.

3 Invertible Case

The purpose of this section is to establish the existence of a "solution" to the feedback equation under a reasonable hypothesis, *invertibility*.

3.1 Invertibility

Proposition 4
If σ is a partial symmetry, then σ can be written as the difference $\sigma^+ - \sigma^-$ of two projections such that $\sigma^+ \cdot \sigma^- = 0$.

Proof : This is a special case of a standard result mentioned in appendix C.3. Indeed σ^+ and σ^- are the orthoprojections of the eigenspaces corresponding to the values $+1$ and -1. Spectral calculus yields $\sigma^+ + \sigma^- = \sigma^2$ and $\sigma^+ = 1/2(\sigma^2 + \sigma)$, $\sigma^- = 1/2(\sigma^2 - \sigma)$. □

Definition 7
The cut-system $(\mathcal{H}, \mathsf{h}, \sigma)$ is invertible *(resp.* upper-semi-invertible *: u.s.i.,* lower-semi-invertible *: l.s.i.) when $\sigma - \sigma^2\mathsf{h}\sigma^2$ (resp. $\sigma^+ - \sigma^+\mathsf{h}\sigma^+$, $\sigma^- + \sigma^-\mathsf{h}\sigma^-$) is invertible as an endomorphism of $\mathcal{S} = \sigma^2$ (resp. of σ^+, σ^-).*

By lemma 3.2.13. of [17] (approximate eigenvectors) : a hermitian operator $\mathsf{f} \in B(\mathcal{H})$ is non-invertible iff there exists a sequence (x_n), with $\|x_n\| = 1$ and $\mathsf{f}(x_n) \to 0$ as $n \to \infty$. Of course, $\|x_n\| = 1$ can be weakened into $\|x_n\| \geq a$ for some $a > 0$, replace x_n with $1/\|x_n\| \cdot x_n$. Therefore, $(\mathcal{H}, \mathsf{h}, \sigma)$ is

Invertible : Iff there is no sequence (x_n) with $\|x_n\| = 1$ such that $\sigma^2(x_n) = x_n$ and $\sigma^2\mathsf{h}(x_n) - \sigma(x_n) \to 0$.

Upper-semi-invertible : Iff there is no sequence (x_n) with $\|x_n\| = 1$ such that $\sigma(x_n) = x_n$ and $\sigma^+\mathsf{h}(x_n) - x_n \to 0$.

Lower-semi-invertible : Iff there is no sequence (x_n) with $\|x_n\| = 1$ such that $\sigma(x_n) = -x_n$ and $\sigma^-\mathsf{h}(x_n) + x_n \to 0$.

Lemma 5.1
If f is hermitian and $\|\mathsf{f}\| \leq 1$, $x_n \in \mathcal{H}$, $\|x_n\| = 1$, then $\mathsf{f}(x_n) - x_n \to 0$ iff $\langle \mathsf{f}(x_n) \mid x_n \rangle \to 1$.

Proof : If $\|\mathsf{f}(x_n) - x_n\| \to 0$, then $\langle \mathsf{f}(x_n) \mid x_n \rangle - 1 = \langle \mathsf{f}(x_n) - x_n \mid x_n \rangle \to 0$, hence $\langle \mathsf{f}(x_n) \mid x_n \rangle \to 1$.

Conversely, if $\langle \mathsf{f}(x_n) \mid x_n \rangle \to 1$, since $\|\mathsf{f}(x_n)\| \leq 1$, Cauchy-Schwarz implies that $\|\mathsf{f}(x_n)\| \to 1$, and $\|\mathsf{f}(x_n) - x_n\|^2 = \|\mathsf{f}(x_n)\|^2 + \|x_n\|^2 - 2\langle \mathsf{f}(x_n) \mid x_n \rangle \to 0$, hence $\|\mathsf{f}(x_n) - x_n\| \to 0$. □

Now, if $x_n \in \mathcal{H}$, $\|x_n\| = 1$, is such that $\sigma^2\mathsf{h}(x_n) - x_n \to 0$, we can apply the lemma to $\mathsf{f} := \sigma^2\mathsf{h}\sigma^2$, and we conclude that $\langle \sigma^2\mathsf{h}(x_n) \mid x_n \rangle \to 1$, hence $\langle \mathsf{h}(x_n) \mid x_n \rangle = \langle \mathsf{h}(x_n) \mid \sigma^2(x_n) \rangle \to 1$. Applying the lemma in the other direction, we conclude that $\mathsf{h}(x_n) - x_n \to 0$. The same can be done with σ^+ and σ^-, and we get the simplified characterisations : $(\mathcal{H}, \mathsf{h}, \sigma)$ is

Invertible : Iff there is no sequence (x_n) with $\|x_n\| = 1$ such that $\sigma^2(x_n) = x_n$ and $\mathsf{h}(x_n) - \sigma(x_n) \to 0$.

Upper-semi-invertible : Iff there is no sequence (x_n) with $\|x_n\| = 1$ such that $\sigma(x_n) = x_n$ and $\mathsf{h}(x_n) - x_n \to 0$.

Lower-semi-invertible : Iff there is no sequence (x_n) with $\|x_n\| = 1$ such that $\sigma(x_n) = -x_n$ and $\mathsf{h}(x_n) + x_n \to 0$.

Proposition 5
$(\mathcal{H}, \mathsf{h}, \sigma)$ is invertible iff it is both u.s.i. and l.s.i..

Proof : From the preliminary work just done, it is plain that invertibility implies upper- and lower-semi-invertibilities.

Conversely, assume $(\mathcal{H}, \mathsf{h}, \sigma)$ not invertible, and let (z_n) be such that $\|z_n\| = 1$, $\sigma^2(z_n) = z_n$ and $\mathsf{h}(z_n) - \sigma(z_n) \to 0$. From $\|\mathsf{h}(z_n)\| - \|\sigma(z_n)\| \to 0$, we get $\|\mathsf{h}(z_n)\| \to 1$ and $\langle \mathsf{h}^2(z_n) \mid z_n \rangle = \|\mathsf{h}(z_n)\|^2 \to 1$; by lemma 5.1, $\mathsf{h}^2(z_n) - z_n \to 0$. With $x_n := z_n + \mathsf{h}(z_n)$, $y_n := z_n - \mathsf{h}(z_n)$, $\mathsf{h}(x_n) - x_n \to 0$, $\mathsf{h}(y_n) + y_n \to 0$, $\sigma(x_n) - x_n \to 0$, $\sigma(y_n) + y_n \to 0$. Let $X_n := \sigma(x_n) + x_n$, $Y_n := \sigma(y_n) - y_n$, then $\sigma(X_n) = X_n$ and $\mathsf{h}(X_n) - X_n \to 0$, $\sigma(Y_n) = -Y_n$ and $\mathsf{h}(Y_n) + Y_n \to 0$. Then one of the two sequences $(x_n), (y_n)$ has a subsequence of norm ≥ 1, say x_{n_k} ; then $\|X_{n_k}\| \geq 1$, and $(\mathcal{H}, \mathsf{h}, \sigma)$ is not an u.s.i. system. $\square$

Proposition 6
If $(\mathcal{H}, \mathsf{h}, \sigma) \leq (\mathcal{H}, \mathsf{k}, \sigma)$, then :

(i) *If $(\mathcal{H}, \mathsf{k}, \sigma)$ is u.s.i., so is $(\mathcal{H}, \mathsf{h}, \sigma)$.*

(ii) *If $(\mathcal{H}, \mathsf{k}, \sigma)$ is l.s.i., then $(\mathcal{H}, \lambda\mathsf{h} + (1-\lambda)\mathsf{k}, \sigma)$ is l.s.i. for $0 < \lambda < 1$.*

(iii) *If $(\mathcal{H}, \mathsf{k}, \sigma)$ is l.s.i. and $(\mathcal{H}, \mathsf{h}, \sigma)$ is u.s.i., then $(\mathcal{H}, \lambda\mathsf{h} + (1-\lambda)\mathsf{k}, \sigma)$ is invertible for $0 < \lambda < 1$.*

(iv) *If $(\mathcal{H}, \mathsf{k}, \sigma)$ is invertible, then $(\mathcal{H}, \lambda\mathsf{h} + (1-\lambda)\mathsf{k}, \sigma)$ is invertible for $0 < \lambda < 1$.*

Proof : *(i)* Take (x_n) with $\|x_n\| = 1$ such that $\sigma(x_n) = x_n$; if $\mathsf{h}(x_n) - x_n \to 0$, then lemma 5.1 yields $\langle \mathsf{h}(x_n) \mid x_n \rangle \to 1$, and since $\mathsf{h} \leq \mathsf{k}$, $\langle \mathsf{h}(x_n) \mid x_n \rangle \leq \langle \mathsf{k}(x_n) \mid x_n \rangle \leq 1$ and $\langle \mathsf{k}(x_n) \mid x_n \rangle \to 1$; the same lemma yields $\mathsf{k}(x_n) - x_n \to 0$. So, the upper-semi-invertibility of k implies the upper-semi-invertibility of h.

(ii) Let $\mathsf{g} := \lambda\mathsf{h} + (1-\lambda)\mathsf{k}$, and take (x_n) with $\|x_n\| = 1$ such that $\sigma(x_n) = -x_n$; if $\mathsf{g}(x_n) + x_n \to 0$, then $\langle \mathsf{g}(x_n) \mid x_n \rangle \to -1$, and necessarily $\langle \mathsf{k}(x_n) \mid x_n \rangle \to -1$.

(iii) Combination of (ii) with its dual version.

(iv) $(\mathcal{H}, \lambda\mathsf{h} + (1-\lambda)\mathsf{k}, \sigma)$ is l.s.i. because by (ii) and u.s.i. because $\lambda\mathsf{h} + (1-\lambda)\mathsf{k} \leq \mathsf{k}$, and (i).

□

3.2 The Normal Form

In what follows, we shall deal with cut-systems, often called $(\mathcal{H}, \mathsf{h}, \sigma)$, or $(\mathcal{H}, \mathsf{k}, \sigma)$. It is convenient to adopt a matrix-like notations ("blocks", see appendix C.7) corresponding to the direct sum decomposition $\mathcal{H} = \mathcal{R} \oplus \mathcal{S}$, with $\mathcal{S} = \sigma^2$; we shall implicitly assume that $\mathsf{h} = \begin{bmatrix} \mathsf{A} & \mathsf{B}^* \\ \mathsf{B} & \mathsf{C} \end{bmatrix}$; in the concrete conditions we shall deal with, k will share the same coefficients A, B, i.e., $\mathsf{k} = \begin{bmatrix} \mathsf{A} & \mathsf{B}^* \\ \mathsf{B} & \mathsf{D} \end{bmatrix}$.

Lemma 1.1
If f is an invertible hermitian and p is positive, then the function $(\mathsf{f} - \lambda\mathsf{p})^{-1}$ is defined and monotonous on an open neighbourhood of 0 in $\mathbb{R}$.

Proof : $(\mathsf{f} - \lambda\mathsf{p})^{-1} = \mathsf{f}^{-1} + \lambda\mathsf{f}^{-1}\mathsf{p}\mathsf{f}^{-1} + \lambda^2\mathsf{f}^{-1}\mathsf{p}\mathsf{f}^{-1}\mathsf{p}\mathsf{f}^{-1} + \ldots$. The power series converges for $|\lambda| < \|\mathsf{f}^{-1}\mathsf{p}\|^{-1}$. We can group the non constant terms : let $\mathsf{p_n} := \lambda^{2n+1} \cdot \mathsf{f}^{-1} \cdot (\mathsf{p}\mathsf{f}^{-1})^n \cdot \sqrt{\mathsf{p}} \cdot (\mathrm{I} + \lambda\sqrt{\mathsf{p}}\,\mathsf{f}^{-1}\sqrt{\mathsf{p}}) \cdot \sqrt{\mathsf{p}} \cdot (\mathsf{f}^{-1}\mathsf{p})^n \cdot \mathsf{f}^{-1}$, so that $(\mathsf{f} - \lambda\mathsf{p})^{-1} = \mathsf{f}^{-1} + \sum \mathsf{p}_n$. For $|\lambda|$ small enough, $\mathrm{I} + \lambda\sqrt{\mathsf{p}}\,\mathsf{f}^{-1}\sqrt{\mathsf{p}}$ is positive and all p_n are positive. From this, it is easy to conclude. □

Theorem 1 (Monotonicity)
The map $(\mathcal{H}, \mathsf{h}, \sigma) \rightsquigarrow (\sigma - \mathsf{C})^{-1}$ from invertible cut-systems to $Her(\mathcal{S})$ is monotonous (increasing) and order-continuous w.r.t. the parameter C.

Proof : Keep in mind, although it does not quite matter here, that the parameters A, B of the block decomposition are kept constant. If the cut-systems $(\mathcal{H}, \mathsf{h}, \sigma) \leq (\mathcal{H}, \mathsf{k}, \sigma)$ are both invertible, then, for $\lambda \in [0, 1]$, $(\mathcal{H}, \lambda\mathsf{h} + (1-\lambda)\mathsf{k}, \sigma)$ is invertible (proposition 6). Let $\mathsf{p} := \mathsf{D} - \mathsf{C} \in Her^+(\mathcal{S})$; the function $\varphi : \lambda \rightsquigarrow (\sigma - \mathsf{C} + \lambda\mathsf{p})^{-1}$ is a continuous map from $[0, 1]$ to the Banach space $Her(\mathcal{S})$. By lemma 1.1, this map is locally increasing, and it must be globally increasing. We conclude that $\varphi(0) \leq \varphi(1)$, i.e., $(\sigma - \mathsf{C})^{-1} \leq (\sigma - \mathsf{D})^{-1}$.

Assume now that $(\mathcal{H}, \mathsf{h}_i, \sigma)(i \in I)$ is an increasing *net* of invertible systems, with an invertible l.u.b. $(\mathcal{H}, \mathsf{h} := \sup_{i\in I} \mathsf{h}_i, \sigma)$. By proposition 18 of appendix C.4, $\sigma - \mathsf{C}_i$ and $(\sigma - \mathsf{C}_i)^{-1}$ respectively converge to $\sigma - \mathsf{C}$ and some E in the strong-operator topology ; by monotonicity,
$\mathsf{E} := \sup_{i\in I}(\sigma - \mathsf{C}_i)^{-1} \leq (\sigma - \mathsf{C})^{-1}$. Now observe that that the $\sigma - \mathsf{C}_i$ have their norms bounded by 2 ; multiplication is strong operator-continuous, provided the left argument remains bounded : this implies
$\sigma - \mathsf{C}_i \cdot (\sigma - \mathsf{C}_i)^{-1} \to (\sigma - \mathsf{C}) \cdot \mathsf{E}$. Hence $(\sigma - \mathsf{C}) \cdot \mathsf{E} = \mathcal{S}$, and, since $\sigma - \mathsf{C}$ is invertible, $\mathsf{E} = \sup_{i\in I}(\sigma - \mathsf{C}_i)^{-1} = (\sigma - \mathsf{C})^{-1}$.

Downwards continuity is proved in the same way. □

Definition 8 (Normal Form)
If $(\mathcal{H}, \mathsf{h}, \sigma)$ is invertible, its normal form *is defined by the equation :*

$$\sigma[\![\mathsf{h}]\!] := \mathsf{A} + \mathsf{B}^*(\sigma - \mathsf{C})^{-1}\mathsf{B} \tag{3}$$

The normal form corresponds to the "visible part" of the feedback equation :

Theorem 2 (Normal Form)
If $(\mathcal{H}, \mathsf{h}, \sigma)$ is invertible, then the feedback equation admits the normal form $\sigma[\![\mathsf{h}]\!]$ as unique solution. $\sigma[\![\mathsf{h}]\!]$ is of norm at most 1, and it is norm-continuous, monotonous (increasing) and order-continuous w.r.t. the input h.

Proof : We want to solve the equation : $\begin{bmatrix} \mathsf{A} & \mathsf{B}^* \\ \mathsf{B} & \mathsf{C} - \sigma \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ 0 \end{bmatrix}$. Invertibility quite means that $\sigma - \mathsf{C}$ is invertible. It is then obvious that
$x' = (\mathsf{A} + \mathsf{B}^*(\sigma - \mathsf{C})^{-1}\mathsf{B})(x)$, $y = ((\sigma - \mathsf{C})^{-1}\mathsf{B})(x)$ is a solution to the equation. This is indeed *the* solution : if $\begin{bmatrix} \mathsf{A} & \mathsf{B}^* \\ \mathsf{B} & \mathsf{C} - \sigma \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} x' \\ 0 \end{bmatrix}$, then $(\mathsf{C} - \sigma)(y) = 0$ and $x' = 0$. Hence, the (visible part of) the solution is given by
$\sigma[\![\mathsf{h}]\!] := \mathsf{A} + \mathsf{B}^*(\sigma - \mathsf{C})^{-1}\mathsf{B}$.

It is plain that $\sigma[\![\mathsf{h}]\!]$ is hermitian and that the dependency $\mathsf{h} \rightsquigarrow \sigma[\![\mathsf{h}]\!]$ is norm-continuous.

Finally, from $\mathsf{h}(x \oplus y) = x' \oplus \sigma(y)$, we get
$\|x'\|^2 + \|y\|^2 = \|x'\|^2 + \|\sigma(y)\|^2 \leq \|x\|^2 + \|y\|^2$, hence $\|x'\|^2 \leq \|x\|^2$: we just established that $\|\sigma[\![\mathsf{h}]\!]\| \leq 1$.

Let us now consider the behaviour w.r.t. the pointwise order. First observe that, if "coefficients" A, B in the "block" of $\mathsf{h}, \mathsf{k}, \ldots$ are kept constant, then, by theorem 1, we get order-monotonicity and order-continuity.

The general case is reduced to this case by means of the "Tortoise Principle" of appendix B, and the reduction being extremely simple, the only problem is to avoid pedantism when saying something obvious : hence there might be some (slight) abuses of notations. The idea is to add two copies of $\mathcal{R}$, say

$\mathcal{R}_2, \mathcal{R}_1$, hence $\mathcal{H}$ is replaced with $\mathcal{K} := \mathcal{R}_2 \oplus \mathcal{R}_1 \oplus \mathcal{H}$. If u is an isometry between $\mathcal{R}$ and $\mathcal{R}_1$, we can "extend" our feedback σ into $\tau(x_2 \oplus x_1 \oplus x \oplus z) := 0 \oplus \mathsf{u}(x) \oplus \mathsf{u}^*(x_1) \oplus 0$. Similarly, if v is an isometry between $\mathcal{R}_1$ and $\mathcal{R}_2$, we can "extend" h into $\Psi(\mathsf{h})(x_2 \oplus x_1 \oplus y) := \mathsf{v}(x_1) \oplus \mathsf{v}^*(x_1) \oplus \mathsf{h}(y)$. There are a few obvious facts about this replacement :

(i) The map $\mathsf{h} \rightsquigarrow \Psi(\mathsf{h})$ is monotonous and order-continuous.

(ii) The system $(\mathcal{K}, \Psi(\mathsf{h}), \tau)$ is invertible ; the best remains to solve the equation "manually", i.e., by "equality pushing". The normal form is explicitly given by :

$$\tau[\![\Psi(\mathsf{h})]\!] = \mathsf{v}\mathsf{u}\sigma[\![\mathsf{h}]\!]\mathsf{u}^*\mathsf{v}^* \tag{4}$$

(iii) The normal form $\tau[\![\Psi(\mathsf{h})]\!]$ is monotonous and order-continuous w.r.t. the input $\Psi(\mathsf{h})$: this is because $\Psi(\mathsf{h})$ can be written (w.r.t. a decomposition $\mathcal{R}_2 \oplus {\mathcal{R}_2}^{\perp}$) as a block : $\begin{bmatrix} 0 & \mathsf{v} & 0 & 0 \\ \mathsf{v}^* & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \end{bmatrix}$; we are therefore back to the case "A, B constant".

(iv) The normal form $\sigma[\![\mathsf{h}]\!]$ is monotonous and order-continuous w.r.t. the input h : combination of the previous observations.

$\square$

Remark 6
Technically speaking, the Tortoise introduces a cut with an identity axiom, so that the whole *variable* part of the net is now "invisible".

3.3 Associativity

Definition 9 (Independence)
Two feedbacks σ, τ are independent when $\sigma\tau = 0$ $(= \tau\sigma)$.

In presence of two *independent* feedbacks σ, τ, there are several possibilities to reach a "normal form" for $(\mathcal{H}, \mathsf{h}, \sigma + \tau)$. Either we directly form $(\sigma + \tau)[\![\mathsf{h}]\!]$, or first normalise $(\mathcal{H}, \mathsf{h}, \sigma)$, yielding the normal form $\sigma[\![\mathsf{h}]\!]$, and then normalise $(\mathcal{R}, \sigma[\![\mathsf{h}]\!], \tau)$, yielding $\tau[\![\sigma[\![\mathsf{h}]\!]]\!]$. We can also do it the other way around, leading to $\sigma[\![\tau[\![\mathsf{h}]\!]]\!]$. The question is whether or not these protocols yield the same output.

The presence of a double feedback occurs naturally when we deal with the static —category-theoretic— interpretation of logic : we must establish associativity of the composition[7] of morphisms, $f \circ (g \circ h) = (f \circ g) \circ h$.

[7]See also appendix B.2.

Now, composition corresponds to the cut-rule, i.e., to a feedback ; here we are given two compositions, i.e., two independent feedbacks σ, τ, together with a hermitian k corresponding to the three morphisms f, g, h "put together". $f \circ (g \circ h)$ corresponds to $\sigma[\![\tau[\![\mathsf{k}]\!]]\!]$, whereas $(f \circ g) \circ h$ corresponds to $\tau[\![\sigma[\![\mathsf{k}]\!]]\!]$. To sum up, the question of equating the three protocols above is nothing but the soundness of the categorical approach to logic : this is why we speak of *associativity*. In traditional rewriting technology, associativity comes from the possibility of performing the rewritings in any order, i.e., from the familiar Church-Rosser property.

Theorem 3 (Associativity)
Assume that σ, τ are independent and write $\mathcal{H} = \mathcal{R} \oplus \mathcal{S} \oplus \mathcal{T}$. Then $(\mathcal{H}, \mathsf{h}, \sigma + \tau)$ is invertible iff $(\mathcal{H}, \mathsf{h}, \tau)$ and $(\mathcal{R} \oplus \mathcal{S}, \tau[\![\mathsf{h}]\!], \sigma)$ are invertible. Moreover

$$(\sigma + \tau)[\![\mathsf{h}]\!] = \sigma[\![\tau[\![\mathsf{h}]\!]]\!] \tag{5}$$

Proof : First assume that $(\mathcal{H}, \mathsf{h}, \tau)$ and $(\mathcal{R} \oplus \mathcal{S}, \tau[\![\mathsf{h}]\!], \sigma)$ are invertible ; the left hand side corresponds to the equation, $x \in \mathcal{R}$ being given :

$$\mathsf{h}(x \oplus y \oplus z) = x' \oplus \sigma(y) \oplus \tau(z) \tag{6}$$

whose "visible part" is $x' = (\sigma + \tau)[\![\mathsf{h}]\!](x)$. This equation can be solved in two steps : first, given $x \in \mathcal{R}, y' \in \mathcal{S}$, solve :

$$\mathsf{h}(x \oplus y' \oplus z) = x' \oplus y'' \oplus \tau(z) \tag{7}$$

whose visible part is $x' \oplus y'' = \tau[\![\mathsf{h}]\!](x \oplus y)$. Then, add the constraint $y'' = \sigma(y')$, which amounts at solving :

$$\tau[\![\mathsf{h}]\!](x \oplus y) = x' \oplus \sigma(y) \tag{8}$$

and whose visible part is given by $x' = \sigma[\![\tau[\![\mathsf{h}]\!]]\!](x)$. We just established equation (5) "manually". We need a little more care concerning invertibility matters.

(i) If $(\mathcal{H}, \mathsf{h}, \tau)$ is not invertible, take a sequence $(x_n) \in \mathcal{T}$ of norm 1 such that $\mathsf{h}(x_n) - \tau(x_n) \to 0$; since $\sigma(x_n) = 0$, we just found approximate eigenvectors for $\mathsf{h} - (\sigma + \tau)$.

(ii) If $(\mathcal{H}, \mathsf{h}, \tau)$ is invertible but $(\mathcal{R} \oplus \mathcal{S}, \tau[\![\mathsf{h}]\!], \sigma)$ is not invertible, take a sequence $(y_n) \in \mathcal{S}$ of norm 1 such that $\tau[\![\mathsf{h}]\!](y_n) - \sigma(y_n) \to 0$. There exists z_n such that $\mathsf{h}(y_n \oplus z_n) = \tau[\![\mathsf{h}]\!](y_n) \oplus \tau(z_n)$. Then $(y_n \oplus z_n)$ is such that $\mathsf{h}(y_n \oplus z_n) - (\sigma + \tau)(y_n \oplus z_n) \to 0$, and, once more, $(\mathcal{H}, \mathsf{h}, \sigma + \tau)$ is not invertible. Summing up, we just proved that the invertibility of the full system implies the invertibility of the partial systems corresponding to a two-step "normalisation".

(iii) W.r.t. the decomposition $\mathcal{H} = \mathcal{R} \oplus \mathcal{S} \oplus \mathcal{T}$, write $\mathsf{h} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \mathsf{A} & \mathsf{B}^* \\ \cdot & \mathsf{B} & \mathsf{C} \end{bmatrix}$

Assuming that $\tau - \mathsf{C}$ and $\sigma - \tau[\![\mathsf{h}]\!]$ are invertible, we want to show that $\mathsf{k} := \begin{bmatrix} \sigma - \mathsf{A} & -\mathsf{B}^* \\ -\mathsf{B} & \tau - \mathsf{C} \end{bmatrix}$ is invertible. Define :

$$\mathsf{g} := \begin{bmatrix} (\sigma - \tau[\![\mathsf{h}]\!])^{-1} & (\sigma - \tau[\![\mathsf{h}]\!])^{-1}\mathsf{B}^*(\tau - \mathsf{C})^{-1} \\ (\tau - \mathsf{C})^{-1}\mathsf{B}(\sigma - \tau[\![\mathsf{h}]\!])^{-1} & (\tau - \mathsf{C})^{-1}\mathsf{B}(\sigma - \tau[\![\mathsf{h}]\!])^{-1}\mathsf{B}^*(\tau - \mathsf{C})^{-1} + (\tau - \mathsf{C})^{-1} \end{bmatrix}$$

A straightforward (but painful) computation yields $\mathsf{gk} = \begin{bmatrix} \mathcal{S} & 0 \\ 0 & \mathcal{T} \end{bmatrix}$. This proves the left-invertibility of k ; since k is hermitian, it is invertible : from $\mathsf{gk} = \mathrm{I}$, we get $\mathsf{k}^*\mathsf{g}^* = \mathsf{kg} = \mathrm{I}$.

□

Remark 7
Putting things together, i.e., using the theorem, the inverse can be expressed in a more symmetrical way, typically :

$$\mathsf{k}^{-1} = \begin{bmatrix} (\sigma - \tau[\![\mathsf{h}]\!])^{-1} & (\sigma - \tau[\![\mathsf{h}]\!])^{-1}\mathsf{B}^*(\tau - \mathsf{C})^{-1} \\ (\tau - \sigma[\![\mathsf{h}]\!])^{-1}\mathsf{B}(\sigma - \mathsf{A})^{-1} & (\tau - \sigma[\![\mathsf{h}]\!])^{-1} \end{bmatrix}$$

or the adjoint expression

$$\mathsf{k}^{-1} = \begin{bmatrix} (\sigma - \tau[\![\mathsf{h}]\!])^{-1} & (\sigma - \mathsf{A})^{-1}\mathsf{B}^*(\tau - \sigma[\![\mathsf{h}]\!])^{-1} \\ (\tau - \mathsf{C})^{-1}\mathsf{B}(\sigma - \tau[\![\mathsf{h}]\!])^{-1} & (\tau - \sigma[\![\mathsf{h}]\!])^{-1} \end{bmatrix}$$

3.4 Stability and Incarnation

Proposition 7
If $(\mathcal{H}, \mathsf{h}, \sigma) \sqsubset (\mathcal{H}, \mathsf{k}, \sigma)$ and $(\mathcal{H}, \mathsf{k}, \sigma)$ is invertible, then $(\mathcal{H}, \mathsf{h}, \sigma)$ is invertible.

Proof : Completely immediate. □

In the next theorem, all systems are supposed to be invertible.

Theorem 4 (Stability)
The normal form is compatible with stability, more precisely :

(i) *If $(\mathcal{H}, \mathsf{h}, \sigma) \sqsubset (\mathcal{H}, \mathsf{k}, \sigma)$, then $\sigma[\![\mathsf{h}]\!] \sqsubset \sigma[\![\mathsf{k}]\!]$.*

(ii) *If $(\mathcal{H}, \mathsf{h}_i, \sigma) \sqsubset (\mathcal{H}, \mathsf{k}, \sigma)$, then $\sigma[\![\bigsqcap_i \mathsf{h}_i]\!] = \bigsqcap_i \sigma[\![\mathsf{h}_i]\!]$.*

Proof : *(i)* Assume that $\mathsf{h} = \mathsf{k}\mathcal{E}, \sigma\mathcal{E} = \mathcal{E}\sigma$; then $\mathsf{h} = \mathcal{E}\mathsf{k}$ and $\mathcal{E}$ commutes to both of $\mathcal{R}, \mathcal{S}$, and let $\mathcal{F} := \mathcal{R}\mathcal{E}, \mathcal{G} := \mathcal{S}\mathcal{E}$. If $\mathsf{h}(x \oplus y) = x' \oplus \sigma(y)$, then using $\mathsf{h} = \mathsf{k}\mathcal{E}$, we get $\mathsf{k}(\mathcal{F}(x) \oplus \mathcal{G}(y)) = x' \oplus \sigma(y)$; using $\mathsf{h} = \mathcal{E}\mathsf{k}$, we get $x' \oplus \sigma(y) = \mathcal{F}(x') \oplus \mathcal{G}(\sigma(y)) = \mathcal{F}(x') \oplus \sigma(\mathcal{G}(y)))$. Summing up, we get $\mathsf{k}(\mathcal{F}(x) \oplus \mathcal{G}(y)) = x' \oplus \sigma(\mathcal{G}(y))$, i.e., $\sigma[\![\mathsf{k}]\!](\mathcal{F}(x)) = \sigma[\![\mathsf{h}]\!](x)$.

(ii) Immediate.

□

This justifies the following definition :

Definition 10 (Incarnation)
If $(\mathcal{H}, \mathsf{h}, \sigma)$ *is an invertible cut-system, its* incarnation *is the smallest* $(\mathcal{H}, \mathsf{k}, \sigma) \sqsubset (\mathcal{H}, \mathsf{h}, \sigma)$ *such that* $\sigma[\![\mathsf{k}]\!] = \sigma[\![\mathsf{h}]\!]$.

Incarnation, namely the "useful part" of a system, plays a central role in ludics. Remember that for instance the *mystery of incarnation*[8] reduces the Cartesian product to an intersection !

3.5 Normal Forms and Winning

Let us investigate the normal form in the *bipartite* case. The next proposition is quite surprising :

Proposition 8
If a bipartite hermitian is invertible, its inverse is still bipartite.

Proof : Let $x = \mathsf{h}(x' \oplus y')$, with $x, x' \in \mathcal{H}^+, y' \in \mathcal{H}^-$; then

$$\begin{aligned}\langle \mathsf{h}^{-1}(x) \mid x\rangle &= \langle x' \oplus y' \mid x\rangle = \langle x' \mid x\rangle = \langle x' \mid \mathsf{h}(x' \oplus y')\rangle \\ &= \langle x' \mid \mathsf{h}(x')\rangle + \langle x' \mid \mathsf{h}(y')\rangle\end{aligned}$$

and $\langle x' \mid \mathsf{h}(y')\rangle = \langle \mathsf{h}(x') \mid y'\rangle = \langle x \mid y'\rangle - \langle \mathsf{h}(y') \mid y'\rangle = -\langle \mathsf{h}(y') \mid y'\rangle$. Summing up, we find $\langle \mathsf{h}^{-1}(x) \mid x\rangle = \langle \mathsf{h}(x') \mid x'\rangle - \langle \mathsf{h}(y') \mid y'\rangle \geq 0$. In the same way one proves that $\langle \mathsf{h}^{-1}(y) \mid y\rangle \leq 0$ for $y \in \mathcal{H}^-$. □

Let us now introduce an inessential —but useful— tool : assume that, w.r.t. the decomposition $\mathcal{H} = \mathcal{R} \oplus \mathcal{S}$, $\mathsf{h} = \begin{bmatrix} \mathsf{A} & \mathsf{B}^* \\ \mathsf{B} & \mathsf{C} \end{bmatrix}$ and given $x \in \mathcal{R} = \mathcal{S}^\perp$, with $\|x\| \leq 1$, we can define h_x as a hermitian of $\mathbb{C} \oplus \mathcal{S}$: $\mathsf{h}_x := \begin{bmatrix} \langle \mathsf{A}(x) \mid x\rangle & \mathsf{B}(x)^* \\ \mathsf{B}(x) & \mathsf{C} \end{bmatrix}$. Obviously, $\|\mathsf{h}_x\| \leq \|\mathsf{h}\|$.

Proposition 9
If h *is invertible and* $x \in \mathcal{R}$, $\|x\| \leq 1$, *then :* $\sigma[\![\mathsf{h}_x]\!] = \langle \sigma[\![\mathsf{h}]\!](x) \mid x\rangle$.

Proof : Completely obvious, e.g., from the explicit formula for $\sigma[\![\mathsf{h}]\!]$:

$$\begin{aligned}\langle (\mathsf{A} + \mathsf{B}^*(\sigma - \mathsf{C})^{-1}\mathsf{B})(x) \mid x\rangle &= \langle \mathsf{A}(x) \mid x\rangle + \langle ((\sigma - \mathsf{C})^{-1}\mathsf{B})(x) \mid \mathsf{B}(x)\rangle = \\ &= \langle \mathsf{A}(x) \mid x\rangle + \mathsf{B}(x)^* \cdot (\sigma - \mathsf{C})^{-1} \cdot \mathsf{B}(x)\end{aligned}$$

□

[8] See [13].

Theorem 5 (Winning)
The normal form of a bipartite invertible closed system is bipartite. Furthermore, if the the system is winning, so is its normal form.

Proof : Assume that $\mathcal{R} = \mathcal{R}^+ \oplus \mathcal{R}^-$; we must show that, for $x \in \mathcal{R}^+$ (resp. $x \in \mathcal{R}^-$) $\langle \sigma[\![\mathsf{h}]\!](x) \mid x\rangle \geq 0$ (resp. $\langle \sigma[\![\mathsf{h}]\!](x) \mid x\rangle \leq 0$). Let us treat the case where $x \in \mathcal{R}^+$, and let us assume that $\|x\| \leq 1$. By proposition 9, we can reduce the problem to the case of h_x, i.e., to the case where $\mathcal{H} = \mathbb{C} \oplus \mathcal{S}$, the component $\mathbb{C}$ being declared positive. Let us enlarge[9] $\mathcal{H}$ into $\mathcal{K} := \mathbb{C} \oplus \mathbb{C} \oplus \mathcal{H}$, the first $\mathbb{C}$ being positive, the second one being negative. Then, starting with the bipartite $\mathsf{h} = \begin{bmatrix} a & y^* \\ y & \mathsf{C} \end{bmatrix}$ (in particular $a \geq 0$), consider $\mathsf{k} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & a & y^* \\ 0 & 0 & y & \mathsf{C} \end{bmatrix}$, still bipartite. If $\tau = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, then σ, τ are independent and $(\mathcal{K}, \mathsf{k}, \sigma + \tau)$ is invertible ; indeed $\tau[\![\sigma[\![\mathsf{k}]\!]]\!] = \sigma[\![\mathsf{h}]\!]$. Let $\mathsf{g} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -a & -y^* \\ 0 & -y & \sigma - \mathsf{C} \end{bmatrix}$, and $\mathsf{g}^{-1} = \begin{bmatrix} b & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$, then $\sigma[\![\mathsf{h}]\!] = \tau[\![\sigma[\![\mathsf{k}]\!]]\!] = (\sigma + \tau)[\![\mathsf{k}]\!] = b$; $-\mathsf{g}$ (and its inverse $-\mathsf{g}^{-1}$, by proposition 8) is bipartite. b is located in the second copy of $\mathbb{C}$, declared negative, so $-b \leq 0$ and $\sigma[\![\mathsf{h}]\!] \geq 0$. □

Corollary 5.1
If a bipartite system is winning, so is its normal form.

Proof : We observed (proposition 1) that winning is the same as remaining bipartite when $\mathcal{H}^+$ and $\mathcal{H}^-$ are swapped. If $(\mathcal{H}, \mathsf{h}, \sigma)$ is winning, then its normal form is bipartite w.r.t. the decomposition $\mathcal{R}^+, \mathcal{R}^-$, but also w.r.t. the opposite decomposition, hence it is winning. □

4 Termination

This section is concerned with the "natural" solution of the feedback equation (1). The subsection on deadlocks is devoted to the unicity of the "invisible" component y : essentially, we can assume unicity, i.e., "remove deadlocks". The subsection on computational size makes use of the norm of the *execution operator* and proves an inequality corresponding to associativity.

[9]Another application of the Tortoise Principle, see appendix B.

4.1 Deadlocks

We use the notations of section 3.2 : w.r.t. the decomposition $\mathcal{H} = \mathcal{R} \oplus \mathcal{S}$, $\mathsf{h} = \begin{bmatrix} \mathsf{A} & \mathsf{B}^* \\ \mathsf{B} & \mathsf{C} \end{bmatrix}$.

Definition 11 (Deadlocks)
The cut-system $(\mathcal{H}, \mathsf{h}, \sigma)$ *is* deadlock-free *when* $\sigma - \mathsf{C}$ *is* injective *as an endomorphism of* $\mathcal{S} = \sigma^2$.

Remark 8
The notion of *deadlock-free* algorithm of [10] ($\sigma\mathsf{C}$ weakly nilpotent) is less general.

Invertible systems are deadlock-free, but the converse is not true : in infinite dimension, injectivity does not imply invertibility. But there is a relation between the two notions :

Invertible : Means the absence of a *approximate* eigenvectors (for the value 0), $x_n \in \mathcal{S}$, with $\|x_n\| = 1$, and $\sigma(x_n) - \mathsf{h}(x_n) \to 0$; we know that h can be replaced with C : $\sigma(x_n) - \mathsf{C}(x_n) \to 0$.

Deadlock-free : No longer the approximate notion, the exact one : x_n constant. In particular, $\sigma - \mathsf{C}$ is injective on $\mathcal{S}$ iff $\sigma - \mathsf{h}$ is injective on the same $\mathcal{S}$.

Definition 12 (Deadlocks)
The deadlock *space* $\mathcal{Z} \subset \mathcal{S}$ *is defined by* $\mathcal{Z} := \ker(\sigma - \mathsf{C})$, *and its (non-zero) elements are called* deadlocks. *To be deadlock-free therefore means that there is no deadlock.*

Proposition 10
The deadlock space $\mathcal{Z}$ *(or rather the associated orthoprojection) commutes to* $\sigma, \mathsf{C}, \mathsf{h}$.

Proof : The proof is a sort of simplified version of proposition 5. If $z \in \mathcal{Z}$, then $\mathsf{C}(z) = \sigma(z)$ and $\|\mathsf{C}(z)\| = \|\sigma(z)\| = \|z\|$; since $\|\mathsf{C}\| \leq 1$, $\mathsf{C}^2(z) = z$ (use lemma 5.1 of section 3.1) and let us consider $x := z + \mathsf{C}(z)$, $y := z - \mathsf{C}(z)$. Then $\mathsf{C}(x) = x$, but also $\sigma(x) = \sigma(z) + \sigma\mathsf{C}(z) = \mathsf{C}(z) + \sigma^2(z) = x$; in the same way, $\mathsf{C}(y) = \sigma(y) = -y$. $\mathcal{Z}$ therefore appears as the sum of two orthogonal subspaces, $\mathcal{Z}^+ = \{x; \mathsf{C}(x) = \sigma(x) = x\}$ and
$\mathcal{Z}^- = \{y; \mathsf{C}(y) = \sigma(y) = -y\}$. From this $\mathcal{Z}$ commutes to both of σ and C. Finally, if $z \in \mathcal{Z}$, $\mathsf{h}(z) = \mathsf{B}^*(z) \oplus \mathsf{C}(z)$ and $\|\mathsf{C}(z)\| = \|z\|$ together with $\|\mathsf{h}(z)\| \leq \|z\|$ force $\mathsf{B}^*(z) = 0$. This shows that $\mathsf{B}^*\mathcal{Z} = \mathsf{0}$ and $\mathcal{Z}$ commutes with h. $\square$

Remark 9
The previous proposition explains why the kernel of $\sigma - \mathsf{C}$ (indeed, of $\sigma - \mathsf{h}$) is styled the *deadlock space* : nothing enters, nothing exits. Deadlocks are excluded from logical systems and even from λ-calculi[10]. But they are rather friendly : don't bother them, they don't bother you. In particular, we can always "remove" a deadlock by replacing h with $(\mathrm{I} - \mathcal{Z})\mathsf{h}$ without any essential prejudice to the feedback equation.

If $0 \in \mathrm{sp}(\sigma - \mathsf{C})$[11] we must be cautious : in case 0 is isolated, this corresponds to a deadlock that we can easily ignore. The situation is quite different when 0 is an accumulation point of $\mathrm{sp}(\sigma - \mathsf{C})$[12]. We are no longer dealing with deadlocks, but with infinite computations, and things are not that easy !

4.2 Computational Size

Let $(\mathcal{H}, \mathsf{h}, \sigma)$ be a cut-system and let $\mathcal{Z}$ be its deadlock space. Two different solutions of the feedback equation (1) $x' \oplus y$, $x'' \oplus y'$ w.r.t. the same input x are such that $x' = x''$: from

$$\mathsf{h}(0 \oplus y - y') = x' - x'' \oplus \sigma(y - y') \tag{9}$$

and $\|h\| \leq 1$, $\|\sigma(y - y')\| = \|y - y'\|$, we get $\|x' - x''\| = 0$, hence $x' = x''$[13]. But obviously two different solutions $x' \oplus y$, x', y' can still differ on their invisible parts y, y', in which case $y - y'$ is a deadlock.

Using proposition 10, we easily obtain a sort of "unicity" :

(i) If $x' \oplus y$ is a solution to the feedback equation w.r.t. the input x, so is $x' \oplus \mathcal{Z}(y)$.

(ii) $x' \oplus \mathcal{Z}(y)$ is —among all solutions $x' \oplus y'$ corresponding to the input x— the one of smallest norm.

(iii) This smallest choice amounts at "removing deadlocks", i.e., at replacing the system $(\mathcal{H}, \mathsf{h}, \sigma)$ with $(\mathcal{H}, (\mathrm{I} - \mathcal{Z})\mathsf{h}, \sigma)$.

Definition 13 (Termination)
The cut-system $(\mathcal{H}, \mathsf{h}, \sigma)$ is terminating *when the feedback equation (1) has a solution $x' \oplus y$ for every input $x \in \mathcal{R}$. In which case we define the* execution operator *$\mathrm{ex}(\mathsf{h}, \sigma)$ as the bounded operator from $\mathcal{R}$ to $\mathcal{H}$ assigning to $x \in \mathcal{R}$ the vector $x \oplus \mathcal{Z}(y)$*[14].

[10] In proof-nets they correspond to *short trips*, or cycles in the Danos-Regnier criterion [5].

[11] Seen as an operator on $\mathcal{S}$.

[12] Upper-semi-invertibility means that 0 is not limit of strictly positive points of the spectrum.

[13] This was first noticed during the proof of the normal form theorem 2, when showing that the normal form is of norm at most 1.

[14] This is not a misprint, I didn't mean $x' \oplus \mathcal{Z}(y)$, see the proof of theorem 6 below.

Remark 10
The typical terminating case is invertibility ; in which case (with the notations of definition 8), $\mathrm{ex}(\mathsf{h},\sigma) = \mathcal{R} \oplus (\sigma - \mathsf{C})^{-1}\mathsf{B}$. But there are many other cases of termination, for instance when $\mathsf{B} = \mathsf{0}$.

Definition 14 (Computational Size)
If $(\mathcal{H},\mathsf{h},\sigma)$ is terminating, its computational size *is defined by*

$$\mathrm{size}(\mathsf{h},\sigma) := \|\mathrm{ex}(\mathsf{h},\sigma)\| \tag{10}$$

Termination may be a rather accidental property ; this is why the following theorem is less powerful than its original model, theorem 3. In what follows, we use the (undefined) notation $\tau[\![\mathsf{h}]\!]$ in the obvious sense ; but it might be inconsistent with the general definition given later.

Theorem 6 (Associativity of Size)
Assume that σ,τ are independent, and that $(\mathcal{H},\mathsf{h},\tau)$ and $(\mathcal{R}\oplus\mathcal{S},\tau[\![\mathsf{h}]\!],\sigma)$ are terminating ; then $(\mathcal{H},\mathsf{h},\sigma+\tau)$ is terminating too. Moreover :

$$\mathrm{size}(\mathsf{h},\sigma+\tau) \leq \mathrm{size}(\tau[\![\mathsf{h}]\!],\sigma)\cdot\mathrm{size}(\mathsf{h},\tau) \tag{11}$$

Proof : First assume all systems deadlock-free. We reproduce the argument in the beginning of the proof of theorem 3. Rewrite equation (7) as :

$$\mathsf{h}(x\oplus y'\oplus\psi(x\oplus y')) = x'\oplus y''\oplus\tau(\psi(x\oplus y')) \tag{12}$$

and equation (8) as :

$$\tau[\![\mathsf{h}]\!](x\oplus\varphi(x)) = x'\oplus\sigma(\varphi(x)) \tag{13}$$

Both can be combined to yield :

$$\mathsf{h}(x\oplus\varphi(x)\oplus\psi(x\oplus\varphi(x))) = x'\oplus\sigma(\varphi(x))\oplus\tau(\psi(x\oplus\varphi(x))) \tag{14}$$

Now observe that : $x\oplus\varphi(x) = \mathrm{ex}(\tau[\![\mathsf{h}]\!],\sigma)(x)$, $x\oplus y\oplus\psi(x\oplus y) = \mathrm{ex}(\mathsf{h},\tau)(x)$, $x\oplus\varphi(x)\oplus\psi(x\oplus\varphi(x)) = \mathrm{ex}(\mathsf{h},\sigma+\tau)(x)$. Hence :

$$\mathrm{ex}(\mathsf{h},\sigma+\tau) = \mathrm{ex}(\mathsf{h},\tau)\cdot\mathrm{ex}(\tau[\![\mathsf{h}]\!],\sigma) \tag{15}$$

and from this (11) follows.

In general, "remove the deadlocks" in $(\mathcal{H},\mathsf{h},\tau)$ and $(\mathcal{R}\oplus\mathcal{S},\tau[\![\mathsf{h}]\!],\sigma)$. Then we get an equality close to (15), but for the the point that we are not sure that the left hand side corresponds to the smallest solution. Anyway the inequality (11) still holds. $\square$

Remark 11
Our definition of size comes from the naive power series expansion of the solution of (1) :

$$\sigma[\![\mathsf{h}]\!] = \mathcal{R}(\mathsf{h} + \mathsf{h}\sigma\mathsf{h} + \mathsf{h}\sigma\mathsf{h}\sigma\mathsf{h} + \ldots)\mathcal{R} \tag{16}$$

a formula which is for instance correct when $\sigma\mathsf{h}$ is nilpotent :

$$\mathrm{ex}(\mathsf{h}, \sigma) = \mathcal{R} + \mathsf{B}^*\sigma\mathsf{B} + \mathsf{B}^*\sigma\mathsf{C}\sigma\mathsf{B} + \mathsf{B}^*\sigma\mathsf{C}\sigma\mathsf{C}\sigma\mathsf{B} + \ldots + \mathsf{B}^*\sigma(\mathsf{C}\sigma)^{\mathsf{n}-2}\mathsf{B} \tag{17}$$

where n is the greatest integer such that $(\sigma\mathsf{C})^n \neq \mathsf{0}$ (the "order of nilpotency" of $\sigma\mathsf{C}$). Then $\mathrm{size}(\mathsf{h}, \sigma) \leq n$; in practice, e.g., for a converging normalisation in —say– system $\mathbb{F}$, $\mathrm{size}(\mathsf{h}, \sigma) \sim \sqrt{n}$.

It might be more natural to replace the size with its logarithm : Danos observed in his Thesis (see, e.g., [6]) that, in λ-calculus, this "order of nilpotency" is indeed exponential in the number of reduction steps needed to normalise the term interpreted by $(\mathcal{H}, \mathsf{h}, \sigma)$. With such an alternative definition, theorem 6 would involve a sum instead of a product.

5 Semi-invertible case

5.1 Order Approximations

When we speak of "the l.u.b. of a net", we implicitly assume that we are speaking of directed increasing net (with non-empty index set) ; symmetrically for "the g.l.b. of a net". Observe that l.s.i. are closed under l.u.b., u.s.i. are closed under g.l.b. : this is immediate from proposition 6 (i).

Proposition 11
Invertible systems are dense w.r.t. the pointwise order. More precisely :

(i) *Every system is the g.l.b. of a net of l.s.i. systems.*

(ii) *Every system is the l.u.b. of a net of u.s.i. systems.*

(iii) *Every u.s.i. system is the g.l.b. of a net of invertible systems.*

(iv) *Every l.s.i. system is the l.u.b. of a net of invertible systems.*

Proof : *(i)* For $0 < \mu < 1$, let $\mathsf{h}^\mu := \mu\mathsf{h} + (1-\mu)\mathrm{I}$; since $\mathsf{h} \leq \mathrm{I}$, and $(\mathcal{H}, \mathrm{I}, \sigma)$ is l.s.i., the convex combinations h^μ is l.s.i., by by proposition 6 (ii). The h^μ form a decreasing net, with g.l.b. h.

(ii) Symmetrical : use the increasing net $\mathsf{h}_\lambda := \lambda\mathsf{h} + (\lambda - 1)\mathrm{I}$.

(iii) If h is assumed to be u.s.i., then the h^μ are invertible, by proposition 6 (i) or (iii) ; and h is the g.l.b. of a net of invertible systems.

(iv) Symmetrical.

□

5.2 Lower-semi-invertible Case

Theorem 7 (Normal Form)
There exists a unique extension of the normal form to l.s.i. systems which commutes with least upper bounds. This extension is order-monotonous ; furthermore, the theorems established in the invertible case, and styled "associativity", "winning", "stability", still hold.

Proof : With the notations of proposition 11, define :

$$\sigma[\![\mathsf{h}]\!] := \sup_{\lambda} \sigma[\![\mathsf{h}_\lambda]\!] \tag{18}$$

Since the normal form is order-continuous in the invertible case, equation (18) holds in this case, hence our definition extends the original one. Moreover, this extension is monotonous : if $\mathsf{h} \leq \mathsf{k}$, then $\mathsf{h}_\lambda \leq \mathsf{k}_\lambda$, and $\sup_\lambda \sigma[\![\mathsf{h}_\lambda]\!] \leq \sup_\lambda \sigma[\![\mathsf{k}_\lambda]\!]$. The next point is that this definition commutes with suprema : if $\mathsf{h} = \sup_i \mathsf{h}[i]$, then $\sigma[\![\mathsf{h}]\!] = \sup_\lambda \sigma[\![\mathsf{h}_\lambda]\!] = \sup_i \sup_\lambda \sigma[\![\mathsf{h}[i]_\lambda]\!] = \sup_\lambda \sup_i \sigma[\![\mathsf{h}[i]_\lambda]\!] = \sup_i \sigma[\![\mathsf{h}[i]]\!]$: besides a triviality on double suprema, one uses the order continuity of the normal form in the invertible case, i.e., that $\sup_i \sigma[\![\mathsf{h}[i]_\lambda]\!] = \sigma[\![\mathsf{h}_\lambda]\!]$. Unicity is a trivial consequence of commutation to suprema. Let us now check the the extension of the main theorems :

Associativity : If $(\mathcal{H}, \mathsf{h}, \sigma + \tau)$ is l.s.i., then the $(\mathcal{H}, \mathsf{h}_\lambda, \sigma + \tau)$ are invertible. From this, $\tau[\![\mathsf{h}]\!] = \sup_\lambda \tau[\![\mathsf{h}_\lambda]\!]$ is l.s.i., and $(\mathcal{R} \oplus \mathcal{S}, \tau[\![\mathsf{h}]\!], \sigma)$ is l.s.i., as supremum of l.s.i. systems. Finally,
$\sigma[\![\tau[\![\mathsf{h}]\!]]\!] = \sup_\lambda \sigma[\![\tau[\![\mathsf{h}_\lambda]\!]]\!] = \sup_\lambda (\sigma + \tau)[\![\mathsf{h}_\lambda]\!] = (\sigma + \tau)[\![\mathsf{h}]\!]$.

Conversely, assume that $\tau[\![\mathsf{h}]\!]$ is l.s.i., but $(\mathcal{H}, \mathsf{h}, \sigma + \tau)$ is not l.s.i. ; then some $(\mathcal{H}, \mathsf{h}_\lambda, \sigma + \tau)$ is not invertible, but, since $\tau[\![\mathsf{h}_\lambda]\!]$ is invertible, theorem 3 shows that $(\mathcal{R} \oplus \mathcal{S}, \tau[\![\mathsf{h}_\lambda]\!], \sigma)$ is not invertible. Since
$\mathsf{h}_\lambda \leq \lambda(\mathcal{R} + \mathcal{S})$, we get $\tau[\![\mathsf{h}_\lambda]\!] \leq \tau[\![\lambda \mathrm{I}]\!] \leq \lambda \mathrm{I}$, hence $(\mathcal{R} \oplus \mathcal{S}, \tau[\![\mathsf{h}_\lambda]\!], \sigma)$ is u.s.i. by proposition 6 (i), and therefore not l.s.i.

Winning : If h is bipartite (resp. winning), so are the (h_λ), and so are the $\sigma[\![\mathsf{h}_\lambda]\!]$, and their supremum $\sigma[\![\mathsf{h}]\!]$.

Stability : Proved in the same tautological way, relying on a property that I like to state independently, see next proposition.

□

Proposition 12
Assume that $(\mathcal{H}, \mathsf{h}, \sigma) \sqsubset (\mathcal{H}, \mathsf{k}, \sigma)$, and that $(\mathcal{H}, \mathsf{k}, \sigma)$ is l.s.i. ; then $(\mathcal{H}, \mathsf{h}, \sigma)$ is l.s.i. too.

Proof : Assume that $\mathsf{h} = \mathsf{k}\mathcal{E}$ and $\mathcal{E}\sigma = \sigma\mathcal{E}$; then $\mathcal{E}\mathsf{h} = \mathsf{k}\mathcal{E}$. If $\mathsf{h}(x_n) + x_n \to 0$, with $\sigma(x_n) = -x_n$, and $\|x_n\| = 1$, then $\mathsf{k}(\mathcal{E}(x_n)) + \mathcal{E}(x_n) = \mathcal{E}(\mathsf{h}(x_n) + x_n) \to 0$, and $\sigma(\mathcal{E}(x_n)) = \mathcal{E}(\sigma(x_n)) = -\mathcal{E}(x_n)$. Moreover, since $\|\mathsf{k}\mathcal{E}(x_n)\| = \|\mathsf{h}(x_n)\| \to 1$ and $\|\mathsf{k}\| \leq 1$, we conclude that $\|\mathcal{E}(x_n)\| \to 1$. The sequence $\mathcal{E}(x_n)$ is an approximate eigenvector sequence for $-\sigma^- - \mathsf{k}$. □

5.3 Semi-invertible Case

Of course, the normal form can be symmetrically extended to u.s.i. systems.

Theorem 8 (Normal Form)
There exists a unique extension of the normal form to u.s.i. systems which commutes with greatest lower bounds. This extension is order-monotonous ; furthermore, the theorems established in the invertible case, and styled "associativity", "stability", "winning", still hold.

Proof : With the notations of proposition 11, we define :

$$\sigma[\![\mathsf{h}]\!] := \inf_{\mu} \sigma[\![\mathsf{h}^{\mu}]\!] \tag{19}$$

etc. □

We have therefore two extensions of the normal form ; these extensions are consistent, since there is no conflict (the intersection of the two domains, l.s.i. and u.s.i., consists of invertibles).

Proposition 13
The normal form is monotonous.

Proof : $\mathsf{h} \leq \mathsf{k}$, we must show that $\sigma[\![\mathsf{h}]\!] \leq \sigma[\![\mathsf{k}]\!]$. This is already taken care of when h, k are "on the same side". In view of proposition 6 (i), it remains to consider the case where h is u.s.i. and k is l.s.i.. But then, the interpolant $\mathsf{g} := 1/2(\mathsf{h} + \mathsf{k})$ is invertible by proposition 6 (iii) : hence $\sigma[\![\mathsf{h}]\!] \leq \sigma[\![\mathsf{g}]\!] \leq \sigma[\![\mathsf{h}]\!]$. □

No clear order-continuity can be stated, this is due to the fact that one side commutes to sups, the other to infs[15]. But order order-continuity can still be used as a tool. For instance, one can prove the following :

[15] And this result is optimal, see section 6.6.

Proposition 14

$$(-\sigma)[\![-\mathsf{h}]\!] = -\sigma[\![\mathsf{h}]\!] \tag{20}$$

Proof : Immediate : first checked in the invertible case, then extended to the —say— l.s.i. case, using an (increasing) order-continuity argument. □

Stability remains, essentially because there is no conflict l.s.i./u.s.i., the same for winning. But *associativity* is problematic. This is the central problem and also the key to the general case, hence let us keep this point for the ultimate section.

6 General Case

6.1 Lopsided Feedbacks

Definition 15 (Lopsided Feedbacks)
A feedback σ is lopsided *iff it is either a projection, i.e., $\sigma^2 = \sigma$ (*positive *feedback), or the opposite of a projection, i.e., $\sigma^2 = -\sigma$ (*negative *feedback).*

Roughly speaking, the three following words socialise : *positive* feedbacks, *l.s.i.* nets, and *l.u.b.*. If σ is positive, then all cut-systems $(\mathcal{H}, \mathsf{h}, \sigma)$ are l.s.i., in particular, we have a very satisfactory notion of *normal form*, expressed by theorem 7 ; therefore any system is the l.u.b. of a (directed, increasing) net of invertibles. The socialisation of the cocktail "positive + l.s.i. + l.u.b." is expressed by the equation :

$$\sigma[\![\sup_i \mathsf{h}_i]\!] = \sup_i \sigma[\![\mathsf{h}_i]\!] \tag{21}$$

which holds for any (increasing, directed) net.

Symmetrically, the words *negative*, u.s.i., g.l.b. associate well. In case of a negative feedback, all systems are u.s.i., etc. and

$$\sigma[\![\inf_i \mathsf{h}_i]\!] = \inf_i \sigma[\![\mathsf{h}_i]\!] \tag{22}$$

6.2 Associativity : a First Attempt

Let us come back to the semi-invertible case, section 5.3. If $(\mathcal{H}, \mathsf{h}, \sigma + \tau)$ is semi-invertible, it is —say–, l.s.i. ; then both of $(\mathcal{H}, \mathsf{h}, \tau)$ and $(\mathcal{R} + \mathcal{S}, \tau[\![\mathsf{h}]\!], \sigma)$ are l.s.i., and equation (5) holds. The converse is less obvious, since $(\mathcal{H}, \mathsf{h}, \sigma+\tau)$ and $(\mathcal{R} + \mathcal{S}, \tau[\![\mathsf{h}]\!], \sigma)$ may be respectively u.s.i. and l.s.i.

This case occurs naturally when $\sigma = \pi$ (π for "positive", "projection"), $\tau = \nu$ (ν for "negative"). Every system $(\mathcal{H}, \mathsf{h}, \nu)$ is u.s.i., every system

$(\mathcal{R}\oplus\mathcal{S}, \mathsf{k}, \pi)$ is l.s.i., hence one can form $\pi[\![\nu[\![\mathsf{h}]\!]]\!]$, and for symmetrical reasons, $\nu[\![\pi[\![\mathsf{h}]\!]]\!]$. Without any hypothesis on h, $(\pi+\nu)[\![\mathsf{h}]\!]$ does not make sense, but, with $\pi := \sigma^+$, $\nu := -\sigma^-$, $\pi[\![\nu[\![\mathsf{h}]\!]]\!]$ and $\nu[\![\pi[\![\mathsf{h}]\!]]\!]$ yield two candidates for the normal form $\sigma[\![\mathsf{h}]\!]$. To sum up :

(i) Associativity is the key to the general case.

(ii) It is enough to investigate associativity in the case of *lopsided* feedbacks.

Obviously :

(i) When the feedbacks are positive, everybody is l.c.i. and associativity holds : $\pi'[\![\pi[\![\mathsf{h}]\!]]\!] = \pi[\![\pi'[\![\mathsf{h}]\!]]\!]$ without any hypothesis on h.

(ii) Symmetrically when the feedbacks are negative : $\nu'[\![\nu[\![\mathsf{h}]\!]]\!] = \nu[\![\nu'[\![\mathsf{h}]\!]]\!]$.

(iii) When the feedbacks are of different sign, we know that associativity works under the strong hypothesis that $(\mathcal{H}, \mathsf{h}, \pi+\nu)$ is semi-invertible. Without hypothesis on h, we can prove an inequality.

Lemma 15.1
If π is positive, then $\pi[\![\mathsf{h}]\!] = \inf_\mu \pi[\![\mathsf{h}^\mu]\!]$.

Proof : W.r.t. the decomposition $\mathcal{H} = \mathcal{R}\oplus\pi$, let $\mathsf{h} = \begin{bmatrix} \mathsf{A} & \mathsf{B}^* \\ \mathsf{B} & \mathsf{C} \end{bmatrix}$. Assuming $(\mathcal{H}, \mathsf{h}, \pi)$ invertible, $\pi[\![\mathsf{h}^\mu]\!] = \mu\mathsf{A} + \mu^2\mathsf{B}^*(\pi - \mu\mathsf{C} - (1-\mu)\pi)^{-1}\mathsf{B} + (1-\mu)\mathcal{R}$. Observing that $\pi - \mu\mathsf{C} - (1-\mu)\pi = \mu(\pi - C)$, we eventually get :

$$\pi[\![\mathsf{h}^\mu]\!] = \mu\pi[\![\mathsf{h}]\!] + (1-\mu)\mathcal{R} = \pi[\![\mathsf{h}]\!]^\mu \tag{23}$$

Since both sides of equation (23) commute to l.u.b., we conclude that the equation holds for arbitrary h. From this one easily concludes. □

Lemma 15.2
If ν is negative, then $\nu[\![\mathsf{h}]\!] = \sup_\lambda \nu[\![\mathsf{h}_\lambda]\!]$.

Proof : Symmetrical. □

Proposition 15
If the independent feedbacks π, ν are respectively positive and negative and $(\mathcal{H}, \mathsf{h}, \pi+\nu)$ is a cut-system, then :

$$\pi[\![\nu[\![\mathsf{h}]\!]]\!] \leq \nu[\![\pi[\![\mathsf{h}]\!]]\!] \tag{24}$$

Proof : For the semi-invertible $\mathsf{h}_\lambda, \mathsf{h}^\nu$, associativity holds. Then $\pi[\![\nu[\![\mathsf{h}]\!]]\!] = \sup_\lambda(\pi+\nu)[\![\mathsf{h}_\lambda]\!]$, essentially by lemma 15.2. Symmetrically, lemma 15.1 yields $\nu[\![\pi[\![\mathsf{h}^\mu]\!]]\!] = \nu[\![\pi[\![\mathsf{h}]\!]]\!]$. Using $\mathsf{h}_\lambda \leq \mathsf{h} \leq \mathsf{h}^\mu$, we eventually get (24). □

But there is no hope to prove equality following this pattern. This the ultimate point one can reach by “continuity” techniques.

6.3 Positive Feedbacks

In what follows, π is a positive feedback, and $\mathsf{h} = \begin{bmatrix} \mathsf{A} & \mathsf{B}^* \\ \mathsf{B} & \mathsf{C} \end{bmatrix}$ w.r.t. the direct sum decomposition $\mathcal{H} = \mathcal{R} \oplus \pi$.

Theorem 9 (Minimality)
$\pi[\![\mathsf{h}]\!] - \mathsf{A}$ *is* the *smallest operator* A' *of* $B(\mathcal{R})$ *such that*

$$\begin{bmatrix} \mathsf{A}' & -\mathsf{B}^* \\ -\mathsf{B} & \pi - \mathsf{C} \end{bmatrix} \geq 0 \tag{25}$$

Proof : We first assume $(\mathcal{H}, \mathsf{h}, \pi)$ invertible ; let k be the block corresponding to the choice $\mathsf{A}' = \mathsf{B}^*(\pi - \mathsf{C})^{-1}\mathsf{B}$ in equation (25). Then

$$\begin{aligned} &\langle \mathsf{k}(x \oplus y) \mid x \oplus y \rangle = \\ &\quad \langle \mathsf{B}^*(\pi - \mathsf{C})^{-1}\mathsf{B})(x) \mid x \rangle - \langle \mathsf{B}(x) \mid y \rangle - \langle \mathsf{B}^*(y) \mid x \rangle + \langle (\pi - \mathsf{C})(y) \mid y \rangle \end{aligned} \tag{26}$$

Let $x' := (\pi - \mathsf{C})^{-1/2}\mathsf{B}(x)$, $y' := \sqrt{\pi - C}(y)$. Then the right-hand side of (26) rewrites as $\langle x' \mid x' \rangle - \langle y' \mid x' \rangle - \langle x' \mid y' \rangle + \langle y' \mid y' \rangle = \langle x' - y' \mid x' - y' \rangle$, and is therefore positive. This expression vanishes for $x' = y'$, and this shows that the first term $\langle x' \mid x' \rangle = \langle \mathsf{B}^*(\pi - \mathsf{C})^{-1}\mathsf{B})(x) \mid x \rangle$ actually takes *the* minimum possible value making (26) positive.

Let us quickly conclude : if $\pi - \mathsf{C}$ is not invertible, then $\pi - \mathsf{C} = \inf_\lambda(\pi - \mathsf{C}_\lambda)$. In (26), change C into C_λ, then $\sigma[\![\mathsf{h}]\!]$ remains equation ; solution of the since $\sigma[\![\mathsf{h}]\!] = \sup_\lambda \sigma[\![\mathsf{h}_\lambda]\!]$, it is indeed the smallest solution working for *all* λ, i.e., working for C instead of the C_λ. □

Remark 12
Since $\mathsf{h} \leq \mathrm{I}$, $\mathsf{A}' = \mathcal{R} - \mathsf{A}$ enjoys (25) ; from this we get $\pi[\![\mathsf{h}]\!] \leq \mathcal{R}$, consistently with $\|\pi[\![\mathsf{h}]\!]\| \leq 1$.

The next result relies on the folklore of operators see appendix C.8.

Theorem 10 (Resolution)
$\pi[\![\mathsf{h}]\!] - \mathsf{A} = \psi^*\psi$, *where* ψ *is uniquely determined by the conditions :*

$$\begin{aligned} \sqrt{\pi - \mathsf{C}} \cdot \psi &= \mathsf{B} \\ \mathrm{dom}(\pi - \mathsf{C}) \cdot \psi &= \psi \end{aligned} \tag{27}$$

Proof : Assume that $\sqrt{\pi - \mathsf{C}} \cdot \psi = \mathsf{B}$, and let $\mathsf{A}' := \psi^*\psi$; then equation (25) holds : in the computation of (26), replace $(\pi - \mathsf{C})^{-1/2}\mathsf{B}$ with ψ. The expression vanishes for $x' = y'$, i.e., $\psi(x) = \sqrt{\pi - \mathsf{C}}(y)$. This is possible only if ψ has its range included in $\mathrm{dom}(\sqrt{\pi - \mathsf{C}})$, equivalently, see remark 17 in appendix C.8, only if $\mathrm{dom}(\pi - \mathsf{C}) \cdot \psi = \psi$.

It remains to show the existence of ψ ; in view of proposition 19 (and remark 17), it is enough to show that $\mathsf{BB}^* \leq \mathsf{k} \cdot (\pi - \mathsf{C})$, for a certain real k. Let $x \oplus y \in \mathcal{H}$; since $\mathsf{h} \leq \mathrm{I}$:

$$\langle(\mathrm{I} - \mathsf{h})(x \oplus y) \mid x \oplus y\rangle = \\ \langle(\mathcal{R} - \mathsf{A})(x) \mid x\rangle + \langle\mathsf{B}(x) \mid y\rangle + \langle\mathsf{B}^*(x) \mid y\rangle + \langle(\pi - \mathsf{C})(y) \mid y\rangle \geq 0$$

Now, the same high-school technique used in the proof of the Cauchy-Schwarz inequality yields $\|\langle\mathsf{B}(x) \mid y\rangle\|^2 \leq \langle(\mathcal{R} - \mathsf{A})(x) \mid x\rangle \cdot \langle(\pi - \mathsf{C})(y) \mid y\rangle$. Taking $x := \mathsf{B}^*(y)$, we get $\langle\mathsf{BB}^*(\mathsf{y}) \mid y\rangle^2 \leq \|\mathcal{R} - \mathsf{A}\| \cdot \|\mathsf{B}^*(y)\|^2 \cdot \langle(\pi - \mathsf{C})(y) \mid y\rangle$. Using the familiar $\langle\mathsf{BB}^*(\mathsf{y}) \mid y\rangle = \|\mathsf{B}^*(y)\|^2$, we eventually get :

$$\mathsf{BB}^* \leq \|\mathcal{R} - \mathsf{A}\| \cdot (\pi - \mathsf{C}) \tag{28}$$

we got our inequality, with $k = \|\mathcal{R} - \mathsf{A}\|$. □

Definition 16 (Resolvant)
Let $(\mathcal{H}, \mathsf{h}, \pi)$ be a cut-system with a positive feedback, and let ψ be as in theorem 10. We define the resolvant $\mathrm{res}(\mathsf{h}, \pi) := \begin{bmatrix} \mathsf{A} & \psi^* \\ \psi & 0 \end{bmatrix}$.

Remark 13
It is easily shown that $\|\mathrm{res}(\mathsf{h}, \pi)\| = \|\mathsf{A} + \psi^*\psi\| \leq 1$.

Corollary 10.1
The cut-system $(\mathcal{H}, \mathsf{h}, \pi)$ is equivalent to its resolvant :

$$\pi[\![\mathsf{h}]\!] = \pi[\![\mathrm{res}(\mathsf{h}, \pi)]\!] \tag{29}$$

Proof : $\pi[\![\mathrm{res}(\mathsf{h}, \pi)]\!] = \mathsf{A} + \psi^*\psi$. □

6.4 Negative Feedbacks

The case of a negative feedback ν is symmetrical. One introduces the resolvant $\mathrm{res}(\mathsf{h}, \nu) := -\mathrm{res}(-\mathsf{h}, -\nu)$ and one checks, using proposition 14, that :

$$\nu[\![\mathsf{h}]\!] = \nu[\![\mathrm{res}(\mathsf{h}, \nu)]\!] \tag{30}$$

Proposition 16
If π positive and ν negative are independent, then $\mathrm{res}(\mathrm{res}(\mathsf{h})\nu)\pi = \mathrm{res}(\mathrm{res}(\mathsf{h})\pi)\nu$.

Proof : W.r.t. the decomposition $\mathcal{H} = \mathcal{R} \oplus \pi \oplus (-\nu)$, let $\mathsf{h} = \begin{bmatrix} \mathsf{A} & \mathsf{B}^* & \mathsf{D}^* \\ \mathsf{B} & \mathsf{C} & \mathsf{E}^* \\ \mathsf{D} & \mathsf{E} & \mathsf{F} \end{bmatrix}$.
Let $\mathsf{k}_1 := \mathrm{res}(\mathrm{res}(\mathsf{h})\nu)\pi$ and $\mathsf{k}_2 := \mathrm{res}(\mathrm{res}(\mathsf{h})\pi)\nu$: $\mathsf{k_i} = \begin{bmatrix} \mathsf{A} & \mathsf{b^*}_i & \mathsf{d}_i{}^* \\ \mathsf{b}_i & 0 & \mathsf{e}_i{}^* \\ \mathsf{d}_i & \mathsf{e}_i & 0 \end{bmatrix}$. We

must show that $\mathsf{k}_1 = \mathsf{k}_2$, what we do coefficientwise. All these coefficients are unique in some sense, the unicity being characterised by their range and/or domain see definition 18 in appendix C.8. For instance b_i and d_i are with respective ranges included in $\mathrm{dom}(\pi - \mathsf{C})$ and $\mathrm{dom}(\mathsf{F} - \nu)$. The case of e_i is more complex : its range is included in $\mathrm{dom}(\mathsf{F} - \nu)$, and its domain is included in $\mathrm{dom}(\pi - \mathsf{C})$. So far so good, the coefficients satisfy the same range/domain constraints.

Now let us look at the precise equalities defining our coefficients :

(i) $\sqrt{\pi - \mathsf{C}} \cdot \mathsf{b}_i = \mathsf{B}$, hence $\mathsf{b}_1 = \mathsf{b}_2$.

(ii) $\sqrt{\mathsf{F} - \nu} \cdot \mathsf{d}_i = \mathsf{D}$, hence $\mathsf{d}_1 = \mathsf{d}_2$.

(iii) $(\sqrt{\mathsf{F} - \nu} \cdot \mathsf{e}_1) \cdot \sqrt{\pi - \mathsf{C}} = \mathsf{E}$, $\sqrt{\mathsf{F} - \nu} \cdot (\mathsf{e}_2 \cdot \sqrt{\pi - \mathsf{C}}) = \mathsf{E}$; again, $\mathsf{e}_1 = \mathsf{e}_2$.

□

6.5 The Solution

It is plain from the discussion of section 6.2 that, if we can prove that lopsided feedbacks of opposite sides associate, we can define :

Definition 17 (Normal Form)
Let $(\mathcal{H}, \mathsf{h}, \sigma)$ be a cut-system ; we define its normal form *by :*

$$\sigma[\![\mathsf{h}]\!] := (-\sigma^-)[\![\sigma^+[\![\mathsf{h}]\!]]\!] = \sigma^+[\![(-\sigma^-)[\![\mathsf{h}]\!]]\!] \tag{31}$$

Theorem 11 (Lopsided Associativity)
Assuming that π (positive) and ν (negative) are independent :

$$\pi[\![\nu[\![\mathsf{h}]\!]]\!] = \nu[\![\pi[\![\mathsf{h}]\!]]\!] \tag{32}$$

Proof : $\pi[\![\nu[\![\mathsf{h}]\!]]\!] = \pi[\![\nu[\![\mathrm{res}(\mathsf{h}, \nu)]\!]]\!]$; with $\sigma := \pi + \nu$, since $(\mathcal{H}, \mathrm{res}(\mathsf{h}, \nu), \sigma)$ is l.s.i., $\pi[\![\nu[\![\mathrm{res}(\mathsf{h}, \nu)]\!]]\!] = \nu[\![\pi[\![\mathrm{res}(\mathsf{h}, \nu)]\!]]\!] = \nu[\![\pi[\![\mathrm{res}(\mathrm{res}(\mathsf{h}, \nu), \pi)]\!]]\!] = \sigma[\![\mathrm{res}(\mathrm{res}(\mathsf{h}, \nu), \pi)]\!]$. In the same way, $\nu[\![\pi[\![\mathsf{h}]\!]]\!] = \sigma[\![\mathrm{res}(\mathrm{res}(\mathsf{h}, \pi), \nu)]\!]$. The theorem is therefore a consequence of proposition 16. □

Corollary 11.1
With notations coming from the proof of proposition 16, the normal form is given by :

$$\sigma[\![\mathsf{h}]\!] = \mathsf{A} + \mathsf{b}^*(\mathsf{e}^*\mathsf{e} + \pi)\mathsf{b} - \mathsf{b}^*\mathsf{e}^*(\mathsf{e}\mathsf{e}^* - \nu)^{-1}\mathsf{d} - \mathsf{d}^*(\mathsf{e}\mathsf{e}^* - \nu)^{-1}\mathsf{e}\mathsf{b} - \mathsf{d}^*(\mathsf{e}\mathsf{e}^* - \nu)^{-1}\mathsf{d} \tag{33}$$

Proof : basically one must inverse $\begin{bmatrix} \pi & -\mathsf{e}^* \\ -\mathsf{e} & \nu \end{bmatrix}$, whose square is $\begin{bmatrix} \pi + \mathsf{e}^*\mathsf{e} & 0 \\ 0 & -\nu + \mathsf{e}\mathsf{e}^* \end{bmatrix}$. From this one easily gets (33). □

Remark 14
Since $(\mathsf{e}\mathsf{e}^* - \nu)^{-1}\mathsf{e} = \mathsf{e}(\mathsf{e}^*\mathsf{e} + \pi)^{-1}$, we can also express the normal form by :

$$\sigma[\![\mathsf{h}]\!] = \mathsf{A} + \mathsf{b}^*(\mathsf{e}^*\mathsf{e} + \pi)\mathsf{b} - \mathsf{b}^*(\mathsf{e}^*\mathsf{e} + \pi)^{-1}\mathsf{e}^*\mathsf{d} - \mathsf{d}^*\mathsf{e}(\mathsf{e}^*\mathsf{e} + \pi)^{-1}\mathsf{b} - \mathsf{d}^*(\mathsf{e}\mathsf{e}^* - \nu)^{-1}\mathsf{d} \tag{34}$$

Theorem 12 (Full Normal Form)
The normal form $\cdot[\![\cdot]\!]$ is the only associative and order-monotonous extension of definition 8 (the invertible case) commuting to l.u.b. (resp. g.l.b.) of l.s.i. (resp. u.s.i.) systems. It enjoys the analogues of the stability theorem 4 and the winning theorem 5.

Proof : There are obviously enough constraints to make this extension unique. As to monotonicity, and —say— commutation to l.u.b. of l.s.i. systems, use the definition $\sigma[\![\mathsf{h}]\!] := \sigma^+[\![(-\sigma^-)[\![\mathsf{h}]\!]]\!]$: if $(\mathcal{H}, \mathsf{h}, \sigma)$ is l.s.i., then $(\mathcal{H}, \mathsf{h}, -\sigma^-)$ is invertible (this is the definition). This shows that our definition extends the l.s.i. case, and we are back to section 5.2. Associativity is almost obvious :

$$\begin{aligned}(\sigma + \tau)[\![\mathsf{h}]\!] = (\sigma^- - \tau^-)[\![(\sigma^+ + \tau^+)[\![\mathsf{h}]\!]]\!] &= \tau^-[\![\sigma^-[\![\tau^+[\![\sigma^+[\![\mathsf{h}]\!]]\!]]\!]]\!] \\ &= \tau^-[\![\tau^+[\![\sigma^-[\![\sigma^+[\![\mathsf{h}]\!]]\!]]\!]]\!] = \tau[\![\sigma[\![\mathsf{h}]\!]]\!]\end{aligned}$$

As to stability, if $(\mathcal{H}, \mathsf{h}, \sigma) \sqsubset (\mathcal{H}, \mathsf{k}, \sigma)$, then there exists $\mathcal{E}$ such that $\sigma\mathcal{E} = \mathcal{E}\sigma$ and $\mathsf{h} = \mathsf{k}\mathcal{E}$. But then $(\sigma + \sigma^2)\mathcal{E} = \mathcal{E}(\sigma + \sigma^2)$, i.e., $\sigma^+\mathcal{E} = \mathcal{E}\sigma^+$, hence $(\mathcal{H}, \mathsf{h}, \sigma^+) \sqsubset (\mathcal{H}, \mathsf{k}, \sigma^+)$. From this $\sigma^+[\![\mathsf{h}]\!] \sqsubset \sigma^+[\![\mathsf{k}]\!]$. Since $\sigma^-\mathcal{E} = \mathcal{E}\sigma^-$, we get $(\mathcal{H}, \sigma^+[\![\mathsf{h}]\!], \sigma^-) \sqsubset (\mathcal{H}, \sigma^+[\![\mathsf{k}]\!], \sigma^-)$, i.e., $\sigma[\![\mathsf{h}]\!] \sqsubset \sigma[\![\mathsf{k}]\!]$. We skip the part on conditional infima.

Finally, we already know that bipartism is preserved by the normal form in case of a l.s.i. system. In order to conclude, it is therefore enough to remark that the normal form commutes to g.l.b. of decreasing nets of the form h^μ, see proposition 17 below. □

We like to state independently the last fact mentioned in the proof : this limited amount of general order-continuity is useful !

Proposition 17
$\sigma[\![\mathsf{h}]\!] = \sup_\lambda \sigma[\![\mathsf{h}_\lambda]\!] = \inf_\mu \sigma[\![\mathsf{h}^\mu]\!]$.

Proof : While proving proposition 15, we established the inequality
$\sigma^+[\![\sigma^-[\![\mathsf{h}]\!]]\!] \leq \sup_\lambda \sigma[\![\mathsf{h}_\lambda]\!] \leq \inf_\mu \sigma[\![\mathsf{h}^\mu]\!] \leq \sigma^-[\![\sigma^+[\![\mathsf{h}]\!]]\!]$, but
$\sigma^+[\![\sigma^-[\![\mathsf{h}]\!]]\!] = \sigma[\![\mathsf{h}]\!] = \sigma^-[\![\sigma^+[\![\mathsf{h}]\!]]\!]$. □

Remark 15
The previous commutations *do not* ensure general order-continuity, see next section.

6.6 Order-Continuity : a Counter-Example

The counterexample is obtained by successive simplifications : commutation to g.l.b. in the case of a positive feedback, indeed a hyperplane.

(i) Let $\mathcal{H} := \mathbb{C}\oplus\ell^2$; if $y = (y_i) \in \ell^2$ is of norm $1/4$, then $\|1/2\oplus -2y\|^2 = 1/2$, and $(1/2 \oplus -2y)(1/2 \oplus -2y)^* = \begin{bmatrix} 1/4 & -y^* \\ -y & 4yy^* \end{bmatrix}$ is of norm $1/2$. Then $\begin{bmatrix} 0 & y^* \\ y & -4yy^* \end{bmatrix}$ is of norm $1/2$ as well.

(ii) Let $\pi_n \subset \ell^2$, $\pi_n := \{(x_i)\ ;\forall j \geq n\ x_j = 0\}$ and let $\pi = \ell^2$. Then $\pi/2 - \pi_n/4$ and $\pi/4$ are of norm $\leq 1/2$.

(iii) If $\mathsf{C}_n := \pi/2 - \pi_n/4 - 4yy^*$, $\mathsf{C} := \pi/4 - 4yy^*$, and $\mathsf{h}_n := \begin{bmatrix} 0 & y^* \\ y & \mathsf{C}_n \end{bmatrix}$, then (h_n) is a decreasing net of hermitians of norm ≤ 1, whose g.l.b. is $\mathsf{h} := \begin{bmatrix} 0 & y^* \\ y & \mathsf{C} \end{bmatrix}$.

(iv) Assume that all coefficients y_i are non-zero. Then $y \notin \mathrm{rg}(\pi_\mathsf{n})$. From this $\sup\{a; ayy^* \leq \pi_n/4\} = 0$. Since $\|yy^*\| = 1/16$, $\sup\{a; ayy^* \leq \pi/2+\pi_n/4+4yy^*\} = 12$. But $\pi/2+\pi_n/4+4yy^* = \pi - \mathsf{C}_n$, and $ayy^* \leq \pi - \mathsf{C}_n$ iff $\mathsf{h}_n := \begin{bmatrix} 1/a & y^* \\ y & \pi - \mathsf{C}_n \end{bmatrix}$ is positive. From this we get $\pi[\![\mathsf{h}_n]\!] = 1/12$.

(v) The same computation, but done for h, yields : $\sup\{a; ayy^* \leq 3\pi/4 + 4yy^*\} = 16$. From this $\pi[\![\mathsf{h}_n]\!] = 1/16$.

(vi) $\pi[\![\inf_n \mathsf{h}_n]\!] < \inf_n \pi[\![\mathsf{h}_n]\!]$, that's our counter-example.

Summing up :

Theorem 13 (Order-Discontinuity)
The normal form $\sigma[\![\cdot]\!]$ is not order-continuous.

Remark 16
Remark that the C_n do not commute. If they were commuting, the commutation to infima would reduce to pointwise computation w.r.t. a basis —may be "continuous", in which all the C_n are "diagonal".

A Geometry of Interaction

A.1 Cut-Elimination

The essential[16] principle of reasoning is the use of *lemmas* : in order to prove B, first prove it under the hypothesis A, then prove (the lemma) A ; this is expressed by *Modus Ponens* :

$$\frac{A \quad A \Rightarrow B}{B} \tag{35}$$

reformulated by Gentzen in his *sequent calculus* ; a *sequent* $\Gamma \vdash \Delta$ consists of two finite sequences $\Gamma = A_1, \ldots, A_m$ and $\Delta = B_1, \ldots, B_n$ of formulas separated by the "turnstile" $\vdash$, with the intended meaning that the conjunction of the A_i implies the disjunction of the B_j. Sequent calculus is organised along the "rule you love to hate", the *cut-rule* :

$$\frac{\Gamma \vdash \Delta, A \quad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \tag{36}$$

which contains *Modus Ponens* as the particular case $\Gamma = \Delta = \Gamma' = \emptyset$, $\Delta' = B$. The cut-rule emphasises the fact that A occurs twice, once negatively (a hypothesis) once positively (a conclusion). These two occurrences "cancel" each other ; later on, this cancellation will be rendered by a partial symmetry σ swapping the two occurrences of A.

This cancellation is virtual ; however there is a process known as *cut-elimination*[17] which allows one to replace —under certain hypotheses— a proof with cuts with one without cuts, a *cut-free* proof. This process is long and tedious, and uses many "cooking recipes". A proof without cuts, without lemmas, is usually unreadable, but more explicit ; and, by the way, cut-elimination is related to the problem of finding "elementary proofs" in number-theory[18]. The structure of cut-elimination is better understood through *natural deduction*, or through (isomorphic) functional calculi, such as λ-*calculus*, in which *Modus Ponens* is nothing but the application of a function f which maps A into B to an argument a in A, yielding $f(a) \in B$, compare equation (35) with :

$$\frac{a \in A \quad f \in A \Rightarrow B}{f(a) \in B} \tag{37}$$

The various versions of λ-calculus are governed by the equation :

$$(\lambda x t)u = t[u/x] \tag{38}$$

[16]A basic reference for this section is [15].

[17]With variants, such as *normalisation* in natural deduction or λ-calculi, see below.

[18]And has sometimes been successfully used to this effect.

λxt is the function associating t, an expression containing x, to the argument represented by a variable x ; the equation looks therefore like a mere triviality... but a very powerful triviality, the extant λ-calculi being able to represent computable functions :

Typed λ-calculi : Typically system $\mathbb{F}$ which contains all computable *terminating* functions whose termination can be proved within second-order arithmetic. There is no hope to find a terminating computable function not representable in $\mathbb{F}$, the only exceptions being obtained through *ad hoc* diagonalisations, i.e., by *cheating*.

Pure λ-calculus : In typed λ-calculi, there is a "super-ego", *typing*, which forbids certain "non-logical" combinations —in the same way the choice of names can be used to avoid incest[19]. In pure λ-calculus, a function can be applied to anything, including itself. This calculus contains *all* computable functions, most of them partial, i.e., non-terminating. In the absence of typing, there is no way to tell the wheat from the tares, i.e., to individuate the *total* functions[20].

A.2 Categorical Interpretations

The Church-Rosser theorem[21] states that cut-elimination is *associative*, i.e., that we can apply equation (38) in any order. Concretely, this means that, in λ-calculi, the composition of functions is associative ; the functional intuition is therefore correct. But where to find such functions ?

Set theory is inadequate : the interpretation of self-application would lead to Russell's paradox, and, by the way, one can see pure λ-calculus, which is a "naive" function theory, as the "correct" version of naive set-theory. In both cases, every operation has a fixpoint : in set-theory the fixpoint of negation —constructed by Russell's paradox— is a contradiction ; pure λ-calculus avoids the pitfall by allowing *undefined* objects, i.e., non-terminating computations : for instance, if we try to mimic the fixpoint of negation we obtain a never ending process.

Set-theory being too brutal, people turned their attention towards category theory. Is it possible to replace "functions" with *morphisms* ? The answer is positive, and at work in *Scott domains*. The idea is to make a category out of certain topological spaces, so that our functions are continuous morphisms. The main difficulty is to have the function space $\hom(X, Y)$ to be in turn a

[19] In GoI, typing is responsible for nilpotency.

[20] The problem of making total a partial algorithm is of the same nature as the problem of extending an unbounded operator to the full Hilbert space, a pure nonsense !

[21] A good reference for this section is [2].

topological space of the same nature, not to speak of the *continuity* of the canonical operations, e.g., the composition of morphisms. To make the long story short, the operation succeeded, the equation

$$\mathbb{D} \simeq \hom(\mathbb{D}, \mathbb{D}) \tag{39}$$

which produces a *domain* isomorphic with its function space, has a solution : what was exactly needed for λ-calculus.

... But the patient was dead ; the challenge of making continuous too many canonical morphisms was too heavy, and Scott domains are far astray from standard topology : for instance, they are never Hausdorff.

More recent investigation with Banach spaces [12] help us to understand the problem in standard topological terms : the function space $\hom(X, Y)$ can be seen as the space of *analytical* maps from the *open* unit ball of X to the *closed* unit ball of Y. Composition is problematic, since it would involve the extension of a bounded analytic map defined on $\{x; \|x\| < 1\}$ to $\{x; \|x\| \leq 1\}$, an operation already desperate when $X = \mathbb{C}$. So there is no *real*[22] continuity in cut-elimination.

If we look carefully, Scott domains are indeed ordered sets, and continuity is just commutation to least upper bounds. This aspect of Scott's contribution is beyond criticism : more, it is extremely important. But this does not justify the building of a sort of "counter-topology". In this paper, cut-elimination is explained by the inversion of a hermitian operator on Hilbert space. The solution found in the invertible case is extended by various methods, including l.u.b. and g.l.b., to the general case, but is not even order-continuous.

A.3 Stability

Instead of explaining commutation to l.u.b. by topology, I prefered to use commutation to *direct limits* ; direct limits usually socialise well with *pull-backs*, and this apparently minor change of viewpoint introduced *stability* as commutation to pull-backs, a notion with no topological interpretation. This led to *coherent spaces*[23], partly rediscovering earlier work of Berry [3]. Anticipating on a further discussion, there are two ways of presenting them :

Essentialist : A coherent space X is the pair $(|X|, \frown)$ of a *carrier* (a set) $|X|$ and a *coherence* $\frown$ on $|X|$, i.e., a binary and reflexive relation. A *clique* $a \sqsubset X$ is any subset of the carrier made of pairwise coherent points. The main operations on cliques are directed unions (i.e., direct limits) and conditional intersections (i.e., pull-backs) $a \cap b$, provided $a \cup b$

[22] Neither "actual", nor "compatible with $\mathbb{R}$."

[23] See for instance [15].

is a clique . The main theorems basically rest on the (linear) negation $\sim X := (|X|, \asymp)$ —$\asymp$ meaning “incoherent or equal”— and on the basic remark that a clique and an *anti-clique* intersect on at most one point :

$$a \sqsubset X, b \sqsubset \sim X \Rightarrow \sharp(a \cap b) \leq 1 \tag{40}$$

Existentialist : Instead of working with coherence (cliques), we admit arbitrary subsets of the carrier $|X|$, and we say that two such subsets $a, b \subset |X|$ are *polar* when $\sharp(a \cap b) \leq 1$. We can define a coherent space as a set of subsets of the carrier equal to its bipolar $\sim\sim X$. This alternative definition is shown to be equivalent to the “official” one : observe that, if $X = \sim\sim X$ and $x, y \in |X|$ then either $\{x, y\} \in X$ or $\{x, y\} \in \sim X$, the disjunction being exclusive when $x \neq y$; from this one easily recovers the coherence of X : $x \frown y \Leftrightarrow \{x, y\} \in X$.

This discussion may seem extremely philosophical, i.e., for the common sense, a gilding of the lily. It takes all its significance when one tries to get rid of commutativity, i.e., when one considers the carrier as the basis —among others— of a complex vector space. There is no way of speaking of a coherence relation on a vector space, but there is still the existentialist version, for instance, we could replace subsets with positive hermitians, and the cardinal (the dimension) with the *trace*, so as to define polarity by $\mathrm{tr}(\mathsf{hk}) \leq 1$. This is what we did in *quantum coherent spaces*, see [14]. By the way there are still major variants of coherent spaces, typically the *hypercoherences* of Ehrhard [7], which lack an existentialist approach ; and this is not a meaningless “philosophical” digression.

A.4 Geometry of Interaction

Equation (38) defines a universal algorithmics[24], but there is something strange in this equality, *one side is more equal than the other*. The equation is indeed treated as a *rewriting* :[25]

$$(\lambda x t)u \rightsquigarrow t[u/x] \tag{41}$$

which *a priori* makes no sense in category theory. The question is therefore to decide whether this rewriting is pure engineering, or —in the same way the original “cooking recipes” of cut-elimination admitted a categorical (static) interpretation— if it admits a decent mathematical interpretation.

Geometry of interaction (GoI) is a *dynamic* explanation of logic, based on operator algebras. A (cut-free) proof (of a sequent) is represented by a square

[24] For this section, the reference is the first four papers on Geometry of interaction [8, 9, 10, 11].

[25] See for instance [15].

matrix whose entries are bounded operators on a given Hilbert space $\mathcal{H}$; the dimension of the matrix corresponds to the number of formulas in the *sequent.*

Here, beware of possible misunderstandings : operators are functions on Hilbert space, and they are in turn used to interpret functions (those coming from λ-calculus) ; but the composition of functions has nothing to do with the composition of the associated operators. For instance, the *identity axiom* of logic $A \vdash A$, (which roughly corresponds to the identity function λxx) is interpreted by the anti-diagonal matrix : $\begin{bmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix}$ where I is the identity operator on $\mathcal{H}$. The matrix is 2×2, the two rows/columns corresponding to the two occurrences of A. The anti-diagonal matrix should be viewed as a common electronic device, the *extension cord* : the two A (the two copies of $\mathcal{H}$) correspond to the two plugs through which some alternative current (an element of $\mathcal{H}$) may enter/exit. The matrix says that everything coming from the left exits on the right without change, and similarly from right to left : in real life, an extension cord works in this way, in both directions, even if our choice of plugs male/female tends to make them unidirectional ; but this is only a "super-ego" designed to avoid accidents[26].

In presence of cuts, the pattern is slightly modified ; typically, if cut-free proofs of $\Gamma \vdash \Delta, A$ and $A, \Gamma' \vdash \Delta'$ are interpreted by matrices M and M' (respectively indexed by Γ, Δ, A and A, Γ', Δ') then (renaming the second A as A'), the proof obtained by applying the cut-rule (36) is the matrix $\mathsf{h} = \begin{bmatrix} M & 0 \\ 0 & M' \end{bmatrix}$ indexed by $\Gamma, \Delta, A, A', \Gamma', \Delta'$; to this matrix is added another matrix, the *feedback* $\sigma = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{I} & 0 \\ 0 & \mathrm{I} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with only two non-zero entries, $\sigma_{AA'} = \sigma_{A'A} = \mathrm{I}$. The name *feedback* suggests that some output of h is given back to h trough σ. Indeed, Geometry of Interaction explains cut-elimination as the I/O diagram of this system.

We shall have plenty of space to conceptualise this, so let us solve the equation in the simplest case, namely that of a *Modus Ponens* with an identity axiom :

$$\frac{\vdash A \quad A \vdash A}{\vdash A} \tag{42}$$

Obviously, if the original proof of A corresponds to a 1×1 matrix $\begin{bmatrix} \mathsf{a} \end{bmatrix}$ then $\mathsf{h} = \begin{bmatrix} \mathsf{a} & 0 & 0 \\ 0 & 0 & \mathrm{I} \\ 0 & \mathrm{I} & 0 \end{bmatrix}$ and $\sigma = \begin{bmatrix} 0 & \mathrm{I} & 0 \\ \mathrm{I} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The I/O equation corresponding to

[26]If the cord were not bi-directional, there would be no need for heavy precautions, a different colour would suffice.

the feedback consists in, given $z \in \mathcal{H}$, finding $x, x', y, y', z' \in \mathcal{H}$ such that

$$\begin{aligned} \mathsf{h}(x \oplus y \oplus z) &= x' \oplus y' \oplus z' \\ \sigma(x' \oplus y' \oplus z') &= x \oplus y \oplus 0 \end{aligned} \tag{43}$$

The solution is obvious : $x' = \mathsf{a}(x), y' = z, z' = y$, $y = x', x = y'$, hence $x = y' = z$, $z' = y = x' = \mathsf{a}(x) = \mathsf{a}(z)$. Viewed from outside, the system yields, given the input z, the output $z' = \mathsf{a}(z)$. If we rename the three occurrences of A, from left to right, as A, A', A'', what we actually did was, given some device a communicating through plug A and some extension cord exchanging plugs A', A'', to physically plug A with A'. The resulting system behaves like a, but now trough the only "bachelor" plug, A''.

A.5 GoI : Main Results

First the good news, it works. The interpretation of logic, or λ-calculus, involves the identity axiom and the cut-rule, that we both explained ; it also involves the interpretation of various logical operations (conjunction, disjunction). It turns out that all these operations are indeed $*$-isomorphisms between matrix algebras ; typically binary connectives make use of an isomorphism replacing a $n+1 \times n+1$ matrix with a $n \times n$ matrix with coefficients in the same $B(\mathcal{H})$, which can be done by means of an isometry $\mathcal{H} \oplus \mathcal{H} \simeq \mathcal{H}$, equivalently two partial isometries[27] p, q of $\mathcal{H}$ such that

$$\begin{aligned} \mathsf{p}^*\mathsf{p} = \mathsf{q}^*\mathsf{q} &= \mathrm{I} \\ \mathsf{p}\mathsf{p}^* + \mathsf{q}\mathsf{q}^* &= \mathrm{I} \end{aligned} \tag{44}$$

The main result of GoI is that cut-elimination, normalisation, correspond to the solution of the feedback equation between h and σ. In fact the correspondence is not quite exact, the two coincide only in certain cases, but these cases are the "important" ones. In fact GoI corresponds to sophisticated reduction techniques, typically Lamping's *optimal reduction* for pure λ-calculus, see [16].

A last idea came from GoI : it clearly distinguishes between "before" and "after", i.e., the two sides of the rewriting (41), in this respect, this is a progress over categories. But more, it gives a meaning to the computation itself : remember, when we solved the basic, case of the feedback equation (43), we had to give values, in function of the input z, to the output z', but also to "internal values", x, x', y, y'. The operator associating $x' \oplus y' \oplus z'$ to the input z plays the role of the computation process. In this respect, the typed (logical) cases treated in [9, 11] differ from the (non-logical) case of pure λ-calculus treated in [10] : in the typed case, the execution operator is bounded, whereas in the pure, untyped case, the execution is generally unbounded.

[27] See C.6.

A.6 Augustinian Considerations

The opposition between *essentialism* and *existentialism*, Thomas and Augustine, is central in logic. Essentialism explains things as "coming from a hat", existentialism is more interactive, and, maybe, more modern. An essentialist version of ethics is "follow the rule, because it is the rule", an existentialist version would be "try to defeat the rule". Logically speaking, essentialism refuses *untyped* notions, i.e., objects that are not born with a pedigree ; existentialism accepts them all, and later puts some labels on them, depending on their behaviour, see [14] for a discussion. "Proofs as functions" and "Proofs as actions", can be seen under the light of this opposition :

★ Logic is surely born essentialist. If you don't understand what this means, just remember Tarski's definition of truth "$A \wedge B$ is true iff A is true *and* B is true" : behind the conjunction $\wedge$ stands a "meta-conjunction". Essentialism, the primality of essence, is the claim that everything preexists as a meta…, and the meta as a metameta, of course !

★ The functional paradigms, typically untyped λ-calculus, would rather present functions as primitive, let us say that they are given by programs, and formulas as comments, *specifications*. It is plain that the essence (the specification) is posterior to existence (the program)[28] ; bad programs do exist, they breakdown, but if only good programs were in use, certain companies wouldn't sell that much !

Even the idea of a category-theoretic interpretation can be read in a Thomist way : equality is the application of diagrammatic essences (e.g., limits, colimits).

A.7 GoI : Limitations

The first truly existentialist explanation of logic came with the *proof-nets* of linear logic, which inverted the tradition : starting with diagrams as sort of "wild graphs", compose edges by shortening of *paths*. The logico-categorical (essentialist) description of vertices as formulas/objects is now posterior to existence (here, the physical drawing of paths in a graph) ; the formula written on a vertex of the graph is a comment on the topological status of this precise vertex inside the graph. This is the meaning of the *correctness criterion* of proof-nets, seen as a topological property of graphs, *acyclicity*, see [5].

[28] The functional explanation of logic, due to Kolmogorov has been violently attacked by another figure of Thomism, Kreisel : in [19] he claimed that everything should be relativised to a *given* formal system. Logic should presuppose logic…

Geometry of interaction is nothing but an infinite-dimensional generalisation of proof-nets, but it no longer meets our implicit Augustinian standards.

Operators : To any proof/λ-expression, GoI associates a pair (h, σ). The feedback equation is solved by proving that $\sigma\mathsf{h}$ is nilpotent, i.e., $(\sigma\mathsf{h})^n = 0$ for some n [9, 11], with a weaker version for pure λ-calculus [10], *weak nipotency* : $\langle(\sigma\mathsf{h})^n(x) \mid x\rangle \to 0$ for all $x \in \mathcal{H}$.

Associativity : It was necessary to establish some structural properties of the solution, typically *associativity*, which deals with iterated feedbacks. Here we started to work *against* the spirit of operator algebras ; in order to ensure that certain compositions of partial isometries are still partial isometries, we were led to very artificial restrictions.

To sum up : when we follow logic, h and σ are always partial symmetries. But one can represent them in such a way that, w.r.t. a basis given in advance, they correspond to partial bijections. In other terms, all this work eventually amounts as a calculus of partial involutions of $\mathbb{N}$! Again the work was non-trivial, but didn't really go into the very heart of operator algebras.

A.8 Quantum Coherent Spaces

Under the influence of *Quantum Computing*, esp. the recent work of Selinger [20], I was able to revisit the static interpretation (coherent spaces) in the spirit of *quantum* interaction. The result is *Quantum Coherent Spaces* [14], QCS for short. To sum up, QCS are rather satisfactory, as an Augustinian approach to logic, even if they are limited to finite dimension. The basic idea is that the points in Scott domains, coherent spaces, ... are like the distinguished basis of a Hilbert space, and that everything coming from logic, λ-calculus is well-behaved, e.g., diagonal w.r.t. this basis. If we forget this distinguished basis, it turns out that everything still makes sense, but it now looks "quantum". That's an existentialist twist : when a function meets an argument which was not designed for it, nothing happens in the essentialist world (forbidden !) ; in the real world, the interaction takes place anyway (measurement, reduction of the wave packet).

The limitation of QCS to finite dimension seems to be absolute. This is why the idea of using GoI (much more flexible) is extremely natural[29]. But, in this new round of GoI, we shall get rid of any artificial (essentialist) restriction, in particular, of any commitment to a particular basis.

[29] In both cases, the identity axiom flips two copies of the same $\mathcal{H}$; in QCS, it flips $\mathcal{H} \otimes \mathcal{H}$, in GoI, it flips $\mathcal{H} \oplus \mathcal{H}$, which is incredibly better !

B The Tortoise Principle

The Tortoise Principle, at work in the proofs of theorems 2 and 5, exchanges a structural simplification of the feedback equation, against an enlargement of the Hilbert space. By the way, the feedback equation itself is an application of the Tortoise Principle, to a simpler, but less manageable, equation : see section B.2 below.

B.1 An Unexpected Contributor

Although he was Professor of Logic in Oxford, Lewis Carroll didn't make any significant contribution to his official field. He is mainly remembered for his photographs of young girls like Alice Liddell, and also for nonsense books like the two *Alice*. His short story "What the Tortoise said to Achilles" is a sort of endless chasing of *Modus Ponens*, our *cut*, our *feedback*. Basically, a *Modus Ponens* between A and $A \Rightarrow B$ is replaced with another one between $A \wedge (A \Rightarrow B)$ and $(A \wedge (A \Rightarrow B)) \Rightarrow B$; this replacement —or rather simplified versions of it— is what I call the "Tortoise Principle". To understand the technical interest for a modern reader, let us follow the convention of using subscripts, to avoid confusions. One originally starts with a *Modus Ponens* between A_1 and $A_2 \Rightarrow B_1$; for us, it means a hermitian f of $\mathcal{F} := \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{G}_1$, together with a feedback exchanging $\mathcal{H}_1$ and $\mathcal{H}_2$; the output (eventual solution of the feedback equation) is a hermitian of $\mathcal{G}_1$. The second *Modus Ponens* between $A_1 \wedge (A_2 \Rightarrow B_1)$ and $(A_3 \wedge (A_4 \Rightarrow B_2)) \Rightarrow B_3$, is a hermitian k of $\mathcal{K} := (\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{G}_1) \oplus (\mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \mathcal{G}_2 \oplus \mathcal{G}_3)$, together with a feedback exchanging $\mathcal{H}_1$ and $\mathcal{H}_3$, $\mathcal{H}_2$ and $\mathcal{H}_4$, $\mathcal{G}_1$ and $\mathcal{G}_2$. The output of the feedback equation is a hermitian of $\mathcal{G}_3$, strictly isomorphic with the output found in the original case.

So what did we gain in this complication ? Look at f : we know very little about it. Now look at k ; if we adopt the block decomposition suggested by our use of parentheses when we defined $\mathcal{K}$, it can be written : $\begin{bmatrix} \mathsf{f} & 0 \\ 0 & \mathsf{h} \end{bmatrix}$. The point is that h is perfectly well-known : it is the interpretation of the standard tautology $(A \wedge (A \Rightarrow B)) \Rightarrow B$. What is unknown, "variable", i.e., f, is now wholly located in the support ${\mathcal{G}_3}^{\perp}$ of the feedback.

If you are not convinced of the interest of this Tortoise principle, try to prove directly that the normal form of a bipartite system is still bipartite ! No doubt, you can make it, by a tedious equality chasing, after correcting a few errors of signs. But, with an appropriate use of the Tortoise, it reduces to showing that the inverse of a bipartite hermitian is bipartite, and the length of the equality chasing in proposition 8 remains decent.

If we count the number of atoms in the original *Modus Ponens*, they are four (we have to count the conclusion) ; Tortoise doubles it to eight, the following step to sixteen, etc. But all the conceptual simplification, and the relevance to logic, is located in the first step. The infinite iteration performed by Carroll —in order to get a sort of mock Zeno's paradox— is pointless, as expected from the master of nonsense : a pleasant —but somewhat superficial author— who stumbled —accident or intuition, who knows ?— on ideas that would only take shape in the mid 1930's —with the work of a genuine logician, Gehrard Gentzen.

B.2 Example : Cut vs. Composition

Indeed the Tortoise Principle has implicitly been used in our very basic formulation of what is a cut-system. If we come back to the original idea of *composition*, we must deal with a sort of general *Modus Ponens*, essentially the syllogism *Barbara* "all R are S, all S are T, hence all R are T" :

$$\frac{R \Rightarrow S \quad S \Rightarrow T}{R \Rightarrow T} \tag{45}$$

If the (given proofs of) the two premises are expressed by means of operators $\mathsf{f} \in B(\mathcal{R} \oplus \mathcal{S})$, $\mathsf{g} \in B(\mathcal{S} \oplus \mathcal{T})$, then composition amounts at solving, $x \in \mathcal{R}$, $z \in \mathcal{T}$ being given :

$$\begin{aligned} \mathsf{f}(x \oplus y) &= x' \oplus y' \\ \mathsf{g}(y' \oplus z) &= y \oplus z' \end{aligned} \tag{46}$$

and the output is the operator which yields $x' \oplus z'$ as a function $\mathsf{k}(x \oplus z)$.

The main advantage of this formulation is that it is close to what we have actually in mind : a duality. But, technically speaking, it is awfully complex : besides the fact that f, g are hermitians of norm at most 1, we know strictly nothing.

Here comes the Tortoise : introduce the space $\mathcal{H} = \mathcal{R} \oplus \mathcal{S} \oplus \mathcal{S} \oplus \mathcal{T}$, and let σ be the partial symmetry swapping the two $\mathcal{S}$,
$\sigma(x \oplus y \oplus y' \oplus z) := 0 \oplus y \oplus y' \oplus 0$. It is immediate that solving (46) is the same as solving the feedback equation for $(\mathcal{H}, \mathsf{f} \oplus \mathsf{g}, \sigma)$. Moreover, the feedback equation is just a particular instance of (46), where $\mathcal{R}, \mathcal{S}, \mathcal{T}, \mathsf{f}, \mathsf{g}$ have been respectively replaced with $\mathcal{R} \oplus \mathcal{T}, \mathcal{S} \oplus \mathcal{S}, 0, \mathsf{f} \oplus \mathsf{g}, \tau$, where τ is the total symmtry swapping the two copies of $\mathcal{S}$.

We obvious lost the symmetric character, for instance we seldom allow changes of feedback[30]. But we gained the fact that the feedback is a partial symmetry, i.e., almost the simplest case of a hermitian operator.

Associativity is the question arising when we add a third equation to (46) (with $\mathsf{h} \in B(\mathcal{T} \oplus \mathcal{U})$) :

$$\mathsf{h}(z' \oplus w) = z \oplus w' \tag{47}$$

[30] Only exception : associativity.

we can either "solve" (46) + (47) in a single step, or do it in two steps, e.g., first solve (46), which yields $\mathsf{k} \in B(\mathcal{R} \oplus \mathcal{T})$, then solve (47)+ (48)

$$\mathsf{k}(x \oplus z) = z' \oplus w' \tag{48}$$

The Tortoise expresses the system of three equations (46) + (47) as a cut-system $(\mathcal{R} \oplus \mathcal{S} \oplus \mathcal{S} \oplus \mathcal{T} \oplus \mathcal{T} \oplus \mathcal{U}, \mathsf{f} + \mathsf{g} + \mathsf{h}, \sigma + \tau)$, where τ swaps the two copies of $\mathcal{T}$, and associativity really translates as $(\sigma+\tau)[\![\mathsf{f} + \mathsf{g} + \mathsf{h}]\!] = \tau[\![\sigma[\![\mathsf{f} + \mathsf{g} + \mathsf{h}]\!]]\!]$.

C Operator-theoretic Basics

These materials are covered by many textbooks, my favourite one being [17].

C.1 Bounded operators

We are working on a complex Hilbert space $\mathcal{H}$, equipped with the sesquilinear form $\langle x \mid y \rangle$, linear in x, anti-linear in y. We are mostly interested in the space $B(\mathcal{H})$ of *bounded*, i.e., continuous, operators (i.e., linear endomorphisms) on $\mathcal{H}$. As usual, u^* denotes the *adjoint* of u, i.e., the unique operator satisfying $\langle \mathsf{u}(x) \mid y \rangle = \langle x \mid \mathsf{u}^*(y) \rangle$ for all $x, y \in \mathcal{H}$. The identity operator of $\mathcal{H}$ is noted I, or even $\mathcal{H}$ in case of ambiguity as to the underlying Hilbert space ; the null operator is noted $\mathsf{0}$.

C.2 Topologies

Several topologies are of interest on the complex vectors space $B(\mathcal{H})$, we list most of them below, in decreasing *strength* ; remember that to be stronger means to have more open (more closed) sets, i.e., less converging nets[31]. The three topologies below make sum and scalar multiplication continuous :

Norm : The norm $\|\mathsf{u}\| = \sup\{\|\mathsf{u}\|(x) \; ; \|x\| \leq 1\}$ makes both product and adjunction continuous, indeed $\|\lambda\mathsf{u}\| = |\lambda| \cdot \|\mathsf{u}\|$, $\|\mathsf{uv}\| \leq \|\mathsf{u}\| \cdot \|\mathsf{v}\|$, $\|\mathsf{u}^*\| = \|\mathsf{u}\|$; $B(\mathcal{H})$ is complete, i.e., it is an involutive *Banach algebra*. But, last but not least, as a consequence of the Cauchy-Schwarz inequality, $\|\mathsf{uu}^*\| = \|\mathsf{u}\|^2$, making $B(\mathcal{H})$ a C^*-*algebra*.

Strong : The net (u_i) *strongly* converges to u when for all $x \in \mathcal{H}$ $\|\mathsf{u}_i(x) - \mathsf{u}(x)\|$ converges to 0. The strong topology seems a bit weird, since adjunction is not strongly continuous. The main point is that composition $\mathsf{u}, \mathsf{v} \rightsquigarrow \mathsf{uv}$ is strongly continuous, provided the argument u remains bounded in norm.

[31] Generalisation of sequences : a net is a family indexed by a non-empty directed ordered set.

Weak : The net (u_i) *weakly* converges to u when for all $x, y \in \mathcal{H}$ $\langle \mathsf{u}_i(x) \mid y\rangle$ converges to $\langle \mathsf{u}(x) \mid y\rangle$. Good news, adjunction is weakly continuous, but composition is only separately continuous, which is not enough in practice. But, as a compensation, the unit ball $B_1(\mathcal{H}) := \{\mathsf{u}; \|\mathsf{u}\| \leq 1\}$ is compact.

For strong convergence, $\|x\| \leq 1$ is enough, and for weak convergence, $\|x\|, \|y\| \leq 1$, and even $x = y$ is enough. The inequality : $|\langle \mathsf{u}(x) \mid y\rangle| \leq \|\mathsf{u}(x)\| \leq \|\mathsf{u}\|$ is responsible for the relative strength of the topologies.

C.3 Normal Operators

An operator is *normal* when it commutes to its adjoint : $\mathsf{u}\mathsf{u}^* = \mathsf{u}^*\mathsf{u}$. A normal operator generates a commutative C^*-algebra, which is isomorphic —through the *spectral calculus*— with the space of continuous complex-valued functions $\mathbb{C}(\mathrm{sp}(\mathsf{u}))$ on the *spectrum* $\mathrm{sp}(\mathsf{u})$. The *spectral calculus* maps u to the inclusion map $\iota_u : \mathrm{sp}(\mathsf{u}) \subset \mathbb{C}$.

The most common operators are normal, among them :

Unitaries : $\mathsf{u}\mathsf{u}^* = \mathsf{u}^*\mathsf{u} = \mathrm{I}$. They correspond to *isometries of Hilbert space* ; by the spectral calculus, they are exactly those normal operators with spectrum in the unit circle $\mathbb{T} := \{z; |z| = 1\}$.

Hermitian : They are such that $\mathsf{h} = \mathsf{h}^*$, they are also called *self-adjoint*. By the spectral calculus, they are exactly those normal operators with spectrum in $\mathbb{R}$. *Positive* hermitians are exactly those normal operators with spectrum in $\mathbb{R}^+$; the typical positive hermitian is any operator $\mathsf{u}\mathsf{u}^*$. Indeed any positive hermitian h is of this form, and u can even be chosen positive, just take $\mathsf{u} = \sqrt{\mathsf{h}}$, which makes sense, since the function $\sqrt{\cdot}$ is defined and continuous on $\mathrm{sp}(\mathsf{h}) \subset \mathbb{R}^+$. Among the standards of the spectral calculus, the decomposition $\mathsf{h} = \mathsf{h}^+ - \mathsf{h}^-$ of a hermitian as the difference of two positive hermitians : apply the spectral calculus to the real functions $x^+ := \sup(x, 0)$ and $x^- := \sup(-x, 0)$, and observe that $\mathsf{h}^+\mathsf{h}^- = \mathsf{h}^-\mathsf{h}^+ = \mathsf{0}$. We use the notation $Her(\mathcal{H})$ for the set of bounded hermitians operating on $\mathcal{H}$; more generally, we can indicate the spectrum, e.g., $Her_{]-1,+1]}(\mathcal{H})$ will denote hermitians such that $\mathrm{sp}(\mathsf{h}) \subset]-1, +1]$, or the norm, e.g., $Her_{\leq 1}(\mathcal{H})$ consists of hermitians of norm ≤ 1, and $Her_{<1}(\mathcal{H})$ of hermitians of norm < 1. Finally, $Her^+(\mathcal{H})$ will stand for *positive* hermitians.

Normality is not that interesting beyond these two cases : for normal operators don't socialise. This is neither the case for unitaries (closed under product, inversion, multiplication by a scalar of modulus 1), nor hermitians (closed under addition, multiplication by a real scalar ; positive hermitians are closed under addition and multiplication by a positive scalar).

C.4 The Pointwise Order

The standard definition of positivity is pointwise :

$$\langle \mathsf{h}(x) \mid x \rangle \geq 0 \qquad (x \in \mathcal{H}) \tag{49}$$

In fact the quadratic form $Q(x) := \langle \mathsf{h}(x) \mid x \rangle$ determines h, and conversely, any bounded and positive quadratic form

$$\begin{array}{ll} 0 \leq Q(x) \leq M\|x\|^2 & (x \in \mathcal{H}) \\ Q(\lambda x) = |\lambda|^2 Q(x) & (x \in \mathcal{H}, \lambda \in \mathbb{C}) \\ Q(x+y) + Q(x-y) = 2(Q(x) + Q(y)) & (x, y \in \mathcal{H}) \end{array} \tag{50}$$

can uniquely be written $Q(x) := \langle \mathsf{h}(x) \mid x \rangle$, with $\mathsf{h} \in Her^+_{\leq M}(\mathcal{H})$.

Positive hermitians induce a partial ordering of $Her(\mathcal{H})$, which is defined pointwise by :

$$\mathsf{h} \leq \mathsf{k} \quad \Leftrightarrow \quad \forall x \in \mathcal{H} \quad \langle \mathsf{h}(x) \mid x \rangle \leq \langle \mathsf{k}(x) \mid x \rangle \tag{51}$$

A (monotone increasing) *net* of hermitians of $Her(\mathcal{H})$ is a family $(\mathsf{h}_i)(i \in I)$ indexed by a (non-empty) directed ordered set I (I is not supposed to be denumerable), and such that

$$i \leq j \Rightarrow \mathsf{h}_i \leq \mathsf{h}_j \tag{52}$$

A bounded net admits a l.u.b. $\mathsf{h} = \sup_{i \in I} \mathsf{h}_i$ defined by

$$\langle \mathsf{h}(x) \mid x \rangle := \sup_{i \in I} \langle \mathsf{h}_i(x) \mid x \rangle \tag{53}$$

The equation makes sense since the $\langle \mathsf{h}_i(x) \mid x \rangle$ are bounded, and it defines a hermitian, i.e., a positive quadratic form, because of the directedness of I. Although h is defined as a *weak* limit, it appears to be a strong limit :

Proposition 18
If $\mathsf{h} = \sup_{i \in I} \mathsf{h}_i$, *then* $\mathsf{h}_i \to \mathsf{h}$ *in the* strong-operator *topology.*

Proof : See lemma 5.1.4.in [17]. □

For obvious reasons, symmetric results hold for *monotone decreasing* nets (existence of g.l.b., strong convergence).

C.5 Projections and Symmetries

An operator which is both hermitian and unitary enjoys $\mathsf{u}^2 = \mathsf{u}$, let us call it a *symmetry* ; symmetries are exactly those normal operators with spectrum in $\{-1,+1\} = \mathbb{R} \cap \mathbb{T}$, i.e., in $Her_{\{-1,+1\}}(\mathcal{H})$. An idempotent hermitian is called a *projection*, and among normal operators, projections are those with spectrum in $\{0,1\}$, i.e., in $Her_{\{0,+1\}}(\mathcal{H})$. On the Hilbert space $\mathcal{H}$, a projection can be identified with its range $\mathcal{R} = \mathrm{rg}(\mathsf{h})$, which is a closed subspace : using the orthogonal decomposition $\mathcal{H} = \mathcal{R} \oplus \mathcal{R}^\perp$, $\mathsf{h}(x \oplus y) = x$, i.e., h acts as the orthoprojection on its range $\mathcal{R}$. This justifies the abusive notational identification between h and its range.

Every symmetry σ is induced by an orthogonal decomposition $\mathcal{H} = \mathcal{R} \oplus \mathcal{S}$: with the abuse of notations just introduced, let $\mathcal{R} := (\mathrm{I}+\sigma)/2$, $\mathcal{S} := (\mathrm{I}-\sigma)/2$, so that $\sigma(x \oplus y) = x - y$. Conversely, a projection $\mathcal{R}$ induces the symmetry $2\mathcal{R} - \mathrm{I}$.

C.6 Partial Isometries

A *partial isometry* is any u such that uu^* is a projection ; partial isometries must be handled with care, since they need not be normal. Since uu^* is normal, a spectral characterisation is $\mathrm{sp}(\mathsf{uu}^*) \subset \{0,1\}$, which yields[32]
$\mathrm{sp}(\mathsf{u}^*\mathsf{u}) \subset \mathrm{sp}(\mathsf{uu}^*) \cup \{0\} \subset \{0,1\}$: hence $\mathsf{u}^*\mathsf{u}$ is also a projection.

A partial isometry establishes an isomorphism between the spaces (i.e., projections) $\mathsf{u}^*\mathsf{u}$ and uu^*. But one can hardly compose them : if u, v are partial isometries, then uv is a partial isometry exactly when the projections vv^* and $\mathsf{u}^*\mathsf{u}$ commute.

A *partial symmetry* is a hermitian partial isometry. Partial symmetries are those normal operators σ such that $\mathrm{sp}(\sigma) \subset \{-1,0,1\}$, i.e., belong to $Her_{\{-1,0,+1\}}(\mathcal{H})$; in other terms, among the normal operators, those enjoying $\sigma^3 = \sigma$. Then σ^2 is a projection $\mathcal{R}$, and σ restricted to $\mathcal{R}$ is a symmetry. Any partial symmetry σ can be uniquely written as the difference of two projections $\sigma^+ - \sigma^-$, with $\sigma^+ \cdot \sigma^- = \mathsf{0}$.

C.7 Blocks and Matrices

We explain our conventions about matrices.

Blocks : In case of a (Hilbert) direct sum decomposition $\mathcal{H} = \bigoplus_1^n \mathcal{H}_i$, we can write any operator f on $\mathcal{H}$, bounded or unbounded, as the sum $\sum_{ij} \mathcal{H}_i \mathsf{f} \mathcal{H}_j$. What one can write, for instance, when $n = 2$ as

[32] See [17], proposition 3.2.8.

$\mathsf{f} = \begin{pmatrix} \mathcal{H}_1\mathsf{f}\mathcal{H}_1 & \mathcal{H}_1\mathsf{f}\mathcal{H}_2 \\ \mathcal{H}_2\mathsf{f}\mathcal{H}_1 & \mathcal{H}_2\mathsf{f}\mathcal{H}_2 \end{pmatrix}$. Such blocks compose in the usual way, i.e., $\mathsf{h}_{ik} = \sum_j \mathsf{f}_{ij}\mathsf{g}_{jk}$. However, they are not the real thing, we did hardly more than provide a useful, readable, notation.

Matrices : They correspond to an isomorphism $B(\mathcal{H}) \sim B(\mathcal{K}) \otimes M_n(\mathbb{C})$. In other terms, in $\begin{bmatrix} \mathsf{f}_{11} & \mathsf{f}_{12} \\ \mathsf{f}_{21} & \mathsf{f}_{22} \end{bmatrix}$ (observe the different style of brackets), all coefficients belong to $B(\mathcal{K})$, not to $B(\mathcal{H})$. This is the real thing.

The two notions can be related in a particular case, namely when we are given partial isometries α_{ij}, such that $\alpha_{ii} = \mathcal{H}_i$, $\alpha_{ji} = {\alpha_{ij}}^*$, $\alpha_{ik} = \alpha_{ij}\alpha_{jk}$. Then f can be represented by the actual matrix (with coefficients in $B(\mathcal{H}_1)$), $\mathsf{f}_{ij} = \alpha_{1i}\mathsf{f}\alpha_{j1}$, e.g., when $n = 2$, $\mathsf{f} = \begin{bmatrix} \alpha_{11}\mathsf{f}\alpha_{11} & \alpha_{11}\mathsf{f}\alpha_{21} \\ \alpha_{12}\mathsf{f}\alpha_{11} & \alpha_{12}\mathsf{f}\alpha_{21} \end{bmatrix}$.

C.8 Inclusion of Ranges

What follows is basically folklore, a variation on the *polar decomposition* of operators, see [18], p. 401.

Definition 18 (Domain)
Let $\mathsf{u} \in B(\mathcal{H})$ *; we define* $\mathrm{dom}(\mathsf{u}) := (\ker(\mathsf{u}))^{\perp}$, *and* $\mathrm{clrg}(\mathsf{u})$ *as* $\mathrm{dom}(\mathsf{u}^*)$, *so that the two notions coincide in the hermitian case.*

$\mathrm{clrg}(\mathsf{u})$ is the closure of $\mathrm{rg}(\mathsf{u})$, i.e., $\mathrm{rg}(\mathsf{u})^{\perp\perp}$.

Proposition 19
Assume that $0 \leq \mathsf{f} \leq \mathsf{h}$ *; then* $\mathrm{rg}(\sqrt{\mathsf{f}}) \subset \mathrm{rg}(\sqrt{\mathsf{h}})$ *; indeed* $\sqrt{\mathsf{f}} = \sqrt{\mathsf{h}} \cdot \varphi$ *for an appropriate* φ *of norm at most* 1.

Proof : Using $\langle \mathsf{f}(x) \mid x \rangle = \|\sqrt{\mathsf{f}}(x)\|^2$, etc., we easily obtain $\|\sqrt{\mathsf{f}}(x)\| \leq \|\sqrt{\mathsf{h}}(x)\|$. Therefore we can define the linear map ψ from $\mathrm{rg}(\sqrt{\mathsf{h}})$ to $\mathrm{rg}(\sqrt{\mathsf{f}})$, by $\psi(\sqrt{\mathsf{h}}(x)) := \sqrt{\mathsf{f}}(x)$. It is immediate that :

(i) ψ is well-defined ;

(ii) $\|\psi(y)\| \leq \|y\|$.

In two steps, we can :

(i) First extend ψ by norm-continuity to the closed subspace $\mathrm{dom}(\sqrt{\mathsf{h}})$;

(ii) Next extend it to the full $\mathcal{H}$ by making it null on the orthocomplement $\ker(\sqrt{\mathsf{h}})$ $(= \ker(\mathsf{h}))$ of $\mathrm{dom}(\sqrt{\mathsf{h}})$ $(= \mathrm{dom}(\mathsf{h}))$.

If this extension is called φ^*, it is plain that $\|\varphi^*\| \leq 1$ and $\sqrt{\mathsf{f}} = \varphi^* \cdot \sqrt{\mathsf{h}}$. From this, $\sqrt{\mathsf{f}} = \sqrt{\mathsf{h}} \cdot \varphi$ and we are done. $\square$

Remark 17
More generally, if $\mathsf{uu}^* \leq \mathsf{h}$, there exists a ψ of norm at most 1 such that $\mathsf{u} = \sqrt{\mathsf{h}} \cdot \psi$. This ψ is made unique by the requirement $\mathrm{rg}(\psi) \subset \mathrm{dom}(\mathsf{h})$. By the way, if $\mathsf{u} = \sqrt{\mathsf{h}} \cdot \theta$; then $\psi = \mathrm{dom}(\mathsf{h}) \cdot \theta$.

NON SI NON LA

References

[1] V. M. Abrusci and P. Ruet. **Non-commutative logic I : the multiplicative fragment**. *Annals of Pure and Applied Logic*, 101:29 – 64, 2000.

[2] R. Amadio and P.-L. Curien. **Domains and Lambda-Calculi**. Number 46 in Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 1998.

[3] G. Berry. **Stable models of typed lambda-calculi**. In *Proceedings of the 5th International Colloquium on Automata, Languages and Programming*, number 62 in Lecture Notes in Computer Science. Springer Verlag, 1978.

[4] A. Connes. **Non-commutative geometry**. Academic Press, San Diego, CA, 1994.

[5] V. Danos and L. Regnier. **The structure of multiplicatives**. *Archive for Mathematical Logic*, 28:181 – 203, 1989.

[6] Vincent Danos and Laurent Regnier. **Proof-Nets and the Hilbert Space**. In Jean-Yves Girard, Yves Lafont, and Laurent Regnier, editors, *Advances in Linear Logic*, volume 222 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1995.

[7] T. Ehrhard. **Hypercoherences : a strongly stable model of linear logic**. In Girard, Lafont, and Regnier, editors, *Advances in Linear Logic*, pages 83 – 108, Cambridge, 1995. Cambridge University Press.

[8] J.-Y. Girard. **Towards a geometry of interaction**. In *Categories in Computer Science and Logic*, pages 69 – 108, Providence, 1988. American Mathematical Society. Proceedings of Symposia in Pure Mathematics n°92.

[9] J.-Y. Girard. **Geometry of interaction I : interpretation of system F**. In Ferro, Bonotto, Valentini, and Zanardo, editors, *Logic Colloquium '88*, pages 221 – 260, Amsterdam, 1989. North-Holland.

[10] J.-Y. Girard. **Geometry of interaction II : deadlock-free algorithms**. In Martin-Löf and Mints, editors, *Proceedings of COLOG 88*, volume 417 of *Lecture Notes in Computer Science*, pages 76 – 93, Heidelberg, 1990. Springer-Verlag.

[11] J.-Y. Girard. **Geometry of interaction III : accommodating the additives**. In Girard, Lafont, and Regnier, editors, *Advances in Linear Logic*, pages 329 – 389, Cambridge, 1995. Cambridge University Press.

[12] J.-Y. Girard. **Coherent Banach Spaces : a continuous denotational semantics**. *Theoretical Computer Science*, 227:275 – 297, 1999.

[13] J.-Y. Girard. **Locus Solum.** *Mathematical Structures in Computer Science*, 11:301 – 506, 2001.

[14] J.-Y. Girard. **Between logic and quantic : a tract**. Technical report, Institut de Mathématiques de Luminy, Octobre 2003.

[15] J.-Y. Girard, Y. Lafont, and P. Taylor. **Proofs and types**, volume 7 of *Cambridge tracts in theoretical computer science.* Cambridge University Press, Cambridge, 1990.

[16] G. Gonthier, M. Abadi, and J.-J. Levy. **The geometry of optimal lambda-reduction**. In *ACM* Press, editor, *POPL'92*, Boston, 1992. Birkhäuser.

[17] R. V. Kadison and J. R. Ringrose. **Fundamentals of the theory of operator algebras, vol. I**. Pure and applied mathematics. Academic Press, 1983. (*vol. III contains the solutions of exercises of vol. I*).

[18] R. V. Kadison and J. R. Ringrose. **Fundamentals of the theory of operator algebras, vol. II**. Pure and applied mathematics. Academic Press, Orlando, Florida, 32887, 1986. ISBN 0-12-393302-1.

[19] G. Kreisel. **Mathematical logic**. In T. L. Saaty, editor, *Lectures in modern mathematics, vol III*, pages 99 – 105. Wiley *&* Sons, New York, 1965.

[20] P. Selinger. **Towards a quantum computing language**. *Mathematical Structures in Computer Science*, 2004.