UPENN 7 MARS 2001

FROM THE RULES OF LOGIC TO THE LOGIC OF RULES

Jean-Yves Girard

FORMALISMS

En mathématiques le XX$^{\text{ème}}$ siècle commence vers 1890. . .

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- Physics made of islets linked by hazardous passerelles. In constrast to unity de principe of la mathématique. Analysis and algebra (calculus on letters, variables, equations) do not contradict each other.
- Central role of natural numbers ; Peano's arithmetic PA, one of the very first examples of a formal system.

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2 PEANO'S ARITHMETIC PA

Terms : $\mathbf{0}$ $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ $\mathbf{St}$ $\mathbf{t} + \mathbf{t}'$ $\mathbf{t} \times \mathbf{t}'$

`(zero, variables, successor (+1), sum, product)`

Example : $\mathbf{SSSSSS0}$ **represents** $\mathbf{6}$**. . . (compare with** IIIIII**).**

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Propositions :

$$\mathbf{t = t' \quad \neg P \quad P \vee P' \quad P \wedge P' \quad P \Rightarrow P' \quad \forall x P \quad \exists x P}$$

(equals, not, or, and, implies, for all, there is)

$$\mathbf{\forall x \forall y \forall z (x \neq 0 \wedge y \neq 0 \wedge z \neq 0) \Rightarrow x^3 + y^3 \neq z^3}$$

with $\mathbf{t \neq u :: \neg(t = u), t^3 :: (t \times (t \times t))}$

(A case of Fermat's last theorem).

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Axioms : $\mathbf{P \Rightarrow P}$ $\mathbf{x = x}$ $\ldots$ (logic)

$$\mathbf{x + 0 = x \quad x + Sy = S(x + y) \quad x \times 0 = 0 \quad x \times Sy = (x \times y) + x}$$

$$\mathbf{Sx \neq 0 \qquad Sx = Sy \Rightarrow x = y}$$

(arithmetic).

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$\mathbf{Sx \neq 0}$ $\mathbf{Sx = Sy \Rightarrow x = y}$ **(arithmetic).**

Proof rules :

$$\frac{\mathbf{P} \quad \mathbf{P \Rightarrow Q}}{\mathbf{Q}}$$

(Modus Ponens)

$$\frac{\mathbf{P[0]} \quad \mathbf{P[x] \Rightarrow P[Sx]}}{\mathbf{P[y]}}$$

(Recurrence or Induction)

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Theorems : $\mathbf{SSS0 + SS0 = SSSSS0}$ $\forall\mathbf{x}\,(\mathbf{0 + x = x})$

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  `Syntax error`
  `A fatal error appeared at 0028:C000BCED in the VXD VMM(01) + 0000ACED`.

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- Negative answer :

NOT TO KNOW $\neq$ TO KNOW NOT

RECESSIVE $\neq$ EXPANSIVE

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PARADOXES

Est-ce grave, Docteur ?

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- Infinite list of all programs in a given language :

  Size : Sorted in increasing size,

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- List of all infinite lists of zeros and ones ? Impossible because of diagonal argument. Let $L_1, L_2, L_3, \ldots$ be a list of all infinite lists ; dispose them one above another and then...

4 Cantor's diagonal

0 0 1 1 1 0 0 1 0 0 0 1 1 1 1 0 0 0 0 1 1 1 0 . . .

0 1 1 0 1 0 1 0 0 1 1 1 1 0 0 1 0 1 1 1 0 0 0 . . .

0 1 1 1 1 0 1 1 1 1 1 0 1 1 1 0 0 0 0 0 0 1 1 . . .

1 1 1 1 1 1 1 1 1 1 0 1 1 1 1 1 1 0 1 1 1 1 1 . . .

1 0 1 1 1 0 0 1 0 0 0 1 1 1 1 0 0 1 0 1 1 1 0 . . .

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- Argument « for » : unfilled optional cases, read as no.
- Objection : `Return`, which means « I do not answer ». Otherwise wait for the bus... This is in fact the halting problem. Rigourous proof (Turing) by means of the Cantor machinery.
- Language is therefore expansive, he only produces positive informations. Think of a search for files.
- Mathematical formalism is expansive too : it accumulates theorems. In sharp contrast with medicine.

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- **Hilbert (1900) : prove the consistency (non-contradiction) of arithmetic. Recidive around 1920 with a finitistic programme. Reduction to the sole formal paradoxes.**

FORMALISTS

Zorro est arrivé. . .

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- **For instance one could show that a theorem has necessarily an even number of symbols ; if** P **is provable,** $\neg P$ **is « odd » hence not provable... Too naive !**

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- In mathematics, verifying up to precision $1/N$ means checking up to integer N : Hilbert, Popper, même combat !
- Consistency is recessive : « so far no contradiction ».
- A property is expansive when its negation is recessive. Example « provability » vs. « consistency ».

INCOMPLETENESS

Non, c'était Gödel. . .

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- **Properties of the formalism represented by propositions of Peano's arithmetic PA ; e.g., « P is provable » becomes $\mathrm{Thm}_{\mathbf{PA}}[\ulcorner P \urcorner]$, « PA is consistent » becomes $\mathrm{Con}_{\mathbf{PA}}$. No immediate arithmetical meaning.**

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- Proposition $G \sim \neg\mathrm{Thm}_{\mathbf{PA}}[\ulcorner G \urcorner]$: « I am not provable » The liar's antinomy « I am lying » ?. . .

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- Recessive $\neq$ Expansive
  True $\neq$ Prouvable
  P Not provable $\neq \neg\mathrm{P}$ Provable.

11 NEGATIONS AND NEGATIONISTS

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- **Shoot Gödel by refuting the theorem ?** « **Tout est faux il suffit d'attendre.** »

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- Hilbert's formalism eventually dies of overscientism : to prove the consistency in mathematics !
  Vous l'avez voulu, Georges Dandin !

INTUITIONNISMS

Pendant que Dupond et Dupont progressaient hardiment. . .

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- **Linear logic (1985) : symmetry program/environment. Procedural logic, no longer realist.**

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- $\Omega \leq \mathfrak{D} \leq$ ✠. The (non)-design Ω (Faith) and the design ✠ (Daimon) as paradigms of recessivity and expansivity.

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- Ludics, single locative description as intersection

  General case : Intersection type.

  Mystery of incarnation : When behaviours disjoint,

$$|\mathbf{G} \cap \mathbf{H}| = |\mathbf{G}| \times |\mathbf{H}|$$

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P VS. NP

(à suivre)

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`http://iml.univ-mrs.fr/~girard/Articles.html`