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BETWEEN LOGIC AND QUANTIC

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I-SOME LOGICAL DIVIDES

I.1 Realism vs. Procedurality

- Logic could refer to its :

  Meta : something preexisting, truth values, models, etc.

  Own procedures : proof-search, cut-elimination.

- Procedurality is more demanding than realism :
  - The meta is exterritorial.
  - The logic of the rules should match the rules of logic.

- Quantum is procedural, Copenhagen school.

- Quantum logic is realistic, b.t.w., where are they now ?

- Instead of a logical taming of quantum, we try quantic renewal of this scholastic activity : foundations.

I.2 POLARITY

- **Natural deduction :** the negative fragment, $\Rightarrow, \&, \forall$.
  Proof-search : invertible $⅋, \&, \forall$, vs. synchronous $\otimes, \oplus, \exists$.
  Ludics : I play (active) vs. you play (passive).
- **Positive/Negative :** active/passive, direct/inverse limits.
  Sum/Supremum : $\mathcal{L}^1/\mathcal{L}^\infty$.
  Object/Subject : semantics/syntax, wave/measurement.
  Explicit/Implicit : synchronous/invertible.
- Commutation if same polarity : Andreoli's focusing.
- Positive post-commute, e.g., $\exists\forall \multimap \forall\exists$.

I.3 PERFECTION VS. IMPERFECTION

- In Russian distinction between I (can) speak Russian and I (am) speak(ing) Russian, rendered by different verbs, imperfective/perfective. Same in French, English, etc., but only for past tenses, imparfait/passé simple (a.k.a. parfait).
- Main novelty of linear logic, it allows perfect connectives : $\otimes, \parr, \multimap, \&, \oplus, \forall, \exists$.
- Perfection cannot be handled by truth values : in a perfect implication $\mathbf{A} \multimap \mathbf{B}$, the premise is destroyed.
- Perfection roughly lives in a finite world.
- Perfection is « quantum », imperfection is « classical ».
- Exponentials $!, ?$ are imperfect, « diamonds are forever ». The basic brick of infinity, e.g., in Dedekind's definition of natural numbers :

$$\forall \mathbf{X}\, [!(\mathbf{X} \multimap \mathbf{X}) \multimap (\mathbf{X} \multimap \mathbf{X})]$$

II-COMMUTATIVE COHERENT SPACES

II.1 COHERENT SPACES REVISITED

- Let $\mathbb{X}$ be a **set** ; $\mathbf{a}, \mathbf{b} \subset \mathbb{X}$ are **polar** when their intersection is at most a singleton :
$$\mathbf{a} \underset{\sim}{\downarrow} \mathbf{b} \Leftrightarrow \sharp(\mathbf{a} \cap \mathbf{b}) \leq 1$$
- A **coherent space** (with **carrier** $\mathbb{X}$) is any $\mathbf{X} \subset \wp(\mathbb{X})$ s.t. $\mathbf{X} = \sim\sim\mathbf{X}$.
- $\mathbf{a} \in \mathbf{X}$ iff $\{\mathbf{x}, \mathbf{y}\} \in \mathbf{X}$ for all $\mathbf{x}, \mathbf{y} \in \mathbf{a}$.
- If $\mathbf{x}, \mathbf{y} \in \mathbb{X}$, then $\{\mathbf{x}, \mathbf{y}\} \in \mathbf{X} \Leftrightarrow \mathbf{x} = \mathbf{y} \vee \{\mathbf{x}, \mathbf{y}\} \notin \sim\mathbf{X}$.
- **Perfect** (linear) implication : if $\mathbf{F} \subset \mathbb{X} \times \mathbb{Y}$, if $\mathbf{a} \subset \mathbb{X}$, $\mathbf{b} \subset \mathbb{Y}$, then cannot find $\mathbf{F}[\mathbf{a}]$ enjoying **cut-elimination** :
$$\sharp(\mathbf{F} \cap \mathbf{a} \times \mathbf{b}) = \sharp(\mathbf{F}[\mathbf{a}] \cap \mathbf{b})$$
- But $\mathbf{F} \in \mathbf{X} \multimap \mathbf{Y}$, $\mathbf{a} \in \mathbf{X}$, $\mathbf{b} \in \sim\mathbf{Y}$, force $\sharp(\ldots) \leq 1$. Define
$$\mathbf{F}[\mathbf{a}] := \{\mathbf{y} \in \mathbb{Y}; \exists \mathbf{x} \in \mathbf{a} \ (\mathbf{x}, \mathbf{y}) \in \mathbf{F}\}$$
- **Identity axiom** $\mathbf{X} \multimap \mathbf{X}$ handled by diagonal $\mathbf{\Delta} := \{(\mathbf{x}, \mathbf{x}); \mathbf{x} \in \mathbb{X}\}$.

II.2 Probabilistic coherent spaces

- Replace $\mathbf{a} \subset \mathbb{X}$ with $\mathbb{X} \xrightarrow{\mathbf{f}} \mathbb{R}^+$
- Replace $\mathbf{a} \underset{\frown}{\downarrow} \mathbf{b}$ with :

$$\mathbf{f} \underset{\frown}{\downarrow} \mathbf{g} \Leftrightarrow \sum_{\mathbf{x} \in \mathbb{X}} \mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) \leq \mathbf{1}$$

- Cut-elimination now works in full generality :

$$\sum_{\mathbf{x},\mathbf{y} \in \mathbb{X} \times \mathbb{Y}} \mathbf{\Phi}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{f}(\mathbf{x}) \mathbf{g}(\mathbf{y}) = \sum\nolimits_{\mathbf{y} \in \mathbb{Y}} \mathbf{\Phi}[\mathbf{f}](\mathbf{y}) \cdot \mathbf{g}(\mathbf{y})$$

  with $\mathbf{\Phi}[\mathbf{f}](\mathbf{y}) := \sum_{\mathbf{x} \in \mathbb{X}} \mathbf{\Phi}(\mathbf{x}, \mathbf{y}).\mathbf{f}(\mathbf{x})$, at least in the perfect case.
- Identity axiom is now the characteristic function of the diagonal.
- There is always an output, even when logic is not respected : existence is anterior to essence.

II.3 THE GENERAL COMMUTATIVE CASE (?)

- Work with (essentially) bounded real valued functions on (measure space) $\mathbb{X}$, i.e., $\mathcal{L}^\infty(\mathbb{X})$.
- One must specifiy coherence of $\mathbf{X}$: closed balanced convex $\mathbf{X_1} \subset \mathcal{L}^\infty(\mathbb{X})$.
- Negation defined by polarity :
$$\sim\mathbf{X_1} := \{\mathbf{g}; \forall \mathbf{f} \in \mathbf{X_1} \int_{\mathbb{X}} |\ \mathbf{fg}\ | \leq \mathbf{1}\}$$
- Hahn-Banach : $\sim\sim\mathbf{X} = \mathbf{X}$.
- Only coherent positive functions of $\mathbf{X}$ are retained.
- Non-positive elements take care of uniformity, e.g., force $\mathbf{F}(\mathbf{a}) = \mathbf{F}(\mathbf{b})$ for all $\mathbf{F}$ by saying that $\boldsymbol{\lambda}(\mathbf{a} - \mathbf{b}) \in \sim\mathbf{X_1}$ for $\boldsymbol{\lambda} \in \mathbb{N}$.
- Cut-elimination uses integral formula $\mathbf{\Phi}[\mathbf{f}](\mathbf{y}) := \int_{\mathbb{X}} \mathbf{\Phi}(\mathbf{x}, \mathbf{y}).\mathbf{f}(\mathbf{x})$; unfortunately, the diagonal is likely to be of measure $\mathbf{0}$: no identity axiom ! Works with ℓ^∞, but not $\mathcal{L}^\infty$, i.e., stays discrete.

III-QUANTUM COHERENT SPACES

III.1 THE NON-COMMUTATIVE CASE

- Connes : dereify by means of non-commutative algebras.
- The non-commutative (complex) analogues of $\mathcal{L}^\infty$ are von Neumann algebras : $\mathcal{R} \subset \mathcal{B}(\mathbb{H})$, self-adjoint and equal to its bicommutant. The latter condition is the same as closure under l.u.b. of projections : « non-commutative Boolean algebras ».
- Mutatis mutandis, reals become hermitians $\mathbf{u} = \mathbf{u}^*$, positive reals become positive hermitians $\langle \mathbf{u}(\mathbf{x}) \mid \mathbf{x} \rangle \geq \mathbf{0}$, the integral becomes a trace...
- However trace hardly exists (only for $\mathbf{I_n}, \mathbf{II_1}$). Moreover, we shall not be happy with the sole positive hermitians.
- We begin with a toy, restricted to $\mathbf{I_n}$ —roughly finite-dimensional matrix algebras— and perfect connectives.

III.2 QUANTUM COHERENT SPACES

- Finite dimensional Hilbert space $\mathbb{X}$ endowed with :

  Coherence : $\mathbf{X_1} \subset \mathcal{B}(\mathbb{X})$, closed, convex, balanced and self-adjoint.

  Positivity : $\mathbf{X}^+ \subset \mathcal{B}(\mathbb{X})$, self-adjoint closed convex cone.

- Negation $\sim\mathbf{X}$ defined on the same carrier $\mathbb{X}$ using :
  $|\, \mathrm{tr}(\mathbf{uv}) \,| \leq \mathbf{1}$, $\mathrm{tr}(\mathbf{uv}) \geq \mathbf{0}$.

- A perfect logical proposition becomes a QCS $\mathbf{X}$, and a proof of it becomes a coherent « positive » hermitian : $\mathbf{u} \in \mathbf{X_1} \cap \mathbf{X}^+$.

- The rules of perfect logic can be interpreted, but « they stay inside the diagonal » : conservative extension of coherent spaces. . . Much ado about nothing ?

III.3 DEFAULT SETTINGS

- In general the default setting for positivity is $X^+ = \mathcal{B}(\mathbb{X})^+$, in which case $\sim\mathbf{X}^+$ is also the positive hermitians, because :

$$\mathbf{tr}(\mathbf{uv}) = \mathbf{tr}((\sqrt{\mathbf{u}})^2(\sqrt{\mathbf{v}})^2) = \mathbf{tr}((\sqrt{\mathbf{u}}\sqrt{\mathbf{v}})(\sqrt{\mathbf{v}}\sqrt{\mathbf{u}})) \geq \mathbf{0}$$

- A default choice for coherence is the unit ball (connective $\&$), in which case, the dual coherence is the unit ball w.r.t. trace norm $\|\mathbf{u}\|_1 = \mathbf{tr}(\sqrt{\mathbf{uu}^*})$, (connective $\oplus$).

- A fancy choice for coherence : the unit ball w.r.t. Hilbert-Schmidt norm $\|\mathbf{u}\|_2 = \sqrt{\mathbf{tr}(\mathbf{uu}^*)}$. This choice is self-dual (HS norm makes $\mathcal{B}(\mathbb{X})$ a Hilbert space).

III.4 CUT-ELIMINATION

- If $\mathbf{F} \in \mathcal{B}(\mathbb{X} \otimes \mathbb{Y})$, if $\mathbf{a} \in \mathcal{B}(\mathbb{X})$, one defines $\mathbf{F}[\mathbf{a}] \in \mathcal{B}(\mathbb{Y})$ by :
$$\mathbf{tr}(\mathbf{F}[\mathbf{a}] \cdot \mathbf{b}) = \mathbf{tr}(\mathbf{F} \cdot \mathbf{a} \otimes \mathbf{b})$$
- Matricially : $\langle \mathbf{F}[\mathbf{x}\mathbf{w}^*](\mathbf{y}) \mid \mathbf{z} \rangle = \langle \mathbf{F}(\mathbf{x} \otimes \mathbf{y}) \mid \mathbf{w} \otimes \mathbf{z} \rangle$.
- Any linear $\boldsymbol{\Phi}$ from $\mathcal{B}(\mathbb{X})$ to $\mathcal{B}(\mathbb{Y})$ of this form with unique $\mathbf{F}$; $\boldsymbol{\Phi} \rightsquigarrow \mathbf{F}$ linear and self-adjoint : $\boldsymbol{\Phi}^*(\mathbf{a}) := \boldsymbol{\Phi}(\mathbf{a}^*)^* \rightsquigarrow \mathbf{F}^*$.
- Let $\mathbb{X} = \mathbb{Y}$, and $\boldsymbol{\sigma} \in \mathcal{B}(\mathbb{X} \otimes \mathbb{X})$ be the flip, $\boldsymbol{\sigma}(\mathbf{x} \otimes \mathbf{y}) := \mathbf{y} \otimes \mathbf{x}$.
- $$\begin{aligned} \langle \boldsymbol{\sigma}[\mathbf{x}\mathbf{w}^*](\mathbf{y}) \mid \mathbf{z} \rangle &= \langle \boldsymbol{\sigma}(\mathbf{x} \otimes \mathbf{y}) \mid \mathbf{w} \otimes \mathbf{z} \rangle = \langle \mathbf{y} \otimes \mathbf{x} \mid \mathbf{w} \otimes \mathbf{z} \rangle \\ &= \langle \mathbf{y} \mid \mathbf{w} \rangle \langle \mathbf{x} \mid \mathbf{z} \rangle = \langle \mathbf{x}\mathbf{w}^*(\mathbf{y}) \mid \mathbf{z} \rangle \end{aligned}$$

  from which we conclude that $\boldsymbol{\sigma}[\mathbf{a}] = \mathbf{a}$.
- The flip is « positive » w.r.t. $\mathbf{X} \multimap \mathbf{X}$: sends « positive » w.r.t. $\mathbf{X}$ to « positive » w.r.t. $\mathbf{X}$. But $\mathbf{sp}(\boldsymbol{\sigma}) = \{-1, +1\}$.

III.5 η-EXPANSION AND REDUCTION OF THE WAVE PACKET

- Scholastic question : function = graph ?
- $\mathbf{A} \oplus \mathbf{B} \multimap \mathbf{A} \oplus \mathbf{B}$ admits two proofs, one coming from $\mathbf{A} \multimap \mathbf{A}$, $\mathbf{B} \multimap \mathbf{B}$, the other from the generic $\mathbf{X} \multimap \mathbf{X}$:

$$\iota := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \sigma := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- $\sigma(\begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \bar{\mathbf{b}} & \mathbf{c} \end{bmatrix}) = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \bar{\mathbf{b}} & \mathbf{c} \end{bmatrix}$; $\sigma \notin \mathcal{B}(\mathbb{X})^+$ is the « real » identity.
- General « wave » transformation $\mathbf{A} \rightsquigarrow \mathbf{UAU}^*$ induced by $\sigma \cdot \mathbf{U} \otimes \mathbf{U}^*$.
- $\iota(\begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \bar{\mathbf{b}} & \mathbf{c} \end{bmatrix}) = \begin{bmatrix} \mathbf{a} & 0 \\ 0 & \mathbf{c} \end{bmatrix}$, is a « Procustus's identity ».
- Reduction of the wave packet : true, false with probabilities $\mathbf{a}, \mathbf{c}$.
- Logical questions blurred by commutativity (diagonal matrices).

III.6 BOOLEANS, COERCIONS, INCARNATIONS. . .

- If $\mathbb{X}$ of dimension $\mathbf{2}$, we can use the defaults :

  Positivity : standard positive hermitians in all cases.

  Quantum booleans : unit ball w.r.t. trace norm : qBool.

  Quantum anti-booleans : unit ball w.r.t. usual norm : $\sim\mathrm{qBool}$.

- $\mathrm{qBool} = \bigcup \mathrm{Bool}_{\mathbf{v},\mathbf{f}}$, union over all axes of truth/falsity.

- if $\mathrm{a} \in \mathrm{Bool}_{\mathbf{v},\mathbf{f}} \subset \mathrm{qBool}$, if $\mathrm{b} \in \sim\mathrm{qBool} \subset \sim\mathrm{Bool}_{\mathbf{v},\mathbf{f}}$; then a, b are diagonal in bases $\{\mathrm{v}, \mathrm{f}\}$, $\{\mathrm{v}', \mathrm{f}'\}$. Two looks at tr$(\mathrm{ab})$:

  $\mathrm{qBool}/\sim\mathrm{qBool}$ **:** as if a reduced on base $\{\mathrm{v}', \mathrm{f}'\}$.

  $\mathrm{Bool}_{\mathbf{v},\mathbf{f}}/\sim\mathrm{Bool}_{\mathbf{v},\mathbf{f}}$ **:** as if b incarnated in $\sim\mathrm{Bool}_{\mathbf{v},\mathbf{f}}$.

- Reduction (incarnation) cancels non-diagonal coefficients, on a diagonal basis for b (for a).

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IV-Logic in von Neumann algebras

IV.1 MATRICIAL vN ALGEBRAS

- Need to accomodate some imperfection, by light (?) generalisation of the finite dimensional case.
- Direct limits of matrix algebras make sense as $\mathbf{C}^*$-algebras : involutive Banach algebras s.t. $\|\mathbf{u}\mathbf{u}^*\| = \|\mathbf{u}\|^2$.
- $\mathbf{C}^*$-algebra $\mathcal{C}$ implemented by left actions on itself : $\mathbf{x} \leadsto \mathbf{a}\mathbf{x}$.
- GNS : given state $\boldsymbol{\rho}$, define prehilbertian structure on $\mathcal{C}$: $\langle \mathbf{x} \mid \mathbf{y} \rangle = \boldsymbol{\rho}(\mathbf{x}\mathbf{y}^*)$, with separation/completion $\mathcal{C}_{\boldsymbol{\rho}}$.
- $\mathcal{C}^{\mathbf{cc}}$ vN algebra obtained by considering the left action on $\mathcal{C}_{\boldsymbol{\rho}}$.
- $\mathcal{C}^{\mathbf{c}}$: same, but with the right action on $\mathcal{C}_{\boldsymbol{\rho}}$.

IV.2 FACTORS

- A factor is a connected vN algebra, i.e., with trivial center., i.e., when $\mathcal{R} \cap \mathcal{R}^{\mathbf{c}} = \mathbb{C} \cdot \mathbf{I}$; the commutant of a factor is factor.
- Two projections π, π' are equivalent when there exists $\mathbf{u} \in \mathcal{R}$ s.t. $\mathbf{uu}^* = \pi$, $\mathbf{u}^*\mathbf{u} = \pi'$. A projection is finite when not equivalent to a subprojection.
- In a factor, the preorder « equivalent to a subprojection » is total ; if we forget the null projection, we get three cases :

Type $\mathbf{I}$: there is a minimum projection ; splits into $\mathbf{I_n}$ and $\mathbf{I_\infty}$ depending whether or not the identity is finite.

Type $\mathbf{II}$: no minimum projection but there is a finite one ; $\mathbf{II_1}$ or $\mathbf{II_\infty}$ depending whether or not the identity is finite.

Type $\mathbf{III}$: all projections are infinite.

IV.3 THE FINITE MATRICIAL FACTOR

- A factor of type $\mathbf{II_1}$ is a sort of space with fractal dimension, e.g., when $\pi \sim 1 - \pi$, $\mathbf{dim}(\pi) = 1/2$. Induces a (unique) finite trace.
- The matricial factor of type $\mathbf{II_1}$ is obtained as the GNS completion of $\varinjlim(\mathcal{M}_{2^n}(\mathbb{C}))$ w.r.t. normalised trace $\varinjlim(2^{-n}\,\mathbf{tr}(a))$.
- A sort of static infinite is expected : in the Hilbert hotel you still can get twice more rooms but half-sized ! Might be the right place for (non-fanatic) « finitism ».
- In presence of trace, the rules for imperfect connectives should move to something like —say— LLL, light linear logic.
- Since imperfect means infinite, the natural numbers should be « different », i.e., the functions from natural numbers should be of tame complexity.

IV.4 GEOMETRY OF INTERACTION

- **The operation succeeded —we got our trace— but the patient is dead : we expect $\mathbf{tr}(\boldsymbol{\sigma} \cdot \mathbf{a} \otimes \mathbf{b}) = \mathbf{tr}(\mathbf{a} \cdot \mathbf{b})$, but :**

$$\mathbf{tr}(\boldsymbol{\sigma} \cdot \mathbf{I} \otimes \mathbf{I}) = \mathbf{tr}(\boldsymbol{\sigma}) = \lim_{n \to \infty} 1/n^2[(n(n+1)/2 - n(n-1)/2] = 0$$

$$\neq \mathbf{tr}(\mathbf{I}^2) = \mathbf{tr}(\mathbf{I}) = 1$$

- **Replace trace with determinant :**

$$\mathbf{det}(1 - \mathbf{F} \cdot \mathbf{a} \oplus \mathbf{b}) = \mathbf{det}(1 - \mathbf{F}_{11} \cdot \mathbf{a}).\mathbf{det}(1 - \mathbf{F}[\mathbf{a}] \cdot \mathbf{b})$$

with $\mathbf{F} = \begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{F}_{21} & \mathbf{F}_{22} \end{bmatrix}$ **and** $\mathbf{F}[\mathbf{a}] = \mathbf{F}_{22} + \mathbf{F}_{21} \cdot \mathbf{a} \cdot (1 - \mathbf{F}_{11} \cdot \mathbf{a})^{-1} \cdot \mathbf{F}_{12}$

- **Identity axiom handled by symmetry : $\mathbf{s}(\mathbf{x} \oplus \mathbf{y}) = \mathbf{y} \oplus \mathbf{x}$.**

- **Quid of convergence ?**

$$\ln \mathbf{det}(\mathbf{u}) := \mathbf{tr}(\ln \mathbf{u})$$

IV.5 LUDIONS

- **Polarity suggests to use « non-commutative » bipartite graphs : Hilbert space splits in two. Two sorts of ludions ($\mathbf{A}, \mathbf{a}, \mathbf{C}, \mathbf{c} \geq \mathbf{0}$) :**

$$\mathbf{U} := \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^* & -\mathbf{C} \end{bmatrix} \qquad \text{or} \qquad \mathbf{u} := \begin{bmatrix} -\mathbf{c} & \mathbf{b} \\ \mathbf{b}^* & \mathbf{a} \end{bmatrix}$$

  with $\|\mathbf{U}\|, \|\mathbf{u}\| \leq \mathbf{1}$.

- **« Proofs » vs. « models » (awfully subjective !!!).**

- **det$(\mathbf{1} - \mathbf{Uu}) \in [\mathbf{0}, +\infty]$, in semi-finite factors (type $\neq \mathbf{III}$), matricial or not.**

- **Monotonous in $\mathbf{A}, \mathbf{a}, \mathbf{C}, \mathbf{c}$: with $\mathbf{V} := \begin{bmatrix} \mathbf{1} & -\mathbf{B} \\ -\mathbf{b} & \mathbf{1} \end{bmatrix}$**

$$\mathrm{det}(\mathbf{1} - \mathbf{Uu}) = \mathrm{det}(\mathbf{VV}^*) \cdot \mathrm{det}(\mathbf{1} + \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{a} \end{bmatrix} \cdot \mathbf{V}^{-1} \cdot \begin{bmatrix} \mathbf{c} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \cdot \mathbf{V}^{*-1})$$

IV.6 AUGUSTINIAN CONSIDERATIONS

- If $1 < \mathbf{det}(1 - \mathbf{Uu}) < \infty$ one of $\mathbf{U}, \mathbf{u}$ might be « truer than the other » : when it does not contribute to the result.

$$\mathbf{det}(1 - \mathbf{Uu}) = \mathbf{det}(1 - \mathbf{U_0 u})$$

with $\mathbf{U_0} := \begin{bmatrix} 0 & \mathbf{B} \\ \mathbf{B}^* & 0 \end{bmatrix}$.

- Both $\mathbf{U}, \mathbf{u}$ cannot win : logical consistency.

- If $\mathbf{A} = \mathbf{C} = 0$ (e.g., $\mathbf{U} = \mathbf{s} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$), $\mathbf{U}$ always wins (is a proof).

- Compare :

$$\sharp(\boldsymbol{\Delta} \cap \mathbf{a} \times \mathbf{b}) = \sharp(\boldsymbol{a} \cap \mathbf{b})$$

$$\mathbf{tr}(\boldsymbol{\sigma} \cdot \mathbf{a} \otimes \mathbf{b}) = \mathbf{tr}(\mathbf{a} \cdot \mathbf{b})$$

$$\mathbf{det}(1 - \mathbf{s} \cdot \mathbf{a} \oplus \mathbf{b}) = \mathbf{det}(1 - \mathbf{a} \cdot \mathbf{b})$$

C'EST AU PIED DU MUR QUE L'ON VOIT LE MAÇON

Next step : revisit logic by means of the determinant in the matricial factor of type $\mathbf{II}_1$ (or $\mathbf{II}_\infty$), and see what happens. . .