Luminy, 16/21 Février 2006

FINITISM
HYPERFINITISM
AND
ICONOCLASM

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1-FOREWORD

- Up to 2000 : Locus Solum : A pure waste of paper, I believed that foundations were dead.
- I discovered that the only dead were the fundamentalists, the Jurassic Park.
- Quantum coherent spaces (2003) helped me to reposition the dichotomy subject/object.
- Moving to von Neumann algebra induced a divine surprise.
  - For instance many isomorphic (standard !) versions of $\mathbb{N}$.
  - Non internally isomorphic.
- Sophisticated mathematics far (esp. in spirit) from usual set-theoretic combinatorics.
- Most difficult question : How to use them ?

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I-C*-ALGEBRAS

2-DEFINITION AND EXAMPLES

- Complex involutive Banach algebra such that :

$$\|uu^*\| = \|u\|^2 \tag{1}$$

- Space $\mathbb{C}(X)$ of complex continuous functions on compact X.
  - Indeed the generic commutative example.
  - If $\mathcal{C}$ commutative, take for X the space of characters.
  - B.t.w., character = pure (extremal) state.
  - State : linear form ρ such that $\rho(uu^*) \geqslant 0$, $\rho(I) = 1$.
  - States of $\mathbb{C}(X)$ = probability measures on X.
- Space $\mathcal{B}(\mathbb{H})$ of bounded operators on Hilbert space $\mathbb{H}$.
  - Involution defined by $\langle u^*(x) \mid y\rangle := \langle x \mid u(y)\rangle$.
  - Subalgebras of $\mathcal{B}(\mathbb{H})$ are generic C*-algebras.
  - Non equivalent faithful representations on $\mathbb{H}$.

3-SIMPLICITY

- Morphisms of C*-algebras defined algebraically.
- Indeed bounded, $\|\varphi(u)\| \leqslant \|u\|$:
  - Use $\|uu^*\| = \|u\|^2$ to reduce to positive hermitians uu^*.
  - Use $\|uu^*\| = r(\mathrm{Sp}(uu^*))$ to define the norm algebraically :

$$\|uu^*\| = \sup\{\lambda\,;\; uu^* - \lambda I \quad \text{not invertible}\} \tag{2}$$

- Injective morphisms are isometric, $\|\varphi(u)\| = \|u\|$:
  - Norm shrinks $\Rightarrow$ spectrum shrinks.
  - Norm shrinks $\Rightarrow$ φ not injective.
- A simple algebra (= no closed two-sided ideal) admits only one semi-norm enjoying (1), a « C*semi-norm ».
- Typical example : matrix algebras $\mathcal{M}_n(\mathbb{C})$.
- $\mathcal{B}(\mathbb{H})$ not simple (infinite dimension) : compact operators.

4-The CAR algebra

- Canonical anticommutation relations, between creators $\kappa(a)$ and their adjoints, the annihilators $\zeta(b)$:

$$\kappa(a)\zeta(b) + \kappa(b)\zeta(a) = \delta_{ab} \cdot I \tag{3}$$

$$\kappa(a)\kappa(b) + \kappa(b)\kappa(a) = 0 \tag{4}$$

- a, b range over a set A (or a Hilbert space $\delta_{ab} \rightsquigarrow \langle a \mid b \rangle$).
  - If A is finite, $\mathrm{Car}(A)$ algebraically isomorphic to matrices $n \times n$, with $n := 2^{\sharp(A)}$.
  - By simplicity, unique C*norm on $\mathrm{Car}(A)$ for A finite.
  - The same holds in general : use inductive limits.
- Related topics :
  - The Clifford algebra : use $\kappa(a) + \zeta(a)$.
  - The (exterior) Fock space : represent $\kappa(a)(x) := a \wedge x$.

II-vN ALGEBRAS

5-THE DEFINITION

- Positive hermitians (the uu^*) define an order relation.
- Require completeness w.r.t. bounded (directed) suprema.
- The solution works only for represented C*algebras :
  - No way to decide equality between suprema.
  - Commutative case : no way to tell null sets.
  - As C*-algebras, dual Banach spaces : e.g. $\ell^\infty = (\ell^1)^\sharp$.
    - Intrinsic approach (W*-algebras) not quite successful.
- Subalgebra of $\mathcal{B}(\mathbb{H})$ closed under :

  Strong limits : $u_i \longrightarrow 0$ iff $\|u_i(x)\| \longrightarrow 0$ ($x \in \mathbb{H}$).

  Weak limits : $u_i \longrightarrow 0$ iff $\langle u_i(x) \mid x\rangle \longrightarrow 0$ ($x \in \mathbb{H}$).
- Equivalently : subalgebra equal to its bicommutant.
- Also : the commutant of a self-adjoint subset of $\mathcal{B}(\mathbb{H})$.

6-COMMUTATIVE vN ALGEBRAS

- As a C*-algebra, $\mathcal{A}$ is of the form $\mathbb{C}(X)$.
- X extremely disconnected :
  - The closure of an open set is still open.
- Clopen sets form a σ-algebra :

$$\bigsqcup \mathcal{O}_i := \overline{\bigcup \mathcal{O}_i} \tag{5}$$

- Commutative vN : space $L^\infty(X, \mu)$.
  - Measure μ is up to absolute continuity.
- $\mathbb{C}([0,1])$ extends into a vN modulo a diffuse measure on $[0,1]$.
- In general : C*-algebra + faithful state ρ (i.e., $\rho(uu^*) = 0$ implies $u = 0$.) yields a vN completion.
- The CAR-algebra admits completions of all types $\mathbf{I}, \mathbf{II}, \mathbf{III}$.

7-The GNS construction

- From a C*-algebra $\mathcal{C}$ and a state ρ construct a representation.
- Define $\langle u \mid v\rangle := \rho(v^* u)$; induces a pre-Hilbert space.
- $\mathcal{C}$ acts by left multiplication on the separation/completion of the latter.
- In case ρ is faithful, this representation is isometric.
- The double commutant of the representation is thus a vN completion of $\mathcal{C}$.
- Applies typically to simple algebras.

8-The CAR algebra

- Indeed inductive limit of matrices $2^n \times 2^n$.
- Each of them equipped with normalised trace :
  $\mathrm{tr}(u) := 2^{-n}\mathrm{Tr}(u)$.
- The trace on the inductive limit is a tracial state :

$$\rho(uv) = \rho(vu) \tag{6}$$

- The vN algebra thus obtained is :
  Factor : Trivial center.
  Finite : It has a trace.
  Hyperfinite : Finite matrices are weakly dense.
- Up to isomorphism, only one such vN algebra, the Murray-von Neumann factor $\mathcal{R}$.

III-THE FINITE/HYPERFINITE FACTOR

9-FACTORS

- Connected vN algebras.
- $Z(\mathcal{A}) = (\mathcal{A} \cup \mathcal{A}')'$ is a vN algebra.
- $\mathcal{A} = \int \mathcal{A}(x) d\mu(x)$.
- Each $\mathcal{A}(x)$ is a factor, i.e., a vN algebra with trivial center.
- Classification of vN algebras thus reduces to classification of factors.

10-Comparison of projections

- **Equivalence of projections :**

$$\pi \simeq \pi' \quad \Leftrightarrow \quad \exists u\ (u^*u = \pi \text{ and } uu^* = \pi') \tag{7}$$

- **Ordering of projections (inclusion + equivalence) :**

$$\pi \lesssim \pi' \quad \Leftrightarrow \quad \exists \pi''\ (\pi = \pi\pi'' \text{ and } \pi'' \simeq \pi') \tag{8}$$

- $\mathcal{A}$ **is finite when** $I \lnsim I$ **is wrong.**

$$uu^* = I \Rightarrow u^*u = I \tag{9}$$

- **For factors,** $\lesssim$ **is total :**

  Type I : **Order type** $\{0, \ldots, n\}$ ($\mathbf{I_n}$) **or** $\{0, \ldots, n, \ldots, \infty\}$ ($\mathbf{I_\infty}$).

  Type II : **Order type** $[0, 1]$ ($\mathbf{II_1}$) **or** $[0, +\infty]$ ($\mathbf{II_\infty}$).

  Type III : **Order type** $\{0, +\infty\}$.

11-TRACES

- Finiteness is the same as the existence of a normal (weakly continuous on the unit ball) trace.
- Can be seen as a dimension.
  - E, F have same dimension when $\exists u \quad \mathrm{dom}(u) = E, \mathrm{Im}(u) = F$.
  - E has dimension $1/2$ when $\dim(E) = \dim(E^{\perp})$.
- The completion of the CAR-algebra is finite and infinite-dimensional :
  - Factor of type II_1.
- On a finite factor, the trace is unique.

12-Discrete groups

- G denumerable induces a convolution algebra, obtained by linearisation.
- The convolution :

$$(x_g) * (y_g) := (\sum_{g=g'\cdot g''} x_{g'} \cdot y_{g''}) \tag{10}$$

  is a bilinear map $\ell^2(G) \times \ell^2(G) \leadsto \ell^\infty(G)$.
- Define $\mathcal{A}(G) := \{(x_g); (x_g)* : \ell^2(G) \leadsto \ell^2(G)\}$.
- $\mathcal{A}(G)$ is the commutant of the right convolutions $*(y_g)$.
- If G has infinite conjugacy classes (i.c.c.), then $\mathcal{A}(G)$ is a factor.
- B.t.w., $\mathrm{tr}((x_g)) = x_1$.

13-HYPERFINITISM

- If $G \subset G'$, then $\mathcal{A}(G) \hookrightarrow \mathcal{A}(G')$.
- If G is locally finite, the union $\bigcup_n \mathcal{A}(G_n)$ is weakly dense.
  - Every finite subset of G generates a finite subgroup.
  - Any operator can be weakly approximated by matrices.
- Hyperfinite algebra : an increasing union $\bigcup_n \mathcal{A}_n$ of finite dimensional algebras is weakly dense in $\mathcal{A}$.
- There are hyperfinite algebras of any type (close the CAR algebra w.r.t. appropriate state).
- But only one hyperfinite factor of type II_1. Murray-von Neumann factor $\mathcal{R}$.

IV-GoI

14-GOI IN A vN ALGEBRA

- Old style : interprets proofs by operators.
  - Are galaxies made of stars or is it the other way around ?
    - Foundations always proceed from small to big.
    - This eventually leads to the FOM discussion list.
  - Old GoI (papers 1,2,3) indeed use type $\mathbf{I}$. « The stable form of commutativity » (dixit Connes).
  - Type $\mathbf{I}$: minimal projections $\sim$ points (sets, graphs).
- New style : takes place in the Murray-vN factor $\mathcal{R}$:
  - Finiteness forbids the primitives p, q, d.
    - In a finite algebra, $pp^* = I \Rightarrow p^*p = I$.
  - Hyperfiniteness forbids $t(u \otimes (v \otimes w))t^* = (u \otimes v) \otimes w$.
    - Discrete group generated by t not locally finite.

15-Finite GoI

- A **base** is the pair (ξ, ξ') of two orthogonal projections of the same dimension $\neq 0$ (default $1/2$).
- **Design** of base (ξ, ξ') : $h \in \mathcal{R} \otimes \mathcal{R}$ such that :
  - h hermitian of support $\subset \xi \otimes I$.
  - **Second** tensor component $\mathcal{R}$ is the **dialect**.
- Duality on the same base : given h, k
  - Tensorise h, k with I, swap the last two $\mathcal{R}$, so as to get k'' :
    - $\cdot \otimes \cdot \rightsquigarrow \cdot \otimes \cdot \otimes I$
    - $\cdot \otimes \cdot \rightsquigarrow \cdot \otimes I \otimes \cdot$
  - For $\mathrm{r}(u) < 1$, define $\det(I - u) := e^{\mathrm{tr}(\log(I-u))}$
  - h, k are **polar**, notation $h \underset{\sim}{\downarrow} k$ iff :
$$\mathrm{r}(h'k'') < 1 \qquad \det(I - h'k'') \neq 1 \tag{11}$$
  - **Behaviour** : set B of designs of given base s.t. $\mathrm{B} = \sim\sim\mathrm{B}$.

16-SEQUENTS

- Heavy use of the auxiliary base ξ'.
- Ternary example (ξ, ξ'), (η, η'), (υ, υ') :
  - 3×3 matrix with entries in $\mathcal{R} \otimes \mathcal{R} \otimes \mathcal{R} \otimes \mathcal{R}$.
  - Supports $\xi \otimes \eta' \otimes \upsilon' \otimes I, \eta \otimes \upsilon' \otimes \xi' \otimes I, \upsilon \otimes \xi' \otimes \eta' \otimes I$.
  - All supports have same dimension : no need for p, q.
- Cut on (ξ, ξ') : replace
  - $\cdot \otimes \cdot$ with $\cdot \otimes \eta' \otimes \upsilon' \otimes \cdot \otimes I$
  - $\cdot \otimes \cdot \otimes \cdot \otimes \cdot$ with $\cdot \otimes \cdot \otimes \cdot \otimes I \otimes \cdot$
  - Apply GoI (paper 4).
  - Invariant (determinant) not quite preserved : $\lambda \rightsquigarrow \lambda^{\dim \xi}$.
  - However, duality preserved : if $h \in \mathrm{B}$ then $h \otimes \pi \in \mathrm{B}$.
  - Introspective phenomenon.

17-MULTIPLICATIVES

- The fax (identity axiom) :

$$\begin{bmatrix} 0 & \xi \otimes \xi' \otimes I \\ \xi \otimes \xi' \otimes I & 0 \end{bmatrix} \qquad (12)$$

  - Maps $\cdot \otimes \cdot$ to $\cdot \otimes \xi' \otimes \cdot \otimes I$
  - Not an etaspansion.
  - If $\dim(\xi)$ rational, finite matrix with entries $= 0, 1$.
- Tensor (cotensor) product replaces (ξ, ξ'), (η, η') with $(\xi \otimes \eta' + \xi' \otimes \eta, \xi \otimes \eta + \xi' \otimes \eta')$.
- Basically use an isometry φ between $\xi \otimes \eta'$ and $\xi' \otimes \eta$.
- φ is part of the data.
- $A \multimap A$ based on $(\xi \otimes \xi' + \xi' \otimes \xi, \xi \otimes \xi + \xi' \otimes \xi')$.

18-THE ADDITIVE MIRACLE

- Additive situation : ξ, ξ', η, η' pairwise orthogonal.
- Replace (ξ, ξ'), (η, η') with $(\xi + \eta, \xi' + \eta')$.
- The with rule (how to share contexts) :
  - Premises are 2×2 matrices :
  - Their supports are $\xi \otimes \upsilon' \otimes I, \upsilon \otimes \xi' \otimes I$ and $\eta \otimes \upsilon' \otimes I, \upsilon \otimes \eta' \otimes I$.
  - Just sum them : disjoint supports.
- Violently anti-η. Quantum coherent spaces.
- Summing up, perfect logic (in the linguistic sense) can be interpreted in the hyperfinite factor.

V-The blind spot

19-EXISTENCE VS. ESSENCE

- Jurassic foundations speak of Platonism.
  - But there are things beyond our experience.
  - Real question is that of morphology : laws etc.
  - 2001 : intelligence preexists to its support. Religious . . .
- The real reference is Thomas Aquinus (Aristotle), not Platon.
  - God is perfect in its perfect perfection.
  - The universe is infinite in its infinite infinity.
- To go against that is to go against set-theory, category-theory, one century of foundations, . . .
- The eternal golden braid : infinity, modalities, integers. Everything is true or false, including meaningless formulas.
- « God created integers, everything else is the deed of man ».

20-LINEAR LOGIC

- Main import was to split connectives into :

  Perfect : $\otimes, ⅋, \oplus, \&, \forall, \exists$.

  Imperfect : $!, ?$, the exponentials.
- The perfect part is not essentialist : no « meta-intelligence ».
  - Satisfactory explanations, e.g., ludics.
- The imperfect part is the finger of Thomism.
  - Put enough exponentials to perennialise.
  - Long ago : double negations (Gödel).
- Schizophrenia between :
  - Perfect world unsufficiently expressive.
  - Imperfect world allowing towers of exponentials.

21-JURASSIC PARK

- The peak of scientism, 1900.
  - Various final solutions : societal, musical, logical. . .
  - None of them very. . . subtle.
- What remains of foundations is set theory.
  - Not taken seriously, i.e., for itself.
  - But very convenient, « hygienic ».
- To be compared with equal temperament : $2^{N/12}$.
  - Very convenient, compare with natural scale : $9/8, 10/9, 16/15, 9/8, 10/9, 9/8, 16/15$.
  - But slightly out of tune.
  - Problematic when pushed to extremities (dodecaphonism).
- Set theory problematic in extreme situations (foundations).

22-ICONOCLASM

- **Destruction of (mental) images.**
- **Another finitist paradigm.**
  - **Gödel's theorem : finitism is not finitistic.**
  - **Complexity : mathematical (logical) functions too fast.**
    - **For no real reason, but logical maintenance.**
- **Foundations internalise everything.**
  - **But eventually ends with transfinite metaturtles.**
- **The meta is the impossibility of internalising everything.**
  - **But too late ; happens at meaningless stages.**
- **Since systematic internalisation is eventually wrong, it must be refused from the start.**
- **Accept foundations with most of operations external.**

23-HYPERFINITISM

- The factor $\mathcal{R}$ is remarkably stable :
  - Matrices with entries in $\mathcal{R}$: $\mathcal{M}_2(\mathcal{R}) \sim \mathcal{R}$.
  - Tensor with himself $\mathcal{R} \otimes \mathcal{R} \sim \mathcal{R}$.
  - Crossed product with a locally finite group of external automorphisms.
- Which means that it has many automorphisms.
- Most of them are external.
  - Some of them can be internalised : crossed products.
  - Typically, the twist σ of $\mathcal{R} \otimes \mathcal{R}$ can be added.
  - Since $\sigma^2 = I$, the result still isomorphic to $\mathcal{R}$.
  - But adding $\mathcal{M}_2(\mathcal{R}) \sim \mathcal{R}$ leads to a type III factor.

VI-An iconoclast logic

24-THE ICONOCLAST PROGRAMME

- Finite from inside, infinite from ouside.
- Accept infinity, but not infinite infinity.
  - Impossibility to create fresh objects forever.
- Reduces to search for light exponentials (BLL, LLL, ELL, . . .).
  - Alternative definition producing complexity effects.
  - Cannot be semantically grounded : the blind spot.
  - Use the geometrical constraints of factor $\mathcal{R}$.
- B.t.w., logic in a factor of type $\mathbf{II_1}$ should correspond to ELL.
  - Infinite product $\prod_{n \in \mathbb{N}} G$ crossed by flush :

$$t \cdot (4n, g) = (2n, g) \cdot t \tag{13}$$

$$t \cdot (4n+2, g) = (4n+1, g) \cdot t \tag{14}$$

$$t \cdot (2n+1, g) = (4n+3, g) \cdot t \tag{15}$$

25-PERENNIAL BEHAVIOURS

- $\mathbf{B}$ is perennial when $\mathbf{B} = \sim\sim(C \otimes I)$.
- Perennial behaviours are duplicable.
  - $\mathbf{B} \vdash \mathbf{B} \otimes \mathbf{B}$ inhabited by a sort of fax :
  - Bases $\xi \otimes (\xi \otimes \xi + \xi' \otimes \xi') \otimes I \otimes I$,
    $(\xi \otimes \xi' + \xi' \otimes \xi) \otimes \xi \otimes I \otimes I$.
  - Works because there is no dialectal component $\otimes\cdot$
- Exponentials perennialise :
  - Replace $\cdot\otimes\cdot$ with $\cdot\otimes\cdot\otimes I \otimes I$.
  - Takes place in $\mathcal{R} \otimes ((\mathcal{R} \ldots \otimes \ldots \mathcal{R}) \rtimes G) \otimes \mathcal{R}$.
  - Denumerable tensor product $\mathcal{R} \ldots \otimes \ldots \mathcal{R}$ crossed by a locally finite group G.
  - G acts on integers by swapping bits in hereditary base 2.

26-EXPONENTIALS

- $X \subset \mathbb{N}$ **infinite and co-infinite ; $!_X \mathrm{B}$ stronger when X smaller.**
- $!_X$ **perennialises with $\otimes I$ on components of indices not in 2^X.**
- **Multipromotion available with output : $!_X \Gamma \vdash !_{X \sqcup Y} \mathrm{B}$.**
  - **Need to internalise the swappings of dialects $\cdot \otimes I / I \otimes \cdot$**
- **Various definitions of integers, all externally isomorphic.**

$$\mathrm{nat}_Y := \bigcap_{X,\mathrm{B}} (!_X(\mathrm{B} \multimap \mathrm{B}) \multimap !_{X \sqcup Y}(\mathrm{B} \multimap \mathrm{B})) \tag{16}$$

  - **Some are internally isomorphic, e.g. nat_{2Y} and nat_{2Y+1}.**
  - **In which case, logical equivalence.**
- **Basic functions :**
  Sum : Type $\mathrm{nat}_Y \otimes \mathrm{nat}_Y \multimap \mathrm{nat}_{Y \sqcup Y'}$.
  Product : Type $\mathrm{nat}_Y \otimes \mathrm{nat}_{Y'} \multimap \mathrm{nat}_{Y \sqcup Y'}$.
  Square : Type $!_X \mathrm{nat}_{2Y} \multimap !_{X \sqcup X'} \mathrm{nat}_{2Y \sqcup 2Y+1}$.

27-À SUIVRE

- Observe that there is no need for syntax/semantics.
- Don't bother with a sequent calculus :
  - Finite combinations in G will do everything.
- Dynamics of G : a tower of exponentials.
  - Height = depth of hereditary bits.
- Which complexity classes can be expressed ?