Normativity in logic

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Abstract

Incompleteness — the absence of alternative natural numbers — can be ascribed to a ready-made normativity, inducing a rigid departure syntax/semantics. Geometry of Interaction, set in the non-commutative universe of von Neumann algebras, makes normative assumptions explicit, thus rending possible their internalisation, a possible way out from the semantic *aporia*. As an illustration, we define an alternative "model": logspace integers.

Foundational questions are cognitive: "What can we know?", "How do we know?", "What are our preconceptions?". Thus, the open problems in algorithmic complexity, which address the efficiency of computation, are foundational, although far from the stereotyped problem of consistency. Foundations can — and must — question everything... including questions, the only limit to this interrogation being efficiency. And the rigid departure between syntax and semantics — which is only appropriate in "usual" situations — is the first dogma that foundations should put into question.

Consider natural numbers: several systems, yet only one model, the *standard* $\mathbb{N}$. This extreme poverty (incompleteness) of the semantic universe is a by-product of *normativity*. Indeed, the question "what is standard?" is booby-trapped: it induces a meaningless dichotomy between standard, normal, integers and non-standard, abnormal ones. There is no way to escape this *aporia* while sticking to the rigid distinction between syntax and semantics, where subject and object — both clearly individuated — relate according to a fixed protocol. This normativity is *ready-made*, i.e., hidden and external: it goes without saying, moreover it proceeds from the sky. By making normative requirements explicit, by internalising them as parts of the "semantic", objective, universe, we produce new "models" for natural numbers. In some sense, normativity appears as a mobile curtain separating the object from the subject; by the way, this would not be the only mobile curtain of logic, think of the departure between sets and proper classes.

This supposes a *constructive* viewpoint, i.e., an emphasis on the construction of natural numbers, with eventually a radical change of framework: the replacement of combinatorics with operator algebras. This is rather natural, since quantum physics — which radically puts into question the departure subject/object — dwells in those spaces. *Geometry of Interaction* (GoI) yields, for each $n \in \mathbb{N}$, infinitely many isomorphic *representations* N_n, none of them more "standard" than the others. In order to avoid *interference*, this intrinsic isomorphy must be bridled, whence normativity. Syntactical devices (formal system, typed calculus, complexity class) should therefore correspond to various ways of taming the isomorphy classes of integers, thus inducing a sort of normativity — no longer absolute, ready-made.

Normativity is usually what goes without saying. Take, for instance, the handling of *variables*: outside their range of operationality, i.e., when bound, variables are up to renaming (isomorphism). This discipline avoids accidental coincidences, i.e., interference[1]. Due to the inherent rigidity of syntax, there is no alternative normativity: this explains why the literature on bound variables is so afflictive. In GoI, the various N_n are the same "up to renaming", i.e., up to isomorphy: what will eventually be recognised as a variable, i.e., excluded from the interaction, depends upon the possible interactions, i.e., upon the context. In this way, GoI proposes a sort of "not-yet-frozen syntax".

We shall implement these ideas in the framework of algorithmic complexity, and produce "logspace integers", corresponding to logspace computations. The "model" thus constructed operates a departure of its own between isomorphic objects, some becoming standard, the others non-standard. Besides these relative non-standard integers do exist truly non-standard ones: those in charge of normativity.

The main technical references for this paper are [Girard, 2007, Girard, 2011]. The ideas sketched here will be developed in a forthcoming paper devoted to *transcendantal syntax*. Thanks to Damiano Mazza for his careful reading and to Paulin Jacobé de Naurois for his expertise on LOGSPACE.

1 One reality, several systems

The completeness theorem says that syntax refers to semantics. However, when dealing with natural numbers, *incompleteness* prevails: anybody acquainted with formal logic may easily name half a dozen formal arithmetics, but only one model for them, $\mathbb{N}$. Incompleteness thus means that, although the difference between remarkable systems can be accounted for by models, such models are by no means remarkable, there are but *ad hoc* doohickeys. This concrete incompleteness — the absence of any *convincing* interpretation

[1] This point was never questioned, not even by the most outrageous AI-oriented "logics".

— is a common drawback of all logico-computational approaches to natural numbers: formal arithmetics, typed calculi, complexity classes.

1.1 Formal systems, typed calculi and complexity classes

1.1.1 Formal systems

There is no standard axiomatisation of natural numbers: besides the widely advertised Peano arithmetic $\mathbf{PA}$, coexist the alternative (and somewhat more flexible) $\mathbf{PA}_2$ (second order Peano arithmetic) or $\mathbf{ZF}$ (Zermelo-Fraenkel set theory); not to speak of various subsystems of the former — e.g., weak arithmetics — introduced for proof-theoretic reasons. None of these systems can claim to be "the" system, since, by Gödel's incompleteness, there is always a true arithmetical formula not provable in it. Incompleteness can indeed be turned into a (rather empty) machinery providing fresh axiomatisations: if $\mathbf{T}$ is a sound system of arithmetic, then $\mathbf{T} + Con(\mathbf{T})$ is still sound, but distinct from $\mathbf{T}$.

Incompleteness offers no *explanation* for this plethora of systems whose meaning remains unclear. According to the book, two classical systems are distinct when separated by a model; but the only models distinguishing $\mathbf{PA}$ from $\mathbf{PA} + Con(\mathbf{PA})$ are non-standard ones satisfying $\neg Con(\mathbf{PA})$. Such crazy models — obtained through a completion of $\mathbf{PA} + \neg Con(\mathbf{PA})$ — are nothing but an illegible rewriting of the second incompleteness theorem: the difference between $\mathbf{PA}$ and $\mathbf{PA} + Con(\mathbf{PA})$ accounts for the model, not the other way around! If these systems can only be separated through non-standard models, it is not because one of them is fishy: what is fishy here is the very notion of model!

1.1.2 Typed calculi

Let us adopt a constructive viewpoint: mathematical objects are no longer given to us, since we *construct* them. We can thus imagine a way out from the *aporia* leading to models and non-standard integers: focusing on functions of natural numbers, i.e., on the various ways of constructing them. Formal systems are replaced with typed calculi, e.g., Martin-Löf's theory of types. These calculi enable one to define computable functions from $\mathbb{N}$ to $\{0, 1\}$ and Cantor's diagonalisation — incompleteness *ante litteram* — ensures that none of them is complete: yet another empty machine providing fresh systems.

The reference universe for the constructive approach is category theory. Two systems can be distinguished by the choice of their *morphisms*. This is undoubtedly a progress w.r.t. models: categories, morphisms etc. are usually effective and meaningful, in sharp contrast to non-standard integers — the meaningless and non-effective scions of model theory. However, putting the burden on morphisms is *reculer pour mieux sauter*: there is no reasonably natural notion of morphism from $\mathbb{N}$ to $\{0,1\}$ that will account for the choice of such and such functional system. For instance, the functions of that type definable in system **F** have but one characterisation... that of coming from system **F**, barely a manageable criterion. This comes from the fact that — up to minor details — a morphism from $\mathbb{N}$ to $\{0,1\}$ always translates as a plain subset of $\mathbb{N}$. There are presumably "not enough" integers, but where to find them? Or perhaps something essential — some missing structure — has not been taken into account.

1.1.3 Complexity classes

The two previous approaches are basically equivalent: typed calculi present the effective side of formal systems. Thus, a provably terminating algorithm of $\mathbf{PA}_2$ can be represented in system **F**; conversely, a function of system **F** is provably terminating in $\mathbf{PA}_2$. If we turn our attention towards extremely weak formal systems, typed calculi are bound to represent complexity classes: this is the case for *light logics* like **LLL** or **ELL**, two variants of linear logic corresponding to polytime and elementary complexities.

The question is, so to speak, a refinement of the previous one: how can we manage to "force" the morphisms from $\mathbb{N}$ to $\{0,1\}$ to be polytime or logspace? Where to find the "missing integers" or the "missing structure"? Are there polytime, logspace, integers?

1.2 Alternative integers

We cannot content ourselves with the usual *credo* saying that systems live their own formalist life, thus reducing a system to a list of theorems — a list mostly out of reach, thanks to undecidability. It is reasonable to require that distinct axiomatisations of natural numbers describe distinct "realities", i.e., distinct "notions" of natural numbers.

First observe that not all formal systems are of interest: typically, *unsound* systems such as $\mathbf{PA} + \neg Con(\mathbf{PA})$ should not be accounted for. The same applies to most sound systems, typically $\mathbf{PA} + Con(\mathbf{PA})$, more a "PhD system" than one in which one would like to formalise arithmetic. Summing up, we are not supposed to explain "all" systems: it will be enough to explain a few *meaningful* ones. The same holds for the constructive, functional, aspect of the problem: not all typed systems are of interest. In the same way, one should not seek a systematic account of complexity classes: some of them may be just "PhD classes".

The general pattern is obviously that of *alternative integers*: by this, I mean integers "besides" the usual ones. I avoided the adjective "non-standard" which would corner us to the rut of non-standard models: non-standard integers are non-effective, moreover each belong in a model of its own, outside of which it makes no sense. Even being aware of this basic misconception, the idea of an alternative integer is booby-trapped. Besides usual, *natural*, numbers, there should be abnormal individuals: this distinction normal/abnormal is the result of a hidden *normativity*. Logic is usually unaware of this normativity, more precisely tries to hide it behind general considerations that one is in right to question at the foundational level.

We shall therefore try to expose the hidden normativity of logic, with the secret hope that there is nothing like a standard, optimal, normativity; it will turn out that the same number — say 3 — has infinitely many isomorphic *representations*; w.r.t. an evaluative context, two isomorphic representations may behave differently. The role of normativity is to restrict the choice of possible representations so as to make the evaluation "objective", independent of the representation. Abnormality thus becomes *relative* to the evaluating context. This is not the end of the story: the internalisation of normative constraints produces "watchdogs" which behave like integers w.r.t. evaluations, but are clearly of a different nature.

1.3 A misfire: ordinals

Among the traditional explanations for the diversity of formal arithmetics, one should mention the use of ordinal numbers, typically Gentzen's assignment of the ordinal ϵ_0 to **PA**. Ordinals are normative devices, whose role is to forbid *draws* (see *infra*) in the game-theoretic interpretation of proofs. The good point is that normativity is made explicit; unfortunately, this normativity remains external. The approach is indeed a disappointment:

1. There is no conceptual background for ordinal assignments. The München School produced, in its day, a correspondence between a long list of not-too-meaningful subsystems of $\mathbf{PA}_2$ and a list of not-too-understandable denumerable ordinals. The limited interest of both list was not compensated for by some enlightening explanation as to the *nature* of their relation.

2. The technique works for systems "not too far" from **PA**. It is completely inadequate for the full $\mathbf{PA}_2$ — no understandable ordinal can be found — and for very weak complexity sensitive systems.

3. The stereotyped relation between ordinals and formal systems is modelled on the following: if f, from $\mathbb{N}$ to $\mathbb{N}$, is a provably terminating recursive function of **PA**, then $f \leq \phi_\alpha$ for some $\alpha < \epsilon_0$, where ϕ_α is a hierarchy of recursive functions. This style of relation fails for complexity-sensitive systems: complexity issues usually deal with functions from $\mathbb{N}$ to $\{0, 1\}$, *a priori* bounded by the constant 1!

2 On normativity

2.1 Ready-made normativity

The ready-made conception of logic (associated with the names of Frege and Tarski) supposes that everything has its predetermined place. In this *essentialist* world, the language (syntax) refers to the reality (semantics). However, the same reality — $\mathbb{N}$ — is handled by distinct syntaxes which can only be distinguished by "non-realities", i.e., non-standard models.

Back to the failed promises of categories, observe that the word "morphism" refers to the form, i.e., the essence. But, what is a morphism from $\mathbb{N}$ to $\{0, 1\}$, besides a plain function, what is it supposed to preserve, comply with? Since there is no answer to that question, there only remain discretionary definitions of the kind: "The morphisms from $\mathbb{N}$ to itself are polytime functions" or "The morphisms… are those functions definable in system **F**".

2.2 *A priori* vs. *a posteriori*

A priori normativity constructs objects according to rules, like a construction kit. Typically, in a typed λ-calculus such as system **F**, the constructors must preserve the types, i.e., obey to the law.

An example of normativity *a posteriori* is given by the pure λ-calculus. λ-terms are defined and interact (through application and normalisation) independently of any logical commitment. Now, we can decide to regroup certain λ-terms into sets called *types* and define logical operations between such types. For the same type of system **F**, we thus get two approaches:

1. The typed (system **F**) pattern: $\Lambda X \lambda x^X x$ *is of* type $\forall X(X \Rightarrow X)$.

2. The untyped pattern: λxx *belongs to* the type $\forall X(X \Rightarrow X)$.

The two approaches are related: every typed λ-term of system **F** yields, if we forget everything pertaining to types (here ΛX and the type superscript ^{X}) an untyped term belonging to the same type. In certain cases, the converse is true: for instance, any closed and normal pure λ-term *in* the type $\mathbf{nat} := \forall X((X \Rightarrow X) \Rightarrow (X \Rightarrow X))$ comes from a typed λ-term *of* the same type in **F**. This establishes, for those types, a *completeness* of the former approach (normativity *a priori*) w.r.t. the latter (normativity *a posteriori*).

Normativity *a posteriori* yields more objects: they may belong to a type without being constructible according to the book — i.e., predetermined analytic patterns. This approach is therefore more promising.

However, everything depends upon the *quality* of the interpretation. In spite of its robustness, λ-calculus is not suited for our purpose: we just observed that **nat**, the type of integers, has no "stowaways". The limitation of this — by other standards, excellent — system is presumably due to the fact that it is syntactic *a priori*. What still remains of the disputable departure syntax/semantics is the setting apart of a combinatorial world — that of language. How can we seriously question natural numbers when $\mathbb{N}$ plays, under the disguise of syntax, such a prominent role?

2.3 Games

The interpretation of logic by games, initiated by Gentzen, is subject to two approaches:

2.3.1 Game semantics

A prenex formula $A := \forall m \exists n \forall p \ldots$ with k alternated quantifiers can be described as a *game*: A is true when player `I` has a winning strategy. In this game of finite duration k, winning strategies are Skolem functions: being non-effective, they cannot be considered as proofs. By allowing "remorse", i.e., replays corresponding to the rule of contraction, Gentzen was able to give an effective version of the same game: winning strategies become sorts of infinite proofs in a game of duration[2] ϵ_0. Peano arithmetic can be seen as a construction kit yielding winning strategies — the ones induced by proofs.

Although technically correct, this approach is very reductive and somehow misses the point. The first hint is the remark that, of the two partners, `I`, which tries to prove A, "plays syntax", whereas `II`, which tries to confute A, "plays semantics". Since this approach encompasses the familiar departure syntax/semantics, the expression "game semantics" — which insists upon an obsolete opposition — is, at least, misleading. Indeed, since there is no essential difference between `I` and `II`, syntax and semantics are no longer separated by a Great Wall: nobody forbids `I` from "playing semantics", `II` from "playing syntax", not to speak of intermediate or joint possibilities — for instance, both playing semantics.

2.3.2 Norm as a game

Indeed, the idea of a game is so rich that normativity itself can be the thing at stake! To sum up, game semantics reduces the debate to the question "Is this true?", an *evaluative* query; whereas the alternative approach developed, e.g., in *ludics* poses the more general question "Is this appropriate?", a *deontic* query which encompasses the evaluative questioning about truth.

Indeed, "Is this true?" supposes at least that we know the question at stake; foundationally speaking, we cannot escape the most basic questioning "What is the question?". I am not playing on words, gilding the (meta) lily, and, by the way, questioning the question has deep technical implications. Typically, in pure λ-calculus, we can see a type as a question whose answer

[2]Ordinal "duration" is topsy-turvied: moves are labelled by *decreasing* ordinals: this ensures termination.

lies in its inhabitants: thus, λxx can be seen as an answer to the questions of the form $A \Rightarrow A$, and also $\forall X(X \Rightarrow X)$. This "anteriority" of the answer over the question prompts the issue of *subtyping*, *intersection types*, which might as well be called subquestioning, intersection of questions…

In terms of games, this questioning about questions poses the problem of the *rule of the game*; indeed, specifying a rule is the same thing as choosing the question. Now, proposing a game "without rules" as a solution seems inadequate, since a game without rules is still a game — with a lax rule, so lax that it makes it of little interest! Everything clarifies if we take into account the possibility of a *draw*.

1. In a game-with-a-rule (game semantics), there is no draw: one of the players wins. Typically, ordinals such as ϵ_0 ensure this absence of draw.

2. In a game-without-a-rule (ludics), there can be draws: typically an infinite play, but also a too long (i.e., infinite) delay before a move. The idea is to forbid draws, thus forcing the players into a mutual discipline. This discipline is a sort of rule of the game, no longer proceeding from the sky: it is a by-product of interaction. The contention between players is therefore not primarily about truth (winning) but about norm (agreeing): what is *appropriate*?

Let us illustrate this point by an example (loosely) inspired from ludics: the treatment of the formula $\exists m \in X\, \forall n \in Y\; m \leq n$ $(X, Y \subset \mathbb{N})$. In game semantics, a strategy for $\mathtt{I}$ is of the form $\sigma = \{m_0\} \times Y$ for some $m_0 \in X$, a strategy for $\mathtt{II}$ is a function τ from X to Y; $\sigma \cap \tau$ is a singleton $\{(m, n)\}$: if $m \leq n$, $\mathtt{I}$ wins, otherwise $\mathtt{II}$ wins. Now, let us introduce *designs* (which are to strategies what pure λ-terms are to typed ones): a design is any subset of $\mathbb{N} \times \mathbb{N}$. If σ, τ are designs for $\mathtt{I},\mathtt{II}$, then $\sigma \cap \tau$ need no longer be a singleton, in which case the result of the "play" is a draw. To the game $A := \exists m \in X\, \forall n \in Y\; m \leq n$, let us associate the following sets of designs:

- $A_{\mathtt{I}}$: all designs of the form $\sigma = \{m_0\} \times Y'$, with $m_0 \in X$, $Y' \supset Y$.

- $A_{\mathtt{II}}$: all designs τ such that $\tau \cap (X \times \mathbb{N})$ is a function from X to Y.

It is easy to verify that $\sigma \in A_{\mathtt{I}}$ iff for all $\tau \in A_{\mathtt{II}}$, $\sigma \cap \tau$ is a singleton, i.e., if $\sharp(\sigma \cap \tau) = 1$; in the same way, $\tau \in A_{\mathtt{II}}$ iff for all $\sigma \in A_{\mathtt{I}}$, $\sharp(\sigma \cap \tau) = 1$. In other terms, $A_{\mathtt{I}}$ and $A_{\mathtt{II}}$ refer to each other, and not to some external *rule of the game*. Indeed, the role of the "rule for $\mathtt{I}$" is played by the designs $\tau \in A_{\mathtt{II}}$, and *vice versa*, via the constraint "no draw". The replacement of strategies with designs has thus the following consequences:

Internalisation: the rule of the game no longer proceeds from the sky.

Subtyping: if $B := \exists m \in X'\, \forall n \in Y'\; m \leq n$, with $X \subset X'$, $Y' \subset Y$, then $A_{\mathtt{I}} \subset B_{\mathtt{I}}$ and $B_{\mathtt{II}} \subset A_{\mathtt{II}}$.

Incarnation: strategies are definable as the *minimal* designs of $A_{\mathtt{I}}$ and $A_{\mathtt{II}}$.

2.4 Negation

Player I tries to prove A, player II tries to refute, i.e., to *negate*, A. This is why the logical explanation of the most basic operation on games — the swapping I/II — is precisely negation. Since swapping is involutive, intuitionistic negation does not fit into this pattern; classical negation neither, because structural rules, especially contraction, are not self-dual[3]. This explanation can only be carried out in the framework of *linear logic*, where linear negation quite represents the swapping I/II. Summing up, we see that negation does not merely *refute*, it *forbids*!

We observed in section 2.2 that the restriction to combinatorial methods (this applies to syntax, but also to various games, including ludics) is booby-trapped: how can we put $\mathbb{N}$ into question while presupposing it? This is the reason for a drastic change of paradigm, *Geometry of Interaction* (GoI). By replacing combinatorics with operator algebras (matrix algebras and their generalisation: von Neumann algebras), we put ourself in a more *constructive* situation w.r.t. integers; this is mainly due to the *non-commutativity* at work in those structures.

3 Integers in GoI

3.1 An example: the number 4

In the absence of contraction[4], proofs can be represented by plain matrices. Thus, the proof of $X \Rightarrow X, X \Rightarrow X, X \Rightarrow X, X \Rightarrow X \vdash X \Rightarrow X$ (corresponding to the function $f, g, h, k \rightsquigarrow k \circ h \circ g \circ f$) is interpreted by:

$$L_4 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{1}$$

L_4 is indexed by the ten occurrences $X_1, \ldots, X_{10}$ of X; $L_{ij} = 1$ iff there is an identity axiom — i.e., a *link* — between X_i and X_j, i.e., when one of $(i, j), (j, i)$ belongs to $\{(1, 9), (2, 3), (4, 5), (6, 7), (8, 10)\}$.

The contraction rule — fingernail of infinity in this finite world — enables one to replace the four left occurrences of $X \Rightarrow X$ with a single one, thus yielding $X \Rightarrow X \vdash X \Rightarrow X$, hence $\vdash \forall X((X \Rightarrow X) \Rightarrow (X \Rightarrow X))$.

[3]Witness Gentzen's entangled "cross-cuts".

[4]I.e., in multiplicative-additive linear logic **MALL**.

This proof is Curry-Howard isomorphic to the integer 4 of system **F**: if f is of type $A \Rightarrow A$, then $4\{A\}(f) = f \circ f \circ f \circ f$. In order to perform the contraction, we use the indices $1, 2, 3, 4$ to distinguish the four contracted occurrences, 0 being used for the occurrence of $X \Rightarrow X$ on the right. L_4 can be transformed into a 20×20 matrix (indexed by $\{1, \ldots, 4\}\times\{0, \ldots, 4\}$): the indices $1, \ldots, 10$ respectively become: $1.1, 2.1, 1.2, 2.2, 1.3, 2.3, 1.4, 2.4, 3.0, 4.0$; the absent $1.0, 2.0, 3.1, 3.2, 3.3, 3.4, 4.1, 4.2, 4.3, 4.4$ induce blank lines and columns (zeros). This inflated L_4 can be written as the 4×4 matrix:

$$M_4 := \begin{bmatrix} 0 & v & u & 0 \\ v^* & 0 & 0 & w \\ u^* & 0 & 0 & 0 \\ 0 & w^* & 0 & 0 \end{bmatrix} \tag{2}$$

whose entries are in turn 5×5 matrices:

$$u := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad v := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad w := \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

Let $\pi_0, \ldots, \pi_4$ be the orthoprojections:

$$\pi_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \pi_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \pi_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \ldots \tag{4}$$

then $uu^* = \pi_1, u^*u = ww^* = \pi_0, w^*w = \pi_4, vv^* = \pi_2 + \pi_3 + \pi_4, v^*v = \pi_1 + \pi_2 + \pi_3$.

The *partial isometries* u, v, w are such that:

$$\begin{aligned} u\pi_0 &= \pi_1 u & \qquad u\pi_i &= 0 \quad (i \neq 0) \\ v\pi_i &= \pi_{i+1} v \quad (i = 1, 2, 3) & \qquad v\pi_i &= 0 \quad (i = 0, 4) \\ w\pi_4 &= \pi_0 w & \qquad w\pi_i &= 0 \quad (i \neq 4) \end{aligned}$$

So to speak, u, v, w organise a "round-trip" $\pi_0 \ldots \pi_4$ in $5 = 4 + 1$ steps.

3.2 Representations

We can, more generally, interpret the number n by a 4×4 matrix M_n whose entries are $n + 1 \times n + 1$ matrices:

$$M_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad M_n = \begin{bmatrix} 0 & v_n & u_n & 0 \\ v_n^* & 0 & 0 & w_n^* \\ u_n^* & 0 & 0 & 0 \\ 0 & w_n & 0 & 0 \end{bmatrix} \tag{5}$$

We would like to encode all integers within the same $\mathcal{M}_4(\mathcal{H})$; for this we need an algebra $\mathcal{H}$ together with embeddings $\mathcal{M}_{n+1}(\mathbb{C}) \overset{\phi_{n+1}}{\rightarrowtail} \mathcal{H}$. An obvious choice for $\mathcal{H}$ is a type $\mathbf{II}_1$ von Neumann algebra, the *hyperfinite factor*, in which usual matrix algebras embed (in a non unique way). If $a_n = \phi_{n+1}(u_n), b_n = \phi_{n+1}(v_n), c_n = \phi_{n+1}(w_n)$, define $N_n \in \mathcal{M}_4(\mathcal{H})$:

$$N_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix} \qquad N_n = \begin{bmatrix} 0 & b_n & a_n & 0 \\ b_n^* & 0 & 0 & c_n^* \\ a_n^* & 0 & 0 & 0 \\ 0 & c_n & 0 & 0 \end{bmatrix} \tag{6}$$

i.e., $N_n := \mathcal{M}_4(\phi_{n+1})(M_n)$; $a_n + b_n + c_n$ is a sort of circular permutation of $n+1$ projections $\pi_{n.0}, \ldots, \pi_{n.n}$ such that $\pi_{n.0} + \ldots + \pi_{n.n} = I$. More precisely:

$$\begin{aligned} a_n &= \pi_{n.1}\, a_n\, \pi_{n.0} & (7)\\ b_n &= \pi_{n.2}\, b_n\, \pi_{n.1} + \pi_{n.3}\, b_n\, \pi_{n.2} + \ldots + \pi_{n.n}\, b_n\, \pi_{n.n-1} & (8)\\ c_n &= a_n^*(b_n^*)^{n-1} \qquad \text{hence} & (9)\\ c_n &= \pi_{n.0}\, c_n\, \pi_{n.n} & (10) \end{aligned}$$

Thus the $\pi_{n,i}$ can in turn be recovered as:

$$\begin{aligned} \pi_{n.0} :&= c_n\, c_n^* & (11)\\ \pi_{n.1} :&= a_n\, a_n^* & (12)\\ \pi_{n.i+1} :&= b_n\, \pi_{n.i}\, b_n^* \qquad (1 \leq i < n) & (13) \end{aligned}$$

If $n \in \mathbb{N}$, a matrix $N_n \in \mathcal{M}_4(\mathcal{H})$ of the type (7) and (in case $n \neq 0$) enjoying (7–9) is called a *representation* of n. This is the only sensible definition: since there is no standard embedding $\mathcal{M}_{n+1}(\mathbb{C}) \overset{\phi_{n+1}}{\rightarrowtail} \mathcal{H}$, the entries a_n, b_n, c_n can only be characterised up to isomorphism.

3.3 Measurement as a determinant

For $n \neq 0$, there is a continuum of representations N_n of n, all of them isomorphic: if N, N' are two representations of the same $n \in \mathbb{N}$, there is a unitary $u \in \mathcal{H}$ such that $N' = \mathcal{M}_4(u)(N')$ (if u_0 is any partial isometry from $\pi_{n.0}$ to $\pi'_{n.0}$, define $u_1 := a'_n\, u_0\, a_n^*, u_{i+1} := b'_n\, u_i b_n^*$ $(1 \leq i < n)$ and let $u := u_0 + \ldots + u_n$).

Thus, among all representations of n, none is "more standard" than the others. From an uncouth logicist standpoint, we gave but another definition of natural numbers — the length of a round-trip replacing the cardinality of a set —, nothing really exciting! Now remember that operator algebras were designed to cope with quantum physics, in particular with the process of *observation* and the possible *interference* with the object observed. In other terms, when dealing with an integer, the *measurement* may depend upon the representation in a rather intricate way: this is non-commutativity. One should not consider this possibility as irrelevant to a our discussion:

- The logic tradition, from Frege to category theory, consider objects up to isomorphism; this supposes that the *form* — what isomorphisms preserve — is given in advance. This *essentialism*, which can be advocated in other contexts, is foundationally suspect.

- Complexity theory is about the *difficulty* of computation; it is a theory without operating concepts, reduced to a phenomenology of Turing machines. It is thus legitimate to seek an *explanation* of complexity classes through the neighbouring idea of *observation*.

In Geometry of Interaction, the operator $A \in \mathcal{K}$ is "observed" by an operator $B \in \mathcal{K}$ — the process being, like in games, symmetrical. The output of this observation, the *measurement*, is the real number:

$$\ll A \,|\, B \gg := \det(I - AB) \qquad (= \det(I - BA)) \tag{14}$$

Here we must say something about the scalar $\det(\cdot)$ in a type $\mathbf{II}_1$ von Neumann algebra (like $\mathcal{H}$ or the isomorphic $\mathcal{M}_4(\mathcal{H})$). The idea is that the determinant should be invariant under the embeddings $\mathcal{M}_n(\mathbb{C}) \overset{\phi_n}{\rightarrowtail} \mathcal{H}$: $\det(\phi_n(M)) = \det(M)$ for $M \in \mathcal{M}_n(\mathbb{C})$. To make the long story short, this rests upon the invariance under the $\phi_{n.k} : \mathcal{M}_n(\mathbb{C}) \mapsto \mathcal{M}_{nk}(\mathbb{C})$ which replace each entry with a diagonal $k \times k$-matrix: $\det(\phi_{n.k}(M)) = \det(M)$; this does not hold for the usual determinant $\mathrm{Det}(\cdot)$ of $\mathcal{M}_n(\mathbb{C})$ which must be "normalised" as $\det(u) := |\mathrm{Det}(u)|^{\frac{1}{n}}$: the exponent fixes the problems of dimension, and the absolute value accounts for the impossibility of defining $z^{\frac{1}{n}}$ for all $z \in \mathbb{C}$. Remembering that a vN algebra of type $\mathbf{II}_1$ admits a trace; in $\mathcal{H}$, the trace extends the normalised trace $\mathrm{tr}(u) := \frac{1}{n}\mathrm{Tr}(u)$ of $\mathcal{M}_n(\mathbb{C})$.

$$-\log\det(1-u) = \mathrm{tr}(u) + \frac{\mathrm{tr}(u^2)}{2} + \frac{\mathrm{tr}(u^3)}{3} + \ldots \tag{15}$$

expresses the determinant when u is hermitian (or a product AB of hermitians) and the spectral radius of u is < 1, e.g., when $\|u\| < 1$.

3.4 Interference

We shall thus "observe" our represented integers by means of matrices $\Phi, \Psi, \ldots \in \mathcal{M}_4(\mathcal{H})$. Then the following question arises : when observing N_n are we observing n or a specific representation ? Indeed, although two represention N_n, N'_n of the same n are isomorphic, the measurements $\ll N_n \mid \Phi \gg$ and $\ll N'_n \mid \Phi \gg$ need not be the same; indeed $\ll \Phi \mid N_n \gg$ may be rather unpredictable, and certain measurements should be discarded as "meaningless". This is similar to the usual requirement about bound variables: when combining formulas, make sure that their bound variables are distinct. The act of discarding a measurement is strongly normative and should not be pushed under the carpet. The point about GoI is that we get the objects, so to speak, with their bound variables, although there is no clear renaming technique like in logic. *Objectivity of measurement*:

If N_n, N'_n are representations of $n \in \mathbb{N}$, if Φ is an observation, then
$$\ll \Phi \,|\, N_n \gg \,=\, \ll \Phi \,|\, N'_n \gg .$$

is a normative requirement, calling for restrictions upon the shape of representations and their observations.

3.5 An example : commutation

Consider the "observation":

$$\Phi := \begin{bmatrix} 0 & v & 0 & 0 \\ v^* & 0 & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & w \end{bmatrix} \tag{16}$$

PROPOSITION 1

If u, v, w commute with the entries a_n, b_n, c_n of N_n, then

$$\ll \Phi \,|\, N_n \gg \,=\, (\det(I - v^n w v^{*n} u))^{\frac{1}{2n+2}} \tag{17}$$

Proof : we restrict to the particular case where (15) converges. Observe that $\mathrm{tr}(N_n\Phi)^p = 0$ when p is not a multiple of $2n+2$, hence:

$$-\log\det(1 - N_n\Phi) = \frac{\mathrm{tr}((N_n\Phi)^{2n+2})}{2n+2} + \frac{\mathrm{tr}((N_n\Phi)^{4n+4})}{4n+4} + \frac{\mathrm{tr}((N_n\Phi)^{6n+6})}{6n+6} + \ldots \qquad \text{and}$$

$$(N_n\Phi)^{(2n+2)k} = \begin{bmatrix} A_{n.k} & 0 & 0 & 0 \\ 0 & B_{n.k} & 0 & 0 \\ 0 & 0 & C_{n.k} & 0 \\ 0 & 0 & 0 & D_{n.k} \end{bmatrix} \tag{18}$$

with:

$$\begin{aligned} A_{n.k} &= \pi_{n.1}\, v^{*n-1} u (v^n w v^{*n} u)^{k-1} v^n w v^* + \ldots + \pi_{n.n-1}\, v^* u (v^n w v^{*n} u)^{k-1} v^n w v^{*n-1} \\ B_{n.k} &= \pi_{n.n}\, v^{n-1} w v^{*n} u (v^n w v^{*n} u)^{k-1} v + \ldots + \pi_{n.2}\, v w v^{*n} u (v^n w v^{*n} u)^{k-1} v^{n-1} \\ C_{n.k} &= \pi_{n.1}\, (v^n w v^{*n} u)^k \\ D_{n.k} &= \pi_{n.0}\, (v^{*n} u v^n w)^k \end{aligned}$$

$\mathrm{tr}(C_{n.k}) = \mathrm{tr}(D_{n.k}) = \frac{\mathrm{tr}((v^n w v^{*n} u)^k)}{n+1}$, $\mathrm{tr}(A_{n.k}) = \mathrm{tr}(B_{n.k}) = n \cdot \mathrm{tr}(C_{n.k})$ yield $\mathrm{tr}((N_n\Phi)^{(2n+2)k}) = \frac{\mathrm{tr}(A_{n.k}+B_{n.k}+C_{n.k}+D_{n.k})}{4} = \frac{\mathrm{tr}((v^n w v^{*n} u)^k)}{2}$, hence (17). $\square$

Proposition 1 establishes objectivity of measurement under the hypothesis that the coefficients u, v, w of the observation do commute with the entries a_n, b_n, c_n of the representation: metaphorically, the "bound variables" of N_n and Φ do not interfere. However, this remedy cures the disease by killing the patient: if u, v, w must commute with all a_n, b_n, c_n, they must be scalars, and one hardly sees how to express any non-trivial algorithm in this way! We can loosen the situation by assuming a certain amount of commutation *a priori*: this is the *dialectal* (or idiomatic) maintenance of GoI, at work in [Girard, 2011]. We thus modify the definition of observations:

Definition 1 (Observations)
A dialect *$\mathcal{D}$ is a matrix space $\mathcal{M}_k(\mathbb{C})$; an* observation *(of dialect $\mathcal{D}$) is an element of $\mathcal{M}_4(\mathcal{H}) \otimes \mathcal{D}$, i.e., a 4×4 matrix with entries in $\mathcal{H} \otimes \mathcal{D} = \mathcal{M}_k(\mathcal{H})$. The output of the observation of N_n by Φ is the* measurement*:*

$$\ll \Phi \,|\, N_n \gg := \det(I - \Phi(N_n \otimes I_k)) \qquad (19)$$

Even in this relaxed setting, commutation remains too drastic, i.e., leaves very few interesting observations.

3.6 Normativity by subalgebras

The translation of logical rules done in [Girard, 2011] — especially of the exponentials et work in the encoding of Dedekind integers — induces a restriction on the observations together with a co-restriction on the "objects", i.e., on the representations. This joint restriction ensures *objectivity of measurement* without assuming commutation.

Without entering into something as normative as the logical maintenance of exponentials, we can directly seek joint restrictions of the pairs observation/representation. The simplest idea is that of a restriction to specific subalgebras: we shall seek pairs $(\mathcal{I}, \mathcal{O})$ of subalgebras of $\mathcal{H}$ such that the restriction $N_n \in \mathcal{M}_4(\mathcal{I})$, $\Phi \in \mathcal{M}_4(\mathcal{O}) \otimes \mathcal{D}$ ($\mathcal{D}$ arbitrary), ensures objectivity:

$$\forall N_n, N'_n \in \mathcal{M}_4(\mathcal{I})\ \forall \Phi \in \mathcal{M}_4(\mathcal{O}) \otimes \mathcal{D} \qquad \ll \Phi \,|\, N_n \gg = \ll \Phi \,|\, N'_n \gg \qquad (20)$$

Let us call such a pair $(\mathcal{I}, \mathcal{O})$ a *normative pair.* Among normative pairs, $(\mathcal{H}, \mathbb{C} \cdot I)$: if the entries of Φ belong to the dialect space $\mathcal{D}$, then we are in — rather, isomorphic to — the situation of proposition 1: the objectivity of measurement is ensured for *all* representations of natural numbers. More generally, if $\mathcal{I}, \mathcal{O}$ are such that, whenever $u \in \mathcal{I}, v \in \mathcal{O}$, then $uv = vu$, then (20) holds.

4 Logspace integers

If we seek a non-trivial (i.e., non-commuting) normative pair, then the most natural example is given by the the crossed product to be defined below; the big surprise is that this restriction corresponds to logspace computation!

4.1 A normative pair

Consider[5] the infinite tensor power $\mathcal{K} := \bigotimes_{n>0} \mathcal{H}'$ of ω copies of some $\mathcal{H}'$ isomorphic to $\mathcal{H}$. For future reference, we note $\mathcal{H}_i$ the subalgebra of $\mathcal{K}$ consisting of the $I \otimes \ldots \otimes u \otimes I \otimes \ldots$ (the $\bigotimes u_n$ s.t. $u_n = I$ for $n \neq i$). The group $\mathfrak{S}$ of (finite) permutations of $\mathbb{N}$ operates on $\mathcal{K}$ by $\sigma(\bigotimes u_n) := \bigotimes u_{\sigma(n)}$. The crossed product $\mathcal{K} \rtimes \mathfrak{S}$ *internalises* $\mathfrak{S}$, the action of σ becoming an inner automorphism:

$$\sigma \cdot \bigotimes u_n = (\bigotimes u_{\sigma(n)}) \cdot \sigma \tag{21}$$

Let $\mathcal{H} := \mathcal{K} \rtimes \mathfrak{S}$. Among the remarkable subalgebras of $\mathcal{H}$: the $\mathcal{H}_i$, $\mathcal{K}$ and the algebra $\mathcal{S}$ generated by $\mathfrak{S}$. These subalgebras ar all isomorphic to $\mathcal{H}$, the *unique* hyperfinite factor of type $\mathbf{II}_1$.

PROPOSITION 2
Any automorphism θ_1 of $\mathcal{H}_1$ can be uniquely extended into an automorphism θ of $\mathcal{H}$ which is the identity on $\mathcal{S}$.

Proof : if $\theta_1(u \otimes I \otimes \ldots) = \vartheta(u) \otimes I \otimes \ldots$, define $\theta(\sigma \cdot \bigotimes_n u_n) := \sigma \cdot \bigotimes_n \vartheta(u_n)$. □

COROLLARY 2.1
$(\mathcal{H}_1, \mathcal{S})$ is a normative pair.

Proof : assume that $N_n, N'_n \in \mathcal{M}_4(\mathcal{H}_1)$; then $N'_n = \mathcal{M}_4(\theta_1)(N_n) = \mathcal{M}_4(\theta)(N_n)$ for some automorphism θ_1 of $\mathcal{H}_1$. If $\Phi \in \mathcal{M}_4(\mathcal{S}) \otimes \mathcal{M}_k(\mathbb{C})$, then:

$$\begin{aligned} \det(I - \Phi \cdot (\mathcal{M}_4(\theta_1)(N_n) \otimes I_k)) &= \det(I - \mathcal{M}_{4k}(\theta)(\Phi \cdot (N_n \otimes I_k))) \\ &= \det(I - \Phi \cdot (N_n \otimes I_k)) \end{aligned}$$

since the determinant is invariant under the isomrphism $\mathcal{M}_{4k}(\theta)$.

□

4.2 Logspace operators

We now turn our attention towards computation.

DEFINITION 2 (LOGSPACE OPERATORS)
A logspace operator *is any $\Phi \in \mathcal{M}_4(\mathcal{S}) \otimes \mathcal{D}$, where $\mathcal{D} = \mathcal{M}_k(\mathbb{C})$ is a matrix algebra, such that the entries of the $4k \times 4k$ matrix Φ are finite linear combinations $\sum \lambda_i s_i$ of elements $s_i \in \mathfrak{S}$ with positive coefficients $\lambda_i > 0$.*

Φ being a logspace operator, consider the set:

$$[\Phi] := \{n \in \mathbb{N}\ ;\ \forall N_n \in \mathcal{M}_4(\mathcal{H}_1) \quad \ll \Phi \,|\, N_n \gg = 1\} \tag{22}$$

[5]This section requires some familiarity with vN algebras, especially with crossed products, see, e.g., [Kadison and Ringrose, 1986].

Theorem 1 (Logspace integers)
The set $[\Phi]$*, as a set of tallies (see remark 1,* infra*) is in* NL *(non-deterministic logspace).*

Proof : let us compute $\Phi(N_n \otimes I_k)$ and its iterates. The elements of $\mathfrak{S}$ occurring in the entries of Φ generate a finite subgroup $\mathfrak{S}_\Phi$; let $N \in \mathbb{N}$ be such that $\sigma(i) = i$ for all $\sigma \in \mathfrak{S}_\Phi$ and $i \geq N$. We can, w.l.o.g., place ourself in $\bigotimes_{1,\ldots,N} \mathcal{H} \rtimes \mathfrak{S}[1,\ldots,N]$. Since $(\mathcal{H}_1, \mathcal{S})$ is a normative pair, we can replace the entries of N_n with $n+1 \times n+1$ matrices whose entries are $0, 1$; in particular, $\mathcal{H}$ is replaced with $\mathcal{M}_{n+1}(\mathbb{C})$. Our computation eventually takes place in $\mathcal{M}_4(\mathbb{C}) \otimes \mathcal{M}_k(\mathbb{C}) \otimes (\mathcal{M}_{n+1}(\mathbb{C}) \otimes \ldots \otimes \mathcal{M}_{n+1}(\mathbb{C})) \rtimes \mathfrak{S}[1,\ldots,N]$. Φ and $N_n \otimes I_k$ have thus been reduced to finite-dimensional operators, on a space of dimension $4k(n+1)^N \cdot N!$ whose canonical base can be written

$$\{(ai(j_1,\ldots,j_N);\sigma) \; ; \; 1 \leq a \leq 4, \; 1 \leq i \leq k, \; j_1 \ldots, j_N \leq n, \; \sigma \in \mathfrak{S}[1,\ldots,N]\}$$

- $\Phi((ai(j_1,\ldots,j_N);\sigma))$ is a sum: if τ "occurs" in the entry $\Phi_{a'i',ai}$, then $(a'i'(j_{\tau(1)},\ldots,j_{\tau(N)});\tau\sigma)$ occurs in $\Phi(ai(j_1,\ldots,j_N);\sigma)$ with the same multiplicity.
- $(N_n \otimes I_k)((ai(j_1,\ldots,j_N);\sigma)) = 0$ if the entries $N_{a',j';a,j_1}$ are all null. Otherwise, let $N_{a',j';a,j_1}$ be the only nonzero entry of N_n of this form; then $(N_n \otimes I_k)((ai(j_1,\ldots,j_N);\sigma)) = (a'i(j'_1,j_2,\ldots,j_N);\sigma)$.

$\det(I - \Phi(N_n \otimes I_k)) = 1$ iff $\Phi(N_n \otimes I_k)$ is nilpotent. This is the same as saying that the iterates $(\Phi(N_n \otimes I_k))^p((ai(j_1,\ldots,j_N);\iota))$ are all null for sufficiently great p (e.g., $p = 4k(n+1)^N$), ι denoting the identity permutation. Now, Φ being fixed, it is plain that the process of iteration yielding the $(\Phi(N_n \otimes I_k))^p((ai(j_1,\ldots,j_N);\iota))$ is logspace: indeed it takes place in a universe of size $s(n) = 4k(n+1)^N \cdot N!$ whose elements can be written with approximately $\log(n).N + \log(4kN!)$ digits. Indeed, non-deterministic logspace: when computing $\Phi((ai(j_1,\ldots,j_N);\sigma))$, several choices $\tau \in \Phi_{a'i',ai}$ are available. Nilpotency is therefore coNL, which is the same as NL. □

Remark 1
The theorem should be stated for binaries, see section 4.4 for the exact relation with NL. Theorem 1 is only a prototype which relies on the dumb tally representation of natural numbers; since, as a binary, the tally n encodes the number 2^n, theorem 1 indeed says that $2^{[\Phi]} := \{2^n \; ; \; n \in [\Phi]\}$ is in NL.

The choice of the coefficients λ_i in the entries $\sum \lambda_i s_i$ is irrelevant, as long as they stay positive. In particular, they can be chosen small enough to ensure $\|\Phi\| \leq 1$, an essential requirement of GoI. One can also require them to be rational: this may simplify technical issues.

4.3 Normative vs. non-standard

Normativity occurs because we are specifically interested in measurements, i.e., in observations; and, as logicians, in general properties of observations: "What can we observe?", "Is this style of observation more efficient that that one?"...

At the level of the objects "observed", normativity induces a departure standard/non-standard. We must distinguish between two forms of non-standardness, *relative* and *absolute*.

4.3.1 Relative non-standardness

W.r.t. normativity by the subalgebras $\mathcal{H}_1, \mathcal{S}$, the integers — rather their representations — $N_n \in \mathcal{M}_4(\mathcal{H}_1)$ are standard. This means that $N_n, N'_n \in \mathcal{M}_4(\mathcal{H}_1)$ cannot be distinguished by observations: they are, so to speak, the same "up to bound variables". The other representations should be styled "non-standard"; they are, however, plainly isomorphic to standard integers and their "non-standardness" is only relative to our observational normativity.

Non-standard integers yield additional objects to which the observation may be applied. Due to interference, the measurement thus obtained may be completely meaningless. But, this need not be always the case: for instance, $(\mathcal{H}_2, \mathcal{S})$ is also a normative pair, hence the non-standard integers $N_n \in \mathcal{M}_4(\mathcal{H}_2)$ yield consistent "alternative" measurements: the example is a bit too simple, since the same result can be achieved by applying $\tau_{12}\Phi\tau_{12}$ — where τ_{12} is the transposition $1 \leftrightarrows 2$ — to "standard" N_n.

4.3.2 Absolute non-standardness

Normativity by subalgebras is external: both the object (the natural number "observed") and the subject (the "observation" Φ) are coerced into algebras of their own, so as to inhibit unwanted interferences.

An internal normativity is much more satisfactory; we can internalise normativity on both sides — numbers and observations — the case of observations being the most interesting one. So how can we ensure that $\Phi \in \mathcal{M}_4(\mathcal{S}) \otimes \mathcal{D}$? In a second step, how can we ensure that the entries of Φ are finite linear combinations $\sum \lambda_i s_i$ of elements of $\mathfrak{S}$, with[6] $\lambda_i \in \mathbb{Q}^+$?

The answer lies in the introduction of additional objects, the "watchdogs of normativity". These objects are *formal* linear combinations $\mathfrak{a} = \bigoplus \lambda_i \bullet A_i$ of operators. The measurement $\ll A \,|\, B \gg$ of (14) is generalised into:

$$\ll \bigoplus \lambda_i \bullet A_i \,|\, \bigoplus \mu_j \bullet B_j \gg := \prod_{ij} \ll A_i \,|\, B_j \gg^{\lambda_i \mu_j} \tag{23}$$

[6]The replacement $\mathbb{R}^+ \rightsquigarrow \mathbb{Q}^+$ ensures that there are only denumerably many entries.

In [Girard, 2011], internal normativity takes the form $\ll \mathfrak{a} \,|\, \mathfrak{b} \gg \neq 0, 1$. Assuming that $\ll N_0 \,|\, \Phi \gg \neq 0, 1$, most normative queries can be expressed under the form:

$$\ll N_0 \oplus \lambda \bullet A \ominus \lambda \bullet B \,|\, \Phi \gg \neq 0, 1 \tag{24}$$

For instance, if (24) holds for all $\lambda \in \mathbb{R}$, then $\ll A \,|\, \Phi \gg = \ll B \,|\, \Phi \gg$. Taking $A \in \mathcal{M}_4(\mathcal{K})$, $B := \theta(A)$, where θ_1, θ are as in proposition 2, we can thus express the constraint $\Phi \in \mathcal{M}_4(\mathcal{S}) \otimes \mathcal{D}$. The constraint $N_n \in \mathcal{M}_4(\mathcal{H}_1)$ can in turn be recovered from $\ll N_n \,|\, \Phi \gg = \ll N_n \,|\, \Psi \gg$, for any observations Φ and $\Psi := \mathcal{M}_{4k}(\sigma)(\Phi)$, where σ is any element of $\mathcal{S}$ such that $\sigma(1) = 1$.

Since non "integer-like", the proper linear combinations $\mathfrak{a} = \bigoplus \lambda_i \bullet A_i$ in charge of the law, are intrinsically non-standard. The question is to determine whether or not they can be of some use, i.e., if the measurements $\ll \mathfrak{a} \,|\, \Phi \gg$ are meaningful. The question extends, of course, to those $\mathfrak{a}$ in charge of other "laws" that Φ may or may not break.

4.4 Logspace binaries

In view of our concern for complexity issues, the tallies just discussed must be replaced with the set $\mathbb{S}$ of lists of $0, 1$, which is in bijection with $\mathbb{N}^*$: the list s encodes the binary number $1s$. The empty sequence therefore encodes 1; the maps $s \rightsquigarrow s0$ and $s \rightsquigarrow s1$ respectively encode the functions $n \rightsquigarrow 2n$ and $n \rightsquigarrow 2n+1$. These binaries can be typed in system **F** by
bin $:= \forall X((X \Rightarrow X) \Rightarrow ((X \Rightarrow X) \Rightarrow (X \Rightarrow X)))$, that GoI handles by means of 6×6 matrices (instead of the 4×4 matrices used for **nat**).

To a list s of $0, 1$, we associate representations:

$$B_{<>} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & I & 0 \end{bmatrix} \qquad B_s = \begin{bmatrix} 0 & c_s & 0 & e_s & a_s & 0 \\ c_s^* & 0 & d_s^* & 0 & 0 & g_s^* \\ 0 & d_s & 0 & f_s & b_s & 0 \\ e_s^* & 0 & f_s^* & 0 & 0 & h_s^* \\ a_s^* & 0 & b_s^* & 0 & 0 & 0 \\ 0 & g_s & 0 & h_s & 0 & 0 \end{bmatrix} \tag{25}$$

If $s =< s_1, \ldots, s_n >$ ($s_i \in \{0, 1\}$), then the entries $a_s, \ldots, h_s$ are partial isometries. Indeed, consider the sets:

$$\begin{array}{ll} a_s := \{0\},\ b_s = \emptyset \quad \text{if } s_1 = 0 & b_s := \{0\},\ a_s = \emptyset \quad \text{if } s_1 = 1 \\ c_s := \{i \neq 0, n\ ;\ s_i = s_{i+1} = 0\} & d_s := \{i \neq 0, n\ ;\ s_i = 0, s_{i+1} = 1\} \\ e_s := \{i \neq 0, n\ ;\ s_i = 1, s_{i+1} = 0\} & f_s := \{i \neq 0, n\ ;\ s_i = s_{i+1} = 1\} \\ g_s := \{n\},\ h_s = \emptyset \quad \text{if } s_n = 0 & h_s := \{n\},\ g_s = \emptyset \quad \text{if } s_n = 1 \end{array}$$

The entries $a_s, \ldots, h_s$ are characterised by the existence of projections

$\pi_{s.0}, \ldots, \pi_{s.n}$ such that $I = \pi_{s.0} + \ldots + \pi_{s.n}$ and:

$$
\begin{aligned}
a_s &= \sum_{i \in a_s} \pi_{s.i+1}\, a_s\, \pi_{s.i} & b_s &= \sum_{i \in b_s} \pi_{s.i+1}\, b_s\, \pi_{s.i} \\
c_s &= \sum_{i \in c_s} \pi_{s.i+1}\, c_s\, \pi_{s.i} & d_s &= \sum_{i \in d_s} \pi_{s.i+1}\, d_s\, \pi_{s.i} \\
e_s &= \sum_{i \in e_s} \pi_{s.i+1}\, e_s\, \pi_{s.i} & f_s &= \sum_{i \in f_s} \pi_{s.i+1}\, f_s\, \pi_{s.i} \\
g_s &= \sum_{i \in g_s} \pi_{s.i+1}\, g_s\, \pi_{s.i} & h_s &= \sum_{i \in h_s} \pi_{s.i+1}\, h_s\, \pi_{s.i}
\end{aligned}
$$

$$\pi_{s,0} = (g_s + h_s)(c_s + d_s + e_s + f_s)^{n-1}(a_s + b_s)$$

From which we can define the notions of *representation* of s. A pair $(\mathcal{I}, \mathcal{O})$ of subalgebras of $\mathcal{H}$ ensuring *objectivity*:

$$\forall B_s, B'_s \in \mathcal{M}_6(\mathcal{I})\ \forall \Phi \in \mathcal{M}_6(\mathcal{S}) \otimes \mathcal{D} \qquad \ll \Phi \,|\, B_s \gg = \ll \Phi \,|\, B'_s \gg \tag{26}$$

is called a *normative pair*. Again, the typical normative pair is $(\mathcal{H}_1, \mathcal{S})$.

Definition 3 (Logspace operators)
A logspace operator *is any $\Phi \in \mathcal{M}_6(\mathcal{S}) \otimes \mathcal{D}$, where $\mathcal{D} = \mathcal{M}_k(\mathbb{C})$ is a matrix algebra such that the entries $\Phi_{a.p,b.q}$ Φ (as a $6k \times 6k$ matrix) are finite linear combinations $\sum \lambda_i s_i$ of elements $s_i \in \mathfrak{S}$ with $\lambda_i > 0$.*

Φ being a normative operator, consider the set:

$$[\Phi] := \{s \in \mathbb{S}\ ;\ \forall B_s \in \mathcal{M}_6(\mathcal{H}_1) \qquad \ll \Phi \,|\, B_s \gg = 0\} \tag{27}$$

Then we get the following (immediate) analogue of theorem 1.

Theorem 2 (Logspace integers)
The set $[\Phi]$ is in NL *(non-deterministic logspace).*

Conversely, consider a non-deterministic logspace algorithm F, applying to binaries. F makes use of N "fingers" simultaneously visiting a binary $s \in \mathbb{S}$ of length n, with locations $\sharp 1, \ldots, \sharp n$ occupied by the digits 0 or 1, and an additional location, the *origin* $\sharp 0$; depending on the *configuration* $(i_1, \ldots, i_N; a)$, i.e., the data $(0, 1,$ `origin`$)$ simultaneously read by the N fingers and the current state (represented by $a \in A$, A finite), one can prompt certain *transitions*, which combine three actions: a change of state, a rearranging of the fingers and a move of the "thumb" (finger $\sharp 1$) forwards or backwards (next or previous location, the origin standing after $\sharp n$ and before $\sharp 1$); let $f_1, \ldots, f_r$ be the possible transitions, so that $F = \{f_1, \ldots, f_r\}$. We say that s *is accepted by* F when F, acting on s, has no loops. This definition implies that certain configurations may prompt no transition at all; otherwise, due to the finiteness of the configuration space, the algorithm must loop.

The computation will be encoded in the algebra $\mathcal{M}_{6^N}(\mathcal{H}) \otimes \mathcal{M}_A(\mathbb{C})$, which can be written: $\mathcal{M}_6(\mathcal{H}) \otimes \mathcal{D}$, with $\mathcal{D} := \mathcal{M}_{6^{N-1} \times A}(\mathbb{C})$. In order to encode F by an operator, it will be enough to encode — in a faithful way — each transition $f_1, \ldots, f_r \in F$ by adequate operators $\phi_1, \ldots, \phi_r \in \mathcal{M}_6(\mathcal{S}) \otimes \mathcal{D}$ and define $\Phi := \mu_1\phi_1 + \ldots + \mu_r\phi_r$, with $\mu_1, \ldots, \mu_r > 0$.

The execution of F applied to s will therefore be represented by the iterates $((B_s \otimes I_{\mathcal{D}})\Phi)^p$ ($p \in \mathbb{N}$), which are linear combinations of "monomials", i.e., alternated products $B_s f B_s g B_s \ldots B_s h$ of transitions $f, g, \ldots, h$ and B_s (indeed, $B_s \otimes I_{\mathcal{D}}$). Each monomial is a partial isometry whose final projection is of the form $(m_{i_1 i_1} \otimes \ldots \otimes m_{i_N i_N} \otimes m_{aa}) \cdot (\pi_{s.q_1} \otimes \ldots \otimes \pi_{s.q_N})$, where n is the length of s, $i_1, \ldots, i_N \in \{1, \ldots, 6\}$, $q_1, \ldots, q_N \in \{0, \ldots, n\}$ and $\pi_{s.0}, \ldots, \pi_{s.n}$ are the projections associated with B_s; such a projection is the product of $(i_1, \ldots, i_N; a) := m_{i_1 i_1} \otimes \ldots \otimes m_{i_N i_N} \otimes m_{aa}$, representing the current configuration and $\pi_{s.q_1} \otimes \ldots \otimes \pi_{s.q_N}$ representing the simultaneous location $(q_1, \ldots, q_N)$ of the N digits $q_1, \ldots, q_N$; let us abbreviate it as $((i_1, q_1), \ldots, (i_N, q_N); a)$. Obviously:

if $i = 1, 2$	then $q_i \in c_s \cup d_s$
if $i = 3, 4$	then $q_i \in e_s \cup f_s$
if $i = 5, 6$	then $q_i = 0$

The integers $i_1, \ldots, i_N \in \{1, \ldots, 6\}$ encode the data possibly read by the fingers, respectively $0, 0, 1, 1, \texttt{begin}, \texttt{end}$. This encoding is redundant: each of the basic data "digit 0", "digit 1", "origin" gets two possible encodings, respectively $\{1, 2\}, \{3, 4\}, \{5, 6\}$. Indeed, our representations do combine a "forward trip", leading from $\{2, 4, 5\}$ to $\{1, 3, 6\}$ and an adjoint "backward trip", leading from $\{1, 3, 6\}$ to $\{2, 4, 5\}$. This duality of encoding is thus of a dynamic nature.

To the transition f, prompted by the configuration $(i_1, \ldots, i_N; a)$ we can associate $\phi \in \mathcal{M}_6(\mathcal{S}) \otimes \mathcal{D}$:

$$\phi := (m_{k i_1} \otimes m_{i_{\sigma(2)} i_2} \otimes \ldots \otimes m_{i_{\sigma(N)} i_N} \otimes m_{ba}) \cdot \sigma \tag{28}$$

where $\sigma \in \mathfrak{S}(\{1, \ldots, N\}) \subset \mathfrak{S}$ is the reordering of the fingers, which induces an operator of $\mathcal{S}$, still noted σ, $b \in A$ is the next state. k is defined as "$i_{\sigma(1)}$ up to a change of direction"; in other terms, $k \in \{2, 4, 5\}$ if the thumb "moves forward", $k \in \{1, 3, 6\}$ if the thumb "moves backward" and $\{i_{\sigma(1)}, k\}$ is included in one of the sets $\{1, 2\}, \{3, 4\}, \{5, 6\}$.

$B_s \cdot \phi \cdot ((i_1, q_1), \ldots, (i_N, q_N); a)$ is a partial isometry with final projection $\nu = ((i_k, q_{\sigma(1)\pm 1}), (i_{\sigma(2)}, q_{\sigma(2)}) \ldots, (i_{\sigma(N)}, q_{\sigma(N)}); b)$, where $(k, q_{\sigma(1)\pm 1})$ is the next location of the thumb: in case of a forward move $\nu = ((i_k, q_{\sigma(1)+1}), \ldots)$ with $k \in \{1, 3, 6\}$, in case of a backward move $\nu = ((i_k, q_{\sigma(1)-1}), \ldots)$ with $k \in \{2, 4, 5\}$.

We just proved (or rather sketched):

THEOREM 3
If $X \subset \mathbb{S}$ is in NL*, then $X = [\Phi]$ for a logspace operator Φ.*

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