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PREDICATE CALCULUS AS A LOGICAL EPICYCLE

JEAN-YVES GIRARD

1 — EPICYCLES, A.K.A. THE REALIST PREJUDICE

- XX$^{\text{th}}$ century logic begins *after* incompleteness.

  Herbrand: synthetic *a posteriori,* a.k.a. *usine.*
  BHK: synthetic *a priori*, a.k.a. *usage.*
  Gentzen: relation usine/usage through *cut-elimination.*

- XIX$^{\text{th}}$ century, up to ~ 1925: axomatic and semantic.

  Hilbert: *militarism* (axiomatics). *A priori* $\rightsquigarrow$ consistency.
  Russell: *religion* (of reality). Semantics, a.k.a. *prejudice.*

- *Realism:* cognitive simplicism, yields monsters.

  Epicycles: fantasmatic reality backing *geocentric* prejudice.

- Realism expressed by *classical* reduction to *true/false.*

  Loss of propositional expressivity.
  Compensation: fantasmatic first-order individuals.
  Symptom: no logical handling of *equality.*

I — THE FOUR HORSEMEN OF COGNITION

	Analytic	Synthetic
Explicit	Constat	Usine
Implicit	Performance	Usage

2 — ANALYTICITY : CONSTAT VS. PERFORMANCE

- **Cognition *without* presupposition: everything on the table.**

  Including table: finite (no etc.), no link to external « reality ».
  Verbatim: the style of cowards, *meaningless.*

- **Key ↲ either *constative:* adds new line, *incremental.* Or:**

  Performative: launches program, *destructive.*

- ***Pure* lambda-calculus approximates analyticity.**

  Strong normalisation relates constat and performance.
  Undecidability: performance not constative.
  Church-Rosser relates performance and usage.

- ***External* performance replaced with *self-performance:***

  Plugging of wires of complementary colours.
  Unification: makes wires split into implicit subwires.
  Resolution: clause $\Gamma \vdash A$ becomes $\{\gamma, a\}$.

3 — SYNTHETICITY : USINE VS. USAGE

- Cognition *with* presupposition. Dubious *since* meaningful.
- *L'usine* a.k.a. synthetic *a posteriori:* factory tests.

  Proof-nets: no vicious circle (already in Herbrand).
  Testing: analytic performance; output unquestionable.
- *L'usage,* a.k.a. synthetic *a priori:* use of the product.

  Gentzen: the cut-rule, deductive *since* destructive.
- Fundamental *duality* of meaning: *dinaturals,* hexagons.

  Predictivity: *commitment* usine w.r.t. usage.
  Cut-elimination: performance implementing the reduction.
  Incompleteness: convergence of reduction problematic.
- *Consistency proofs:* no commitment. *Ditto* with *realism:*

  Semantics: identification usine/usage: no testing.
  Reformed BHK: one must choose between testing and use.

4 — DEREALISM

- **First order treatment of $\mathbb{N}$ *axiomatic,* $\neq$ logic.**

  Second order: (Dedekind) induction on T handled by $\exists X$.
  Flexibility: range of (inductive) witnesses T in $A[T/X]$.
  Subf. property: depends on possible T; ditto for 1st order.
  Foundational problems: reduction usage/usine problematic.

- ***Church*** **and** ***Curry*** **both wrong w.r.t. l'usine:**

  Essentialism: objets born synthetic, ***typed.*** No usine.
  Existentialism: objects born analytic, ***untyped.*** Usine ∞.

- ***Derealism:*** **usine stays finite if witness made part of proof.**

  Épure: analytic ***vehicle*** + synthetic ***mould,*** i.e., witness.
  Epidictics: requires/believes moulds to be ***balanced.***
  Balance: rights/duties (cut-elim.) not checkable at usine.

- ***Consistency*** **and Hegel's contradictory foundations:**
  Animæ: « **Incorrect** » **proofs, mingle analytic/synthetic.**

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II — PREDICATE CALCULUS

5 — A CONTROVERSIAL NOTION

- System $\mathbb{F}$ (Oslo, 1970): propositions are (roughly) enough.

  Forgetful functor: keeps computational (analytic) contents.
  Realisability: awkward reduction predicate $\rightsquigarrow$ proposition.

- ***Predicate calculus:*** XIX$^{\text{th}}$ century legacy.

  Axiomatics: cannot avoid « **Barbari** » $\forall x\, A \vdash \exists x\, A$.
  Semantics: models non-empty; but justification empty.

- Dubious principle: besides ***proper*** variables, used for $\vdash \forall$

  Junk variables: dedicated to the sole ***Barbari.***

- Intrusion of reality through ***external*** domain.

  Variables, functions: proceed from the Sky.

- In constrast to propositional quantification:

  Variables: refer to propositions, well-defined by l'usine.
  Functions: refer to connectives.

6 — EQUALITY

- ***Logical*** **primitive mistreated by metaphysical** ***axiomatics:***

  E.g., a predicate: « function » individuals $\rightsquigarrow$ propositions.

- **And/or through** ***semantic*** **pleonasm:**

  BHK: empty, reduces proof of $t = u$ to semantics.
  Semantics: $t = u$ true when ***same*** **denotation: $|t| = |u|$.**

- **$\forall X\,(Xt \Rightarrow Xu)$ (Leibniz) interesting,** ***since*** **totally wrong.**

  2nd order: not expected at elementary level.
  Circular: are those two « c » equal? Prejudiced:
  Relevant properties: those compatible with. . . equality.

- **A logical** ***epicycle,*** **i.e., a realistic contraption.**

  Individuals + predicates: ***all*** **of those which are** ***relevant.***

- **Break epicycle by replacing** ***individual*** **t with** ***proposition*** **t.**

  Meaning: « I am t ». Equality as logical equivalence $t \equiv u$.

III — PREDICATES AS CONNECTIVES

7 — Individuals as multiplicatives

- ***Individuals = proposition*** **forbidden by prejudice:**

  Classical: $t \equiv u \vee u \equiv v \vee v \equiv t$**. Only two individuals.**
  Intuitionistic: $\neg\neg(t \equiv u \vee u \equiv v \vee v \equiv t)$**. Not more than** 2**.**
  Linear: with $(t \multimap u)\ \&\ (u \multimap t)$ **as equality. No obstacle.**

- n**-ary multiplicative: sets of partition of** $\{1, \ldots, n\}$**.**

  Duality: $\mathcal{C} \perp \mathcal{D}$ **iff their incidence graph is a tree** $(n \neq 0)$**.**
  Multiplicative: non-trivial set of partitions equal to bidual.
  Example: $\otimes := \{\{1,2\}\}$ **vs.** $\parr := \{\{1\},\{2\}\}$**.**
  Series/parallel: $\P := \{\{1,2\},\{3,4\}\} + \{\{2,3\},\{4,1\}\}$**.**
  Not sequential: $\P$ **admits proof-nets, no sequent calculus.**

- ***Linear*** **implication between multiplicatives:**

  Same n **: typically,** $\cdot \otimes (\cdot \parr \cdot) \multimap (\cdot \otimes \cdot) \parr \cdot$ **with** $n = 3$**.**
  ♯ partitions: decreases; equal in case of equivalence.
  Equality: equivalence yields two *isomorphisms,* not related.

8 — FUNCTIONS AND PREDICATES

- **Functional *terms* come from same multiplicative matrix:**

  Positive multiplicatives with possible repetitions.
  Example: $x \parr (x \otimes y)$. No constant, no *Barbari,* no regrets.
  Pairing: ensured by $(x \parr y) \otimes (x \parr x \parr y)$.

- ***Predicate* variables $P, Q, \ldots$ as variable *connectives.***

  Pt handled by unknown binary connective K.
  Usage: all possible uses $Kt\tilde{t}$ of individual t and negation $\tilde{t}$.
  Usine: enough to test with $K = \otimes$ and $K = \parr$.
  Equality principle: $t = u \Rightarrow (Pt \multimap Pu)$ OK'ed by l'usine.
  Refused: $t = u \Rightarrow (Pt \multimap Qu)$ and $t = u \multimap (Pt \multimap Pu)$.

- ***Equality* handled by: $(\tilde{t} \parr u) \,\&\, (t \parr \tilde{u})$.**

- **First-order quantification: restriction of « full » case.**

  Existential witnesses: taken among multiplicative terms.

9 — Discussion

- **Logic is second order, including so-called first-order:**

  Propositions: variables, implicit $\forall X$ performed after.
  Usage: externalised by counter-models ($\exists X$ forbidden).
  No testing: dubious advantage of externalisation.

- **Individuals: *tame* second order.**

  Witnesses: multiplicatives, limited loss of subformula pty.
  Balance: rights/duties, usine/usage not really problematic.

- ***Arithmetic:*** **all axioms removed but:**

  Third/fourth Peano axioms: $Sx \neq 0$ and $Sx = Sy \Rightarrow x = y$.

- **The origin of logical doubt (incompleteness, etc.):**

  Épure vs. gabarit: performance $\mathcal{V} + \mathcal{M} + \mathcal{G}$.
  Variance: usine works better with lax $\mathcal{M}$. Usage may fail.
  Example: induction on « ill-formed » $\mathcal{M}$.