PART III

Π^1_2 proof-theory

"F is a functor from ... to ... preserving direct limits and pull-backs"; this is the leitmotiv of Parts III and IV. This sentence makes use of three keywords, "direct limits", "pull-backs", "functors", which have specific roles in the theory:

direct limits: this expresses, in a category-theoretic framework, nothing but the familiar ideas of *continuity*, i.e. the principle of computability from finitary data; by itself, this idea is not a new principle (but its combinations with the two other principles considerably increase its interest).

functor: this expresses, in a category-theoretic framework, *indiscernability* properties; in some sense, the "neighbourhoods" connected with direct limits will all be "alike".

pull-backs: pull-backs, which are a category-theoretic version of the idea of finite intersection, are essential to obtain *unicity* conditions.

A typical combination of these three principles is the *normal form theorem* of Section 8.2.:

– functoriality enables us to represent any point $< F(x)$ by an expression $(z_0\,;\, x_0, ..., x_i, ...\,;\, x)_F$.

– preservation of direct limits enables us to represent such a point by a finite expression $(z_0\,;\, x_0, ..., x_{n-1}\,;\, x)_F$.

– preservation of pull-backs enables us to choose, among all such representations, a distinguished one, the normal form... .

One of the main features of such objects is that they are completely determined by their restriction to a denumerable category of *finite dimensional* objects; in practice, in Part III, these finite dimensional objects will exactly be integers. From that we shall keep a "finitary" control on all our constructions. On the other hand, a system of "finitary" data does not necessarily determine an object of the kind we are seeking: typically, a functor from integers to integers can be extended to ordinals by means of direct limits, but its limit needs not to be an ordinal.... (Similarly, a sequence of rationals does not necessarily define a real....) Everything which is connected with this "finitary" aspect ot Π^1_2-logic (not bothering too much about well-foundedness of extensions) determines an "algebraic" theory, which, in the case of *dilators*, is developed in Chapter 8.

But the functors we are interested in, send ordinals on well-founded structures, and, at some stage, this phenomenon becomes essential! A dilator is therefore a way of speaking of *well-founded classes*, by means of finitary data.... Chapter 9 develops this viewpoint.

Dilators are in some sense the Π^1_2 analogue of the Π^1_1 concept of wf-tree; but, whereas the mathematical structure of wf-trees is practically inexistent, we have used two chapters to study dilators: Chapters 8 and 9 are the analogues of Chapter 5.

The place of Chapter 6 is taken by Chapter 10; we find a (functorial) analogue of the ω-completeness theorem, by means of functorial proofs. The *β-proofs* obey to the leitmotiv: they are functors preserving direct limits and pull-backs.

Chapter 11 is the analogue of Chapter 7 for inductive definitions in the framework of Π^1_2-logic. Chapter 11 combines all the techniques introduced in the previous chapters (8–10). The inductive definitions are analyzed by means of a cut-elimination theorem. This cut-elimination theorem in the framework of Π^1_2-logic has the following features:

– this is a total cut-elimination (i.e. all cuts are removed);

– there is a true subformula property.

The procedure can also be applied to the simplest of all inductive definitions, namely integers; this yields a new ordinal analysis of arithmetic by means of the Howard ordinal η_0. Other relations of arithmetic with η_0 can be found in 9.A (comparison of hierarchies) and 12.A (Gödel's $\mathcal{T}$ and ptykes) and ???. All these results are closely related.

Since we have β-completeness in the case of inductive definition, the methods of Π^1_2-logic can be used to investigate in questions closely connected to inductive definitions: admissible sets. The next admissible, the Σ^1 functions over a successor admissible, can be, in many cases, analyzed by means of recursive dilators. What is remarkable in this, is that generalized recursion (which is not recursive at all) can be reduced, in some sense, to usual recursion... .

CHAPTER 8
DILATORS: ALGEBRAIC THEORY

We begin by recalling a certain number of category theoretic notions:

1. a **category** consists of the following data

   – a class of objects $|\mathcal{C}|$;

   – for any two objects a, b in $|\mathcal{C}|$, a set $\mathsf{Mor}_{\mathcal{C}}(a,b)$, the set of **morphisms** from a to b;

   – a distinguished element $\mathsf{id}_a \in \mathsf{Mor}_{\mathcal{C}}(a,a)$ for all $a \in |\mathcal{C}|$;

   – for all $a, b, c \in |\mathcal{C}|$, a composition map:

   $$\mathsf{Mor}_{\mathcal{C}}(a,b) \times \mathsf{Mor}_{\mathcal{C}}(b,c) \to \mathsf{Mor}_{\mathcal{C}}(a,c)$$

   $$t, u \rightsquigarrow ut$$

   We require that:

   1) $t \in \mathsf{Mor}_{\mathcal{C}}(a,b) \to \mathsf{id}_b t = t\,\mathsf{id}_a = t$

   2) $t \in \mathsf{Mor}_{\mathcal{C}}(a,b)$, $u \in \mathsf{Mor}_{\mathcal{C}}(b,c)$, $v \in \mathsf{Mor}_{\mathcal{C}}(d,c) \to v(ut) = (vu)t$.

   In the sequel we shall use many categories, for instance **ON** (ordinals), **OL** (linear orders), **DIL** (dilators)... . Most of the time we shall not define the composition, nor the identity morphisms, which will be clear from the context... . When $t \in \mathsf{Mor}_{\mathcal{C}}(a,b)$ is such that for some $u \in \mathsf{Mor}_{\mathcal{C}}(b,a)$: $tu = \mathsf{id}_b$, $ut = \mathsf{id}_a$, then t is called an **isomorphism** and $u = t^{-1}$ is unique.

2. If $\mathcal{C}$ is a category, then a **subcategory** $\mathcal{D}$ of $\mathcal{C}$ consists of the following data:

   – a subclass $|\mathcal{D}|$ of $|\mathcal{C}|$

   – for all $a, b \in |\mathcal{D}|$; a subset $\mathsf{Mor}_{\mathcal{D}}(a,b)$ of $\mathsf{Mor}_{\mathcal{C}}(a,b)$ such that:

     – $\mathsf{id}_a \in \mathsf{Mor}_{\mathcal{D}}(a,a)$

     – the composition maps $\mathsf{Mor}_{\mathcal{D}}(a,b) \times \mathsf{Mor}_{\mathcal{D}}(b,c)$ into $\mathsf{Mor}_{\mathcal{D}}(a,c)$.

A typical example is that of a **full** subcategory: $\mathcal{D}$ is a full subcategory of $\mathcal{C}$ when:

$$\forall a \, \forall b \in |\mathcal{D}| \qquad \mathsf{Mor}_{\mathcal{D}}(a,b) = \mathsf{Mor}_{\mathcal{C}}(a,b) \ .$$

3. If $\mathcal{C}$ and $\mathcal{D}$ are categories, then a **functor** F from $\mathcal{C}$ to $\mathcal{D}$ consists of the following data:

– for all $x \in |\mathcal{C}|$ an object $F(x) \in |\mathcal{D}|$

– for all $x, y \in |\mathcal{C}|$ and $f \in \mathsf{Mor}_{\mathcal{C}}(a,b)$, a morphism $F(f) \in \mathsf{Mor}_{\mathcal{D}}\big(F(a), F(b)\big)$ such that the following holds:

  – $F(\mathsf{id}_a) = \mathsf{id}_{F(a)}$

  – $F(ut) = F(u)F(t)$ if $t \in \mathsf{Mor}_{\mathcal{C}}(a,b)$, $u \in \mathsf{Mor}_{\mathcal{C}}(b,c)$.

Functors are rather “large” objects; in practice, our functors will always be determined by their restrictions to rather small categories; for instance, dilators, which are functors from **ON** to itself, will be completely determined by their restriction to the subcategory of finite ordinals!

4. If F and G are functors from $\mathcal{C}$ to $\mathcal{D}$, then a **natural transformation** T from F to G is a family $(T_x)_{x \in |\mathcal{C}|}$ such that:

– $T_x \in \mathsf{Mor}_{\mathcal{D}}\big(F(x), G(x)\big)$ for all $x \in |\mathcal{C}|$

– if $t \in \mathsf{Mor}_{\mathcal{C}}(x,y)$, then the morphisms $G(t)T_x$ and $T_y F(t)$ are equal; this is expressed by saying that the diagram

$$\begin{tikzcd} F(x) \arrow[r, "T_x"] \arrow[d, "F(t)"'] & G(x) \arrow[d, "G(t)"] \\ F(y) \arrow[r, "T_y"'] & G(y) \end{tikzcd}$$

is commutative.

It is convenient to consider T as something close to a functor: define

– if $x \in |\mathcal{C}|$ $\quad T(x) = T_x$

– if $t \in \mathsf{Mor}_{\mathcal{C}}(x, y)$ $\quad T(t) = G(t)T_x = T_y F(t)$.

It is possible to compose natural transformations:

if T is a natural transformation from F to G,
if U is a natural transformation from G to H, then
UT is a natural transformation from F to H:

$$(UT)_x = U_x T_x \ .$$

In practice, it is not so clear for the beginner that natural transformations are of any interest. In fact this notion corresponds to the idea of "substructure", but some practice is needed to understand how this concept must be used. My advice is (since the technical content of natural transformations is very limited) to ignore at the first reading all the results concerning natural transformations... .

5. An **isomorphism** between $\mathcal{C}$ and $\mathcal{D}$ is a pair (F, G) such that:

– F is a functor from $\mathcal{C}$ to $\mathcal{D}$

– G is a functor from $\mathcal{C}$ to $\mathcal{D}$

– $G \circ F$ is the identity functor of $\mathcal{C}$

– $F \circ G$ is the identity functor of $\mathcal{D}$.

In the sequel, we shall construct many isomorphisms of categories (for instance between **DIL** and **SHD**, **BIL** and **ΩDIL**); however, this situation is not very common,[1] and current category-theoretic practice uses a more general notion:

6. An **equivalence** between $\mathcal{C}$ and $\mathcal{D}$ is a 6-tuple (F, G, T, T', U, U') such that:

[1]Here we can build isomorphisms, because we are dealing with rigid objects, which have no automorphisms... .

– F is a functor from $\mathcal{C}$ to $\mathcal{D}$

– G is a functor from $\mathcal{D}$ to $\mathcal{C}$

– T (resp. T', U, U') are natural transformations from the identity functor of $\mathcal{C}$ (resp. $G \circ F$, the identity functor of $\mathcal{D}$, $F \circ G$) to $G \circ F$ (resp. the identity functor of $\mathcal{C}$, $F \circ G$, the identity functor of $\mathcal{D}$) and

$$\forall x \in |\mathcal{C}| \qquad (T'T)_x = \mathsf{id}_x \qquad (TT')_x = \mathsf{id}_{G\left(F(x)\right)}$$

$$\forall x \in |\mathcal{D}| \qquad (U'U)_x = \mathsf{id}_x \qquad (UU')_x = \mathsf{id}_{F\left(G(x)\right)}$$

The next concepts that we shall use are those of *direct limit* and *pull-back*, that will be introduced later on. Our use of categories is very limited, byt perhaps a bit repellent to some readers; of course, there is no absolute evidence that one must present Π^1_2-logic in a category-theoretic framework. The advantage of such a presentation is that general patterns appear more easily, and a certain number of directions are naturally suggested by the category-theoretic framework. The price to pay for that is that perhaps we lose a bit of our intuitive approach to these matters... .

I have chosen what I think to be a medium position; the category-theoretic language is mainly used in definitions, and generally soon replaced by some useful characterization in usual mathematical terms: for instance the abstraction “preservation of direct limits and pull-backs” is replaced by a normal form theorem which has exactly the same contents.

It may certainly be useful for the beginner to try to translate the abstract category-theoretic constructions into more elementary ones; he will then get close to the original intentions and constructions... . Unfortunately, it is no longer possible for me to think (or to expose) the theory in very elementary and technical terms... . In any case the contents are exactly the same.

8.1. Direct limits and pull-backs

8.1.1. Definition.
Assume that x, y are two linear orders; then $I(x, y)$ denotes the set of all strictly increasing functions from x to y (see 5.3.1 (ii)).

8.1.2. Definition.
We shall consider the following categories:

(i) the category **OL** of linear orders: the objects are linear orders

(ii) the category **ON** of ordinals: the objects are ordinals

(iii) when x is an ordinal, the category $\mathbf{ON} < x$; the objects are ordinals $< x$

(iv) when x is an ordinal, the category $\mathbf{ON} \leq x$; tjhe objects are ordinals $\leq x$.

In all these categories, the morphisms from x to y are the elements of $I(x, y)$.

8.1.3. Definition.
If x and y are ordinals such that $x \leq y$, one defines $\mathbf{E}_{xy} \in I(x, y)$ by $\mathbf{E}_{xy}(z) = z$ for all $z \in x$. $\mathbf{E}_{xx}$ is abbreviated into $\mathbf{E}_x$.

8.1.4. Definition.
The following are functors from $\mathbf{ON}^2$ to $\mathbf{ON}$: let x, x', y, y' be ordinals, let $f \in I(x, x')$, $g \in I(y, y')$

(i) the functor **sum**

- $x + y$ is the familiar ordinal sum of x and y (5.5.3 (iii))
- if $z < x$, then $(f + g)(z) = f(z)$, if $z < y$, then $(f + g)(x + z) = x' + g(z)$.

(ii) the functor **product**

– $x \cdot y$ is the familiar ordinal product of x and y (5.5.3 (iv))

– if $t < x$, $u < y$, then $(f \cdot g)(x \cdot u + t) = x' \cdot g(u) + f(t)$.

(iii) the functor **exponential**

– $(1+x)^y$ is defined as in 5.5.3 (v)

– $(1+f)^g\big((1+x)^{u_1} \cdot (1+t_1) + ... + (1+x)^{u_n} \cdot (1+t_n)\big) = (1+x')^{g(u_1)} \cdot \big(1+f(t_1)\big) + ... + (1+x')^{g(u_n)} \cdot \big(1+f(t_n)\big)$

for all $u_1, ..., u_n$, $t_1, ..., t_n$ such that $y > u_1 > ... > u_n$ and $t_1, ..., t_n < x$.

8.1.5. Remarks.

(i) the definition of the exponential for morphisms makes heavy use of the *Cantor Normal Form* 5.5.4.: the function $(1+f)^g$ can be defined because every ordinal $< (1+x)^y$ can be written in Cantor normal form, in a unique way. The fact that $(1+f)^g$ is strictly increasing is immediate.

(ii) these functors can easily be extended into functors from $\mathbf{OL}^2$ to **OL** (using the Definition 5.4.9 together with 5.4.10 in the case of exponentiation). Direct limits provide a general way of extending (particular) functors from **ON** (or $\mathbf{ON}^2$) to **ON** into functors from **OL** (or $\mathbf{OL}^2$) to **OL** (see 8.2.11 (ii)). In fact our definition of $(1+x)^y$ for x, y arbitrary linear orders is exactly what one obtains if one extends the functor exponentiation by means of direct limits... .

8.1.6. Remark.

It can be of some interest to consider the category **OW** of well-orders (morphisms being still given by $I(x, y)$). There is a functor **order type** from **OW** to **ON**, defined by

– $\|x\|$ = unique ordinal isomorphic to x

– if $f \in I(x, y)$, then $\|f\| \in I(\|x\|, \|y\|)$ is defined by $\varphi_y f = \|f\| \, \varphi_x$ (where φ_z is the unique isomorphism from z to $\|z\|$)

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\varphi_x"'] & y \arrow[d, "\varphi_y"] \\ \|x\| \arrow[r, "\|f\|"'] & \|y\| \end{tikzcd}$$

i.e. this diagram is commutative.

We shall often be in the following situation: we are given a functor F from **OL** to **OL** such that F maps **OW** into **OW**. Then it will be possible to consider the "restriction" of F to **ON**, which will be a functor from **ON** to **ON**, defined by: (and denoted by $\|F\|$)

$$G(x) = \|F(x)\| \qquad G(f) = \|F(f)\| \ .$$

There are many functors from **OL** to **OL** which cannot be "restricted" as above. A typical example is the functor $\tilde{\ }$, defined by:

– $\tilde{x}$ is the order opposite to x (same domain, order reversed)

– $\tilde{f} = f$.

Then, when x is an infinite well-order, $\tilde{x}$ is not a well-order. However, when x is a finite (well-) order, so is $\tilde{x}$, hence it is possible to "restrict" $\tilde{\ }$ to $\mathbf{ON} < \omega$. We have "$\tilde{n}$", and "$\tilde{f}$"$(n - 1 - z) = m - 1 - f(z)$, when $f \in I(n, m)$.

8.1.7. <u>Definition</u>.
If x is an ordinal, let $\hat{x} = x + 1$; if x, y are ordinals and $f \in I(x, y)$, define $\hat{f} \in I(\hat{x}, \hat{y})$ by: $\hat{f}(z) = \sup\limits_{t<z} (f(t) + 1)$, equivalently: $\hat{f}(0) = 0$, $\hat{f}(z + 1) = f(z) + 1$, and for z limit, $\hat{f}(z) = \sup\limits_{t<z} f(t)$.

$\hat{\ }$ is clearly a functor from **ON** to **ON**.

8.1.8. Definition.
Let $\mathcal{C}$ be a category, and let I be a non void ordered set; we shall always assume that I is **directed**, i.e. that, given $i, j \in I$ there exists $k \in I$ such that $i, j \leq k$.

A **direct system** in $\mathcal{C}$ **indexed by** I, appears as a family $(x_i, f_{ij})_{i \leq j \in I}$ such that:

(i) for all $i \in I$, x_i is an object of $\mathcal{C}$

(ii) for all $i, j \in I$ such that $i \leq j$, f_{ij} is $\mathcal{C}$-morphism from x_i to x_j

(iii) for all $i \in I$, f_{ii} is the identity of x_i

(iv) for all i, j, k in I such that $i \leq j \leq k$, we have $f_{ik} = f_{jk} f_{ij}$.

$$\begin{tikzcd} x_i \arrow[rr, "f_{ij}"] \arrow[dr, "f_{ik}"'] & & x_j \arrow[dl, "f_{jk}"] \\ & x_k & \end{tikzcd}$$

8.1.9. Definition.
A **direct system of morphisms** (indexed by φ) from the direct system (x_i, f_{ij}) (indexed by I) to the direct system (y_l, f_{lm}) (indexed by L) is a family $(h_i)_{i \in I}$ (denoted $(h_i)_{i \in \varphi}$) such that:

(i) φ is an increasing function from I to L

(ii) for all $i \in I$ h_i is a $\mathcal{C}$-morphism from x_i to $y_{\varphi(i)}$

(iii) if $i, j \in I$ and $i \leq j$ then $h_j f_{ij} = g_{\varphi(i)\varphi(j)} h_i$.

$$\begin{tikzcd} x_i \arrow[r, "f_{ij}"] \arrow[d, "h_i"'] & x_j \arrow[d, "h_j"] \\ y_{\varphi(i)} \arrow[r, "g_{\varphi(i)\varphi(j)}"'] & y_{\varphi(j)} \end{tikzcd}$$

8.1.10. Remark.
It would be possible to form a category with direct systems in $\mathcal{C}$ as objects,

and direct systems of morphisms as morphisms. If $(h_i)_{i\in\varphi}$ and $(k_l)_{l\in\psi}$ are direct systems of morphisms from $(x_i, f_{ij})_{i\leq^1 j\in I}$ to $(y_l, g_{lm})_{l\leq^2 m\in L}$ and from $(y_l, g_{lm})_{l\leq^2 m\in L}$ to $(z_p, d_{pq})_{p\leq^3 q\in P}$, then the **composition** of $(k_l)_{l\in\psi}$ and $(h_i)_{i\in\varphi}$ is defined to be the family $h'_i = k_{\varphi(i)}h_i$. It is immediate that $(h'_i)_{i\in\psi\varphi}$ is a direct system of morphisms from $(x_i, f_{ij})_{i\leq^1 j\in I}$ to $(z_p, d_{pq})_{p\leq^2 q\in P}$.

We shall always try to avoid the *abstract nonsense* of categories of direct systems; however, it is sometimes useful to have them in mind to be able to understand operations like composition of direct systems of morphisms. The notation $i \in \varphi$ is rather shocking, but is has the immense advantage of giving explicitly the function φ.

8.1.11. <u>Definition</u>.
Let (x_i, f_{ij}) be a direct system in $\mathcal{C}$, indexed by I; a family $(x, f_i)_{i\in I}$ is said to be **a direct limit** of (x_i, f_{ij}) iff (i)–(iv) hold:

(i) x is an object of $\mathcal{C}$

(ii) for all $i \in I$ f_i is a $\mathcal{C}$-morphism from x_i to x

(iii) for all $i, j \in I$ such that $i \leq j$, $f_i = f_j f_{ij}$

(iv) if (y, g_i) is any family enjoying conditions (i)–(iii), then one can find a *unique* morphism h from x to y such that for all $i \in I$ $g_i = hf_i$.

$$\begin{tikzcd}
x_i \arrow[dr, "f_i"] \arrow[dd, "f_{ij}"'] \arrow[drr, "g_i"] & & \\
& x \arrow[r, "h"] & y \\
x_j \arrow[ur, "f_j"'] \arrow[urr, "g_j"'] & &
\end{tikzcd}$$

8.1.12. <u>Theorem</u>.
If $\mathcal{C}$ is one of the categories of Def. 8.1.2, then Condition (iv) of direct limits can be restated as:
(iv)′: $= \bigcup\limits_{i\in I} rg(f_i)$, i.e. every point in x can be written as $f_i(z_i)$ for some $i \in I$ and $z_i \in x_i$.

Proof. (iv) $\to$ (iv)$'$: let $X = \bigcup_{i \in I} rg(f_i)$; define y and $k \in I(y,x)$ by $rg(k) = X$, and $g_i \in I(x_i, y)$ by the condition $f_i = kg_i$; since (y, g_i) enjoys (i)–(iii), Condition (iv) ensures the existence of $h \in I(x,y)$ such that $g_0 = hf_i$, hence $f_i = khf_i$ for all $i \in I$; from this $kh(z) = z$ for all $z \in X$, and h maps X onto y: since h is strictly increasing this forces $X = x$.

(iv)$'$ $\to$ (iv): let $z \in x$; we define $h(z)$ as follows: choose $i \in I$ and $z_i \in x_i$ such that $z = f_i(z_i)$ (this is possible by (iv)$'$) and let $h(z) = g_i(z_i)$; this definition is not absurd, because, if $z = f_j(z_j)$, choose $k \geq i, j$; then $g_i(z_i) = g_k\bigl(f_{ik}(z_i)\bigr) = g_k\bigl(f_{jk}(z_j)\bigr) = g_j(z_j)$ (the equality $f_{ik}(z_i) = f_{jk}(z_j)$ comes from $z = f_k\bigl(f_{ik}(z_i)\bigr) = f_k\bigl(f_{jk}(z_j)\bigr)$ and the fact that f_k is strictly increasing...). h is strictly increasing: if $z < z'$, then by the directedness of I one can choose $i \in I$, z_i, z'_i with $z = f_i(z_i)$, $z' = f_i(z'_i)$. Since f_i and g_i are strictly increasing, it follows that $z_i < z'_i$ and $g_i(z_i) < g_i(z'_i)$, i.e. $h(z) < h(z')$. By construction we clearly have $g_i(z_i) = h\bigl(f_i(z_i)\bigr)$, so $g_i = hf_i$. The unicity of such an h is obvious. □

8.1.13. Examples.

(i) The simplest example of a direct limit is that of a supremum: let x be a limit ordinal, and let $I = x$, and for $y \in x$, let $a_y = y$, and for $y \leq z \in x$, let $f_{yz} = \mathbf{E}_{yz}$. Then $(x, \mathbf{E}_{yx})_{y \in x}$ is a direct limit of (a_y, f_{yz}): in that case the direct limit coincides with the supremum.

(ii) If the system (x_i, f_{ij}) is such that all x_i's are equal to some fixed integer n, then the system has a direct limit of the form (n, f_i). (Because all functions f_{ij} are isomorphisms.) But if x_i is constantly equal to some infinite ordinal, nothing can be said as to its direct limit. For instance, if x is a limit ordinal with a denumerable cofinality, it is possible to find a direct system (x_i, f_{ij}) with all $x_i = \omega$ having a direct limit of the form (x, f_i).

(iii) We give now the crucial example of a system (x_i, f_{ij}) with all x_i's finite, and a direct limit of the form $(\omega + 1, f_i)$: I will be the set $\mathbb{N}$ of integers; if $n \in I$, let $x_n = n + 1$, if $n \leq m$, let $f_{nm} = \mathbf{E}_{nm} + \mathbf{E}_1$, i.e. $f_{nm}(z) = z$ for $z < n$ and $f_n(n) = m$; define $f_n \in I(n+1, \omega+1)$ by $f_n = \mathbf{E}_{n\omega} + \mathbf{E}_1$, i.e. $f_n(z) = z$ for $z < n$, $f_n(n) = \omega$.

$$
\begin{array}{lcccccccccc}
x_3 = 4 & & 0 & 1 & 2 & 3 & & & & & \\
& f_{3,6} & \downarrow & \downarrow & \downarrow & & & & & & \\
x_6 = 7 & & 0 & 1 & 2 & 3 & 4 & 5 & 6 & & \\
& f_{6,7} & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & & \\
x_7 = 8 & & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\
& f_7 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
\omega+1 & & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots \quad \omega
\end{array}
$$

(iv) In a similar way, the system $(n+n, \mathbf{E}_{nm}+\mathbf{E}_{nm})$ has $(\omega+\omega, \mathbf{E}_{n\omega}+\mathbf{E}_{n\omega})$ as direct limit; the system $(n \cdot n, \mathbf{E}_{nm} \cdot \mathbf{E}_{nm})$ has $(\omega \cdot \omega, \mathbf{E}_{n\omega} \cdot \mathbf{E}_{n\omega})$ as direct limit... .

8.1.13. Proposition.

(i) If (x, f_i) is a direct limit of (x_i, f_{ij}), then x is unique up to isomorphism. (This is the reason why we shall often speak of x as "the direct limit of (x_i, f_{ij}).) In **ON** and in its subcategories, identity morphisms are the only isomorphisms, hence x is actually unique. (The same situation will hold for the category **DIL** of *dilators*.)

(ii) Suppose that $(x_i, f_{ij})_{i \leq^1 j \in I}$ and $(y_l, g_{lm})_{l \leq^2 m \in L}$ admit direct limits (x, f_i) and (y, g_l) in $\mathcal{C}$; then, if $(h_i)_{i \in \varphi}$ is a direct system of morphisms from (x_i, f_{ij}) to (y_l, g_{lm}), it is possible to define a morphism $h \in I(x, y)$ such that for all $i \in I$: $hf_i = g_{\varphi(i)} h_i$.

h_i

x_i $\qquad$ $y_{\varphi(i)}$

f_{ij} $\qquad$ f_i $\qquad$ $g_{\varphi(i)}$ $\qquad$ $g_{\varphi(i)}$

h_j

x_j $\qquad$ $y_{\varphi(j)}$

f_j $\qquad$ $g_{\varphi(j)}$

x $\qquad$ h $\qquad$ y

This morphism is called the **direct limit** of (h_i).

Proof. (i) If (x, f_i) and (y, g_i) are two direct limits of (x_i, f_{ij}), then by 8.1.11 (iv) one can find h and k such that $g_i = hf_i$ and $f_i = kg_i$ for all $i \in I$. From this one gets $f_i = khf_i$ for all $i \in I$; by condition 8.1.11 (iv), kh is the *unique* morphism such that $f_i = khf_i$ for all $i \in I$. This forces kh to be the identity of x, and similarly hk is the identity of y. So x and y are isomorphic.

(ii) $(y, g_{\varphi(i)}h_i)$ enjoys conditions 8.1.11 (i)–(iii), hence there is a unique h from x to y such that $hf_i = g_{\varphi(i)}h_i$ for all $i \in I$. □

8.1.14. Remark.
"The" direct limit is functorial in the following sense: assume that $(x_i - f_{ij})$, (y_l, g_{lm}), (z_p, d_{pq}) are as in 8.1.10 and have direct limits (x, f_i), (y, g_l), (z, d_p) and assume that (h_i), (k_l), (h'_i) are as in 8.1.10. Then their respective direct limits h, k, h' are such that $h' = k \cdot h$.

8.1.15. Example.
Assume that $(x_i)_{i\in I}$ and $(y_l)_{l\in L}$ are increasing families of ordinals, and that φ is an increasing function from I to L. Assume that the functions $f \in I(x_i, y_{\varphi(i)})$ are such that $i \leq j \to f_j$ extends f_i, i.e. that for $z \in x_i \quad f_j(z) = f_i(z)$. Then it is possible to define a function $g = \bigcup\limits_i f_i$, with $g \in I\big(\mathsf{sup}(x_i), \mathsf{sup}(y_l)\big)$ by $g(z) = f_i(z)$, where i is any index such that $z \in x_i$.

In fact, $\bigcup\limits_i f_i$ is a very simple case of a direct limit of morphisms: recall rhat (with $x = \mathsf{sup}(x_i)$, $y = \mathsf{sup}(y_l)$) $(x, \mathbf{E}_{x_i x})$ (res[. $(y, \mathbf{E}_{y_l y})$) is the direct limit of the system $(x_i, \mathbf{E}_{x_i x_j})$ (resp. $(y_l, \mathbf{E}_{y_l y_m})$); it is immediate that $(f_i)_{i\in\varphi}$ is a direct system of morphisms between $(x_i, \mathbf{E}_{x_i x_j})$ and $(y_l, \mathbf{E}_{y_l y_m})$ and that the function $g = \bigcup\limits_i f_i$ is their direct limit.

8.1.1.6. Theorem.
The category $\mathbf{ON} < \omega$ is dense in the categories $\mathbf{ON}$, $\mathbf{OL}$ (and $\mathbf{ON} < x$ for x infinite). This means that:

(i) if x is any object of one of these categories, then one can find a direct system (x_i, f_{ij}), with all x_i's integers together with functions f_i, such that (x, f_i) is the direct limit of (x_i, f_{ij}).

(ii) if x, y are objects in one of these categories, and (x, f_i) (resp. (y, g_l)) is the direct limit of (x_i, f_{ij}) (resp. (y_l, g_{lm})) with x_i, y_l finite for all i and l, then for any $h \in I(x, y)$, one can find an increasing function φ together with a direct system of morphisms $(h_i)_{i\in\varphi}$ such that h is the direct limit of (h_i).

Proof. (i) Define $I = \{a\,;\, a \text{ finite},\ a \subset x\}$; I (ordered by inclusion) is obviously directed. Define $x_a = \|a\|$ (i.e. the order type of x restricted to a); if $a \subset b$, let e_{ab} be the inclusion map from a into b, and let $f_{ab} = \|e_{ab}\|$. It is immediate that x_a is an integer and (x_a, f_{ab}) is a direct system. Let e_{ax} be the inclusion map from a into x, and let $f_a = \|e_{ax}\|$. It is immediate that (x, f_i) enjoys Conditions 8.1.11 (i)–(iii) of direct limits w.r.t. (x_i, f_{ij}). Condition (iv)$'$ is satisfied as well, since if $z \in x$, $z \in rg(f_{\{z\}})$. Hence (x, f_i) is a direct limit for (x_i, f_{ij}).

(ii) If a is any finite subset of y, one can find an index $a^* \in L$ such that $a \subset rg(g_{a^*})$. It is an easy exercise of set theory to show that the function $a \rightsquigarrow a^*$ can be chosen increasing, i.e. $a \subset b \to a^* \leq^2 b^*$. (Define, using the axiom of choice, a^* by transfinite induction over a well-ordering of the finite subsets of y, which extends the inclusion.) Given $i \in I$, let $\varphi(i) = \big(rg(hf_i)\big)^*$; then φ is an increasing function and we have $rg(hf_i)^* \subset rg(g_{\varphi(i)})$, so one can define $h_i \in I(x_i, y_{\varphi(i)})$ by the condition: $hf_i = g_{\varphi(i)}h_i$. It is immediate that $(h_i)_{i\in\varphi}$ is a direct system of morphisms from (x_i, f_{ij}) to (y_l, g_{lm}) and h is the direct limit of (h_i). $\square$

8.1.17. Notations.

(i) $(x, f_i) = \displaystyle\lim_{\longrightarrow} (x_i, f_{ij})$ expresses that (x, f_i) is the direct limit of (x_i, f_{ij}); one can note the indexing set I by: $\displaystyle\lim_{\substack{\longrightarrow \\ I}}$. If we are only interested in x, we shall use the notation: $x = \displaystyle\lim_{\longrightarrow}{}^{*} (x_i, f_{ij})$.

(ii) $h = \varinjlim (h_i)$ when h is the direct limit of the system (h_i); one can note the indexing function by: $\varinjlim\limits_{\varphi}$.

8.1.18. Theorem.
In **OL**, all direct systems have direct limits.

Proof. Let (x_i, f_{ij}) be a direct system in **OL**; let X be the disjoint union of the x_i's, and define a binary relation R on X by: $(a,i)\,R\,(b,j)$ iff for some $k \geq i,j$, $f_{i,k}(a) \leq^k f_{jk}(b)$ ($\leq^k$ is the order relation of x_k). R is obviously a preorder, and by directedness, R is a total preorder, i.e. any two elements of X are comparable; if S is the equivalence relation associated with R, X/S is celarly linearly ordered by R/S. Let us call this ordered set x, and define $f_i \in I(x_i, x)$ by $f_i(z) =$ eq. class of z modulo S. It is immediate that (x, f_i) enjoys 8.1.11 (i)–(iii) w.r.t. (x_i, f_{ij}). (iv)$'$ holds as well, since if a is an equivalence class modulo S, and $(z,i) \in a$, then $a = f_i(z)$. Hence $(x, f_i) = \varinjlim (x_i, f_{ij})$. □

8.1.19. Remark.
The analogue of 8.1.18 fails for **ON**: if $x = \varinjlim{}^{*} (x_i, f_{ij})$, with all x_i's finite, then $\tilde{x} = \varinjlim{}^{*} (\tilde{x}_i, f_{ij})$, but $\tilde{x}$ is not an ordinal when x is an infinite ordinal: the direct system $(\tilde{x}_i, \tilde{f}_{ij})$ is (isomorphic to) a direct system in **ON**, but it cannot have any direct limit in **ON** since such a limit would be isomorphic to $\tilde{x}$, which is not a well-order. The relation between the concepts of direct limit in **ON** and in **OL** are given by:

8.1.20. Theorem.
Let (x_i, f_{ij}) be a direct system in **ON**, and let x be its direct limit in the category **OL**. Then (x_i, f_{ij}) admits a direct limit in **ON** iff x is a well-order. In that case, the direct limit is exactly $\|x\|$.

Proof. Assume that (in **OL**) $(x, f_i) = \varinjlim (x_i, f_{ij})$.

(i) If x is a well-order, one can replace x by $\|x\|$, f_i by $\|f_i\|$; and Conditions 8.1.11 (i)–(iv) hold for $(\|x\|, \|f_i\|)$ in **OL**, hence in **ON**.

(ii) Conversely, if (x_i, f_{ij}) has a limit (y, g_i) in **ON**, then in the category **OL**, (y, g_i) enjoys 8.1.11 (i)–(iii) w.r.t. (x_i, f_{ij}), hence for some $h \in I(x, y)$ $g_i = hf_i$; but $h \in I(x, y)$ and y is a well-order: by 5.3.4, x is a well-order. □

8.1.21. <u>Theorem</u>.
Assume that (x_i, f_{ij}) and (x, f_i) enjoy conditions 8.1.11 (i)–(iii), with x_i, x in **ON**; then (x_i, f_{ij}) has a direct limit in **ON**.

<u>Proof</u>. We construct x, y, k, g_i exactly as in the proof of the implication (iv) $\to$ (iv)$'$. It is immediate that (y, g_i) enjoys (iv)$'$, and since $k \in I(y, x)$, y is a well-order. By 8.1.20, (x_i, f_{ij}) has a direct limit in **ON**. □

8.1.22. <u>Remark</u>.
Direct limits can be thought of as an effective way of dealing with infinite objects such as ordinals: we can think of an ordinal $x = \underrightarrow{\lim}^{*} (x_i, f_{ij})$ as the (ideal) limit of an approximation process by means of the integers x_i. It is a fact that most operations on ordinals can be handled by means of similar operations on finite approximations, i.e. can be represented by means of a functor *preserving* direct limits. Examples of such functors are *dilators*, *β-proofs*... . Each time we succeed in eliminating an operation on actual infinite objects in favor of a similar one acting on approximations, we have realized something akin to Hilbert's program, namely eliminating infinite objects from our universe.

Let us see how direct systems can be connected with ideas of recursiveness:

(i) first it is natural to consider recursive index sets: I is a recursive subset of $\mathbb{N}$ ordered by a recursive relation.

(ii) then we look at the question of representing the family x_i: in the basic situation, the ordinals x_i will be integers, hence x_i can be represented as $\{e\}\, i$ for some index i. (More generally, if x_i is a recursive well-order, then we require that $\{e\}\, i$ is an index for x_i.)

(iii) a function $f \in I(n, m)$ can be encoded by $\ulcorner f \urcorner = \langle f(0), ..., f(n-1), m \rangle$. Hence it is possible (when x_i's are integers), to represent the family f_{ij} by $\ulcorner f_{ij} \urcorner = \{f\}\,(i, j)$ for some index f. (More generally, we require that $\{f\}\,(i, j)$ is an index for f_{ij}.)

We have clearly defined two concepts:

– the concept of a recursive direct system of integers

– the concept of a recursive direct system of recursive well-orders.

Observe that any recursive well-order can be obtained as the direct limit of a recursive direct system of integers: let $X = (|X|, \leq)$ be a recursive well-order. Let $I = \mathbb{N}$; for $n \in \mathbb{N}$ let $x_n = \|(|X| \cap n, \leq \restriction |X| \cap n)\|$, and if for $n \leq m$, e_{nm} is the inclusion map between $X\restriction n$ and $X\restriction m$, let $f_{nm} = \|e_{nm}\|$. Then X is clearly the direct limit of (x_n, f_{nm}). Conversely we have:

8.1.23. <u>Proposition</u>.
If (x_i, f_{ij}) is a recursive direct system of integers (or of recursive well-orders), then its direct limit in $\mathbf{OL}(x, f_i)$ can be chosen recursive.

<u>Proof</u>. First of all, we construct a recursive function φ from $\mathbb{N}$ to I, with the property that φ is increasing and $rg(\varphi)$ is cofinal in I: let $\varphi(n)$ be the smallest integer which is greater (for the ordering of I) than all the elements of $I \cap \{0, ..., n-1\}$:

$$\varphi(n) = \mu m \Big(m \in |I| \wedge \forall p < n \; (p \in |I| \to p \leq m) \Big) \; .$$

This function is total because I is non-void and directed! The direct limit will be the linear order $X = (|X|, \leq^1)$ defined by:

$$X = \Big\{ \langle n, z \rangle \; ; \; z \in x_{\varphi(n)} \wedge \forall m \forall z' \; \Big(\langle m, z' \rangle < \langle n, z \rangle \; \to$$

$$\Big((m \leq n \wedge z \neq f_{\varphi(m)\varphi(n)}(z') \Big) \vee \Big((m > n \wedge z' \neq f_{\varphi(m)\varphi(n)}(z)) \Big) \Big) \Big\}$$

$$\langle n, z \rangle \leq^1 \langle m, z' \rangle \leftrightarrow f_{\varphi(n)\varphi(p)}(z) \leq^* f_{\varphi(m)\varphi(p)}(z')$$

where $\leq^*$ stands for the order relation of $x_{\varphi(p)}$ and $p = \mathsf{sup}(n, m)$. X is clearly the direct limit of (x_i, f_{ij}); the functions f_i can be defined by: $f_i(z)$ is the smallest pair $\langle n, z' \rangle$ such that $f_{ij}(z) = f_{\varphi(n)_j}(z')$ for some $j \geq i, \varphi(n)$. $\square$

8.1.24. <u>Definition</u>.
Let x_1, x_2, x_3, x be objects of a category $\mathcal{C}$, and let f_1, f_2, f_3 be $\mathcal{C}$-morphisms from respectively x_1, x_2, x_3 to x; then f_3 is said to be a **pull-back** of f_1 and f_2 iff

(i) there exist $\mathcal{C}$-morphisms f_{31} and f_{32} from x_3 to x_1 and x_2 such that: $f_3 = f_1 f_{31} = f_2 f_{32}$

(ii) given any other solution $(x'_3, f'_3, f'_{31}, f'_{32})$ enjoying (i) one can find a *unique* morphism h from x'_3 to x_3 such that

$$f'_{31} = f_{31} h \quad \mathsf{and} \quad f'_{32} = f_{32} h \ .$$

$$\begin{tikzcd}
 & & x_1 \arrow[dr, "f_1"] & \\
x'_3 \arrow[r, "h"] \arrow[urr, "f'_{31}"] \arrow[drr, "f'_{32}"'] & x_3 \arrow[ur, "f_{31}"] \arrow[rr, "f_3"] \arrow[dr, "f_{32}"'] & & x \\
 & & x_2 \arrow[ur, "f_2"'] &
\end{tikzcd}$$

8.1.25. <u>Notation</u>.
$f_3 = f_1 \wedge f_2$ means that f_3 is "the" pull-back of f_1 and f_2. The pull-back is easily shown to be unique up to isomorphism (on the model of 8.1.13 (i)). In the category **ON**, pull-backs will therefore be uniquely determined.

8.1.26. Theorem.
In the categories **ON** and **OL**, $f_3 = f_1 \wedge f_2$ iff $rg(f_3) = rg(f_1) \cap rg(f_2)$. Hence, in these categories pull-backs always exist.

Proof. Any solution f_3 of 8.1.24 (i) enjoys $rg(f_3) \subset rg(f_i)$ for $i = 1, 2$, so $rg(f_3) \subset rg(f_1) \cap rg(f_2)$. Conversely, if f_3 is such that $rg(f_3) \subset rg(f_1) \cap rg(f_2)$, then 8.1.24 (i) holds: if $z \in x_3$, then $f_3(z) \in rg(f_1)$, hence $f_3(z) = f_1(z')$ for some uniquely determined $z' \in x_1$, and one can put $f_{31}(z) = z'$. Assume now that $rg(f_3) = rg(f_1) \cap rg(f_2)$; then, given an arbitrary f_3' such that $rg(f_3') \subset rg(f_1) \cap rg(f_2)$, define $h \in I(x_3', x_3)$ by: $h(z) =$ the unique z' such that $f_3'(z) = f_3(z')$; then it is immediate that h is the only solution of $f_{31}' = f_{31}h$, $f_{31}' = f_{32}h$. Conversely suppose that $f_3 = f_1 \wedge f_2$, and apply 8.1.24 (ii) to f_3' such that $rg(f_3') = rg(f_1) \cap rg(f_2)$: then $f_3' = f_3 h$, so $rg(f_3) \supset rg(f_3')$, hence $rg(f_3) = rg(f_1) \cap rg(f_2)$. $\square$

8.2. <u>The normal form theorem</u>

8.2.1. <u>Remark</u>.

(i) If F is a functor from $\mathcal{C}$ to $\mathcal{D}$ and if (x_i, f_{ij}) is a direct system in $\mathcal{C}$, $\big(F(x_i), F(f_{ij})\big)$ is a direct system in $\mathcal{D}$. But in general, if $(x, f_i) = \varinjlim\ (x_i, f_{ij})$, $\big(F(x), F(f_i)\big)$ need not be the direct limit of $\big(F(x_i), F(f_{ij})\big)$: conditions 8.1.11 (i)–(iii) of direct limits are obviously fulfilled, whereas (iv) is problematic.

(ii) If F is a functor from $\mathcal{C}$ to $\mathcal{D}$ and if $f_3 = f_1 \wedge f_2$, then in general $F(f_3)$ is not equal to $F(f_1) \wedge F(f_2)$; however, condition 8.1.24 (i) is always fulfilled.

8.2.2. <u>Definition</u>.
Let F be a functor from $\mathcal{C}$ to $\mathcal{D}$; then

(i) F **preserves direct limits** iff given any direct system (x_i, f_{ij}) in $\mathcal{C}$, with a direct limit (x, f_i), then $\big(F(x_i), F(f_{ij})\big)$ has a direct limit in $\mathcal{D}$, and this direct limit is equal to $\big(F(x), F(f_i)\big)$.

(ii) F **preserves pull-backs** iff given any f_1, f_2, f_3 such that $f_3 = f_1 \wedge f_2$ in $\mathcal{C}$, then $F(f_1) \wedge F(f_2)$ exists in $\mathcal{D}$ and equals $F(f_3)$.

8.2.3. <u>Theorem</u> (Girard, [5]) Normal Form Theorem.
Let F be a functor from one of the categories $\mathbf{OL}$, $\mathbf{ON}$, $\mathbf{ON} \leq x$, $\mathbf{ON} < x$ to $\mathbf{OL}$ or $\mathbf{ON}$; then F preserves direct limits and pull-backs iff the following holds: for all x, for all $z \in F(x)$, then one can find an integer n and $z_0 \in F(n)$ and $f \in I(n, x)$ such that

(i) $z = F(f)(z_0)$.

(ii) if $z_0 = F(g)(z_1)$ with $g \in I(n; , n)$ and $z_1 \in F(n')$, then $n = n'$ (and $g = \mathbf{E}_n$) and n, z_0, f are uniquely determined by (i) and (ii).

Proof. Assume first that F preserves direct limits and pull-backs: choose x, an object in the domain of F, and then $(x, f_i) = \underrightarrow{\lim}\ (x_i, f_{ij})$ for a well-chosen direct system, with all x_i's finite, by 8.1.16 (i). By preservation of direct limits $\big(F(x), F(f_i)\big) = \underrightarrow{\lim}\ \big(F(x_i), F(f_{ij})\big)$, hence $z \in rg\big(F(f_i)\big)$ for some i: there exists an integer n, $z_0 \in F(n)$ and $f \in I(n, x)$ such that $z = F(f)(z_0)$. If one chooses n minimum with this property, one gets (i) and (ii). We must show that n, z_0 and f are uniquely determined: assume that n, f, z_0, and m, f', z_1 are two distinct solutions of (i)–(ii); necessarily $rg(f) \not\subset rg(f')$ (otherwise $f = f'k$, and $z_0 = F(k)(z_1)$...), hence $f \neq f \wedge f'$; let $g = g \wedge g'$; by preservation of pull-backs, $F(g) = F(f) \wedge F(f')$, i.e. $rg\big(F(g)\big) = rg\big(F(f)\big) \cap rg\big(F(f')\big)$, hence $z \in rg\big(F(g)\big)$; write $g = fh$; then $z = F(g)(z_2) = F(f)F(h)(z_2)$, hence $z_0 = F\big(h(z_2)\big)$ with $h \neq \mathbf{E}_n$: this contradicts property (ii) of n, z_0, f.

Conversely, assume that F is such that a unique representation by means of (i) and (ii) is possible: if $(x, f_i) = \underrightarrow{\lim}\ (x_i, f_{ij})$, we have already remarked (8.2.1 (i)) that $\big(F(x), F(f_i)\big)$ enjoys conditions 8.1.11 (i)–(iii) w.r.t. $\big(F(x_i), F(f_{ij})\big)$; it suffices to show that (iv)$'$ is fulfilled: if $z \in F(x)$, write $z = F(f)(z_0)$ for $f \in I(n, x)$ and $z_0 \in F(n)$. Since $rg(f)$ is finite, one can gind an index i such that $rg(f) \subset rg(f_i)$ (use condition (iv)$'$ for (x, f_i) and the directedness of I), hence $f = f_i h$ for some h. Then $z = F(f_i)F(h)(z_0)$, i.e. $z \in rg\big(F(f_i)\big)$. Assume now that $f_3 = f_1 \wedge f_2$; then by Remark 8.2.1 (ii), we know that $F(f_3)$ enjoys condition 8.1.24 (i) w.r.t. $F(f_1)$ and $F(f_2)$, i.e. that $rg\big(F(f_3)\big) \subset rg\big(F(f_1)\big) \cap rg\big(F(f_2)\big)$. The reverse inclusion is shown below: if $z \in rg\big(F(f_1)\big) \cap rg\big(F(f_2)\big)$, write $z = F(f_1)(z_1) = F(f_2)(z_2)$, and choose n_1, z_1', g_1, and n_2, z_2', g_2 such that conditions (i) and (ii) hold w.r.t. z_1 and z_2: then $z_1 = F(g_1)(z_1')$, $z_2 = F(g_2)(z_2')$; hence $z = F(f_1 g_1)(z_1') = F(f_2 g_2)(z_2')$. Since z_1' is such that $z_1' = F(h)(z_1'') \to h$ is the identity (similarily for z_2'), it follows that (from the unicity of n, z_0, f such that (i) and (ii)) that $z_1' = z_2'$, $n_1 = n_2$, $f_1 g_1 = f_2 g_2$. Clearly $rg(f_1 g_1) \subset rg(f_3)$, hence $f_1 g_1 = f_3 k$ for some k: $z = F(f_3)F(k)(z_1')$, so $z \in rg\big(F(f_3)\big)$. □

8.2.4. Notation.
When $z < F(x)$, we shall represent x by means of the data n, z_0, f such that (i) and (ii): we shall represent z by $(z_0\,;\, x_0, ..., x_{n-1}\,;\, x)_F$, with $x_0 = f(0), ..., x_{n-1} = f(n-1)$. We may forget the index F when F is clear from the context.

Let us sum up under which conditions $(z_0\,;\, x_0, ..., x_{n-1}\,;\, x)_F$ is a **denotation**:

(i) $x_0, ..., x_{n-1}, x$ are ordinals such that $x_0 < ... < x_{n-1} < x$.

(ii) $z_0 \in F(n)$.

(iii) for all $m < n$ and all $f \in I(m, n)$, $z_0 \notin rg\Big(F(f)\Big)$.

One clearly sees that in the denotation, there is a right part, namely the pair $(z_0\,;\, n)$, and another part that can take arbitrary values (the strictly increasing sequence $x_0, ..., x_{n-1}, x$; but its length is fixed).

The behaviour of the functorial constructions w.r.t. denotations arising from Theorem 8.2.3 is given by:

8.2.5. Proposition.

(i) $F(f)\Big((z_0\,;\, x_0, ..., x_{n-1}\,;\, x)_F\Big) = (z_0\,;\, f(x_0), ..., f(x_{n-1})\,;\, y)_F$ when $f \in I(x, y)$.

(ii) if T is a natural transformation from F to G, then $T(x)\Big((z_0\,;\, x_0, ..., x_{n-1}\,;\, x)_F\Big) = (T(n)(z_0)\,;\, x_0, ..., x_{n-1}\,;\, x)_G$.

Proof. (i) If $g \in I(n, x)$ is defined by: $g(0) = x_0, ..., g(n-1) = x_{n-1}$, then $(z_0\,;\, x_0, ..., x_{n-1}\,;\, x)_F$ denotes $F(g)(z_0)$, and $F(f)(z) = F(fg)(z_0)$ is denoted by: $(z_0\,;\, f(x_0), ..., f(x_{n-1})\,;\, y)$.

(ii) Consider the following commutative diagram

$$\begin{tikzcd} F(n) \arrow[r, "T(n)"] \arrow[d, "F(g)"'] & G(n) \arrow[d, "G(g)"] \\ F(x) \arrow[r, "T(x)"'] & G(x) \end{tikzcd}$$

where g is defined as in (i). Then it is clear that $T(x)\big((z_0\,;\,x_0,...,x_{n-1}\,;\,x)_F\big)$ $= G(g)\big(T(n)(z_0)\big)$ hence, it will suffice to prove that $T(n)(z_0) = G(h)(z_1) \rightarrow h = \mathbf{E}_n$. If $h \neq \mathbf{E}_n$, it is possible to find $g_1, g_2 \in I(n, n+1)$ such that $g_1 \neq g_2$, but $g_1 h = g_2 h$. Then $G(g_1)T(n)(z_0) = G(g_1 h)(z_1) = G(g_2 h)(z_1) = G(g_2)T(n)(z_0)$, but $F(g_1)(z_0) = (z_0\,;\,g_1(0),...,g_1(n-1)\,;\,n+1)_F \neq (z_0\,;\,g_2(0),...,g_2(n-1)\,;\,n+1)_F = F(g_2)(z_0)$, hence $T(n+1)F(g_1)(z_0)$ $\neq T(n+1)F(g_2)(z_0)$, a contradiction:

$$\begin{tikzcd} F(n) \arrow[r, "T(n)"] \arrow[d, "F(g_i)"'] & G(n) \arrow[d, "G(g_i)"] \\ F(n+1) \arrow[r, "T(n+1)"'] & G(n+1) \end{tikzcd}$$

□

8.2.6. <u>Theorem</u>.
Assume that F is a functor from $\mathbf{ON} < \omega$ to $\mathbf{OL}$; then, if F preserves pull-backs

(i) F can be extended into a functor from $\mathbf{OL}$ to $\mathbf{OL}$ preserving direct limits and pull-backs; this extension is unique up to isomorphism.

(ii) Moreover, if F is a functor from $\mathbf{ON} < \omega$ to $\mathbf{ON}$ and if the extension computed in (i) is a well-order for all x, then F can be extended into a functor from $\mathbf{ON}$ to $\mathbf{ON}$ preserving direct limits and pull-backs.

<u>Proof</u>. First of all, remark that F preserves direct limits (if $(n, f_i) = \varinjlim (n_i, f_{ij})$, then there is some $i \in I$ such that $n_i = n$, $f_i = \mathbf{E}_n$), hence the normal form theorem can be applied to F.

(i) If x is a linear order, then define $G(x)$ as follows:

– $G(x)$ consists of all formal expressions $(z_0\,;\,u_0,...,u_{n-1}\,;\,x)$ such that $u_0,...,$
u_{n-1} is a strictly increasing sequence in x, $z_0 \in F(n)$, and z_0 cannot be written $F(h)(z_1)$ with $h \neq \mathbf{E}_n$.

– $G(x)$ is linearly ordered as follows: $(z_0\,;\,u_0,...,u_{n-1}\,;\,x) = z$ and $(z_1\,;\,v_0,...,$ $v_{m-1}\,;\,x) = z'$ can be compared by considering the subset $\{w_0,...,w_{p-1}\} =$

$\{u_0, ..., u_{n-1}\} \cup \{v_0, ..., v_{m-1}\}$ of x, and $f \in I(p, x)$ such that $rg(f) = \{w_0, ..., w_{p-1}\}$; then one can write $u_i = fg(i)$, $v_i = fh(i)$, for some $g \in I(n, p)$ and $h \in I(m, p)$; consider $y = (z_0\,; g(0), ..., g(n-1)\,; p)$ and $y' = (z_1\,; h(0), ..., h(m-1)\,; p)$. Then z is less than z' in $G(x)$ iff y is less than y' in $F(p)$.

– if $f \in I(x, y)$, then define $G(f)\Big((z_0\,; u_0, ..., u_{n-1}\,; x)\Big)$ to be $(z_0\,; f(u_0), ..., f(u_{n-1})\,; y)$. One easily checks that $G(f)$ is a strictly increasing function from $G(x)$ to $G(y)$, and G is a functor from **OL** to **OL**.

Now observe that G enjoys the normal form theorem: if $z \in G(x)$, $z = (z_0\,; x_0, ..., x_{n-1}\,; x)$, then clearly $z = F(f)(z'_0)$, with: $f \in I(n, x)$ defined by $f(0) = x_0, ..., f(n-1) = x_{n-1}$, $z'_0 = (z_0\,; 0, ..., n-1\,; n)$. Conditions (i) and (ii) are obviously fulfilled, and this is of course the only solution.

Strictly speaking, G is not an extension of F, since the values $G(n)$ are not exactly equal to $F(n)$; however, $G(n)$ is isomorphic to $F(n)$ (to $(z_0\,; x_0, ..., x_{m-1}\,; n)$, associate $(z_0\,; x_0, ..., x_{m-1}\,; n)_F$!) and so a very small modification of G on the integers enables us to construct an extension of F preserving direct limits and pull-backs... .

The extension is unique up to isomorphism: if $x = \underset{\longrightarrow}{\lim}{}^{*}\,(x_i, f_{ij})$ with all x's finite, then one must have $G(x) = \underset{\longrightarrow}{\lim}{}^{*}\Big(F(x_i), F(f_{ij})\Big)$: recall that direct limits are unique up to isomorphism.

(ii) is immediate: assume that the extension of F computed in (i) actually maps **ON** into **OW**; then $\|\ \| \circ F$ maps **ON** into **ON** and preserves direct limits and pull-backs. (For a pedantic proof of this: observe that $\|\ \|$ preserves direct limits and pull-backs... .) □

8.2.7. Remark.

In order that the functor F from **ON** $< \omega$ to **ON** may be extendable into a functor from **ON** to **ON** preserving direct limits and pull-backs, it is necessary and sufficient that:

(i) F preserves pull-backs.

(ii) The direct limit of $\Big(F(x_i), F(f_{ij})\Big)$ exists in **ON**, where (x_i, f_{ij}) is a direct system in **ON**, with all x_i's finite, and such that $\varinjlim{}^{*}(x_i, f_{ij}) = \aleph_1$.

(Proof. By 8.2.6 (ii), it suffices to show that $G(x)$ is a well-order for all ordinals x; but, if $G(x)$ is not a well-order, let $y^0, ..., y^p, ...$ be a s.d.s. in $G(x)$, and let $(x_0^0, ..., x_{m_0-1}^0), ..., (x_0^p, ..., x_{n_p-1}^p), ...$ be the elements of x occurring in the normal forms of the points $y^0, ..., y^p, ...$; if Y is the set of all these coefficients, then Y is denumerable, and if $g \in I(y, x)$ is such that $rg(g) = Y$, it is immediate that $y^i = F(g)(z^i)$, for a certain s.d.s. z^i in $G(y)$. Then $G(y)$ is not a well-order; but $y < \aleph_1$, hence $G(e_{y\aleph_1}) \in I\big(G(y), G(\aleph_1)\big)$ and since by hypothesis $G(\aleph_1)$ is a well-order, so is $G(y)$, a contradiction. $\square$)

8.2.8. Definition.
A **dilator** is a functor from **ON** to **ON** preserving direct limits and pullbacks.

8.2.9. Examples.

(i) If x is an ordinal, then $\underline{x}$ defined by

$$\underline{x}(y) = x \qquad \underline{x}(f) = E_x$$

is a dilator.

(ii) The functor Id defined by

$$\mathsf{Id}(x) = x \qquad \mathsf{Id}(f) = f$$

is a dilator.

(iii) The functors sum, product, exponential allow us to form new dilators: if F, F' are dilators, so are $F + F'$, $F \cdot F'$, $(1+F)^{F'}$, defined by

$$(f + F')(x) = F(x) + F'(x) \qquad (F + F')(f) = F(f) + F'(f)$$
$$(F \cdot F')(x) = F(x) \cdot F'(x) \qquad (F \cdot F')(f) = F(f) \cdot F'(f)$$
$$(1+F)^{F'}(x) = \big(1 + F(x)\big)^{F'(x)} \quad (1+F)^{F'}(f) = \big(1 + F(f)\big)^{F'(f)}$$

These combinations are dilators because of the

8.2.10. <u>Theorem</u>.
The functors sum, product, exponential preserve direct limits and pull-backs.

<u>Proof</u>. The theorem is an obvious consequence of an analogue of the normal form theorem 8.2.3 for binary functors. It suffices to show that if F is any of the functors sum, product, exponential, then any $z = F(x,y)$ can be written as $F(f,g)(z_0)$, for some $f \in I(n,x)$, $g \in I(m,y)$ and $z_0 \in F(n,m)$ such that $z_0 = F(f',g')(z_1) \rightarrow f' = \mathbf{E}_n$ and $g' = \mathbf{E}_m$, moreover, one must show that z_0, n, m, f, g are uniquely determined by these conditions. We content ourselves with exhibiting z_0, n, m, f, g, and the unicity is left to the reader:

(i) if $z < x + y$, and: $+z < x$; let $n = 1$, $m = 0$, $z_0 = 0$, $f \in I(1,x)$ be defined by $f(0) = z$, $g = \mathbf{E}_{0y}$. $+z \geq x$; write $z = x + z'$; let $n = 0$, $m = 1$, $z_0 = 0$, $f = \mathbf{E}_{0x}$, $g \in I(1,y)$ with $g(0) = z'$.

(ii) if $z < x \cdot y$, write $z = x \cdot u + v$, with $u < y$, $v < x$; let $n = m = 1$, $z_0 = 0$, $f \in I(1,x)$ be defined by $f(0) = v$, $g \in I(1,y)$ be defined by $g(0) = u$.

(iii) if $z < (1{+}x)^y$, write $z = (1{+}x)^{y_{p-1}}(1{+}u_{p-1}){+}...{+}(1{+}x)^{y_0}(1{+}u_0)$ with $u_0, ..., u_{p-1} < x$, $y_0 < ... < y_{p-1} < y$; let $m = p$ and let $g \in I(p,y)$ be defined by $g(0) = y_0, ..., g(p-1) = y_{p-1}$; let n and $f \in I(n,x)$ be such that $rg(f) = \{u_0, ..., u_{p-1}\}$, and let $k_0, ..., k_{p-1}$ be such that $f(k_0) = u_0, ..., f(k_{p-1}) = u_{p-1}$; then define $z_0 = (1+n)^{p-1} \cdot (1+k_{p-1}) + ... + (1+n)^0 \cdot (1+k_0)$. □

8.2.11. <u>Remarks</u>.

(i) One of the most remarkable features of direct limits is their good behaviour w.r.t. questions of double limits, function spaces (in contrast with topological continuity). Results established for unary functors can usually be extended without any problems to the case of binary functors.

(ii) For instance, any functor from $(\mathbf{ON} < \omega)^2$ to $\mathbf{ON}$ preserving pull-backs can be extended into a functor from $\mathbf{OL}^2$ to $\mathbf{OL}$ preserving direct limits and pull-backs. The obvious idea is to use two variable normal forms. If one applies this to sum, product, exponential, this enables us to define these operations on arbitrary linear orders. It turns out that this way of extending these functors to $\mathbf{OL}^2$ exactly coincides with our definition of Chapter 5.

8.2.12. Examples.

(i) $\tilde{}$, as a functor from $\mathbf{OL}$ to $\mathbf{OL}$, preserves direct limits and pull-backs.

(ii) $\hat{}$ is a functor from $\mathbf{ON}$ to $\mathbf{ON}$; but is preserves neither direct limits, nor pull-backs.

(Proof. The functor $\underline{1} + \mathsf{Id}$ coincides with $\hat{}$ on the category $\mathbf{ON} < \omega$; this functor preserves direct limits, hence, if $\hat{}$ would preserve direct limits one should have $\hat{} = \underline{1} + \mathsf{Id}$, by unicity of the extension by means of direct limits; but $\hat{\omega} = \omega + 1$ whereas $(\underline{1} + \mathsf{Id})(\omega) = \omega$. So $\hat{}$ does not preserve direct limits (this can be seen from the definition of $\hat{f}$, which is not "finitary"). Let $f, g \in I(\omega, \omega)$ be defined by $f(n) = 2n$, $g(n) = 2n + 1$. Then $f \wedge g = \mathbf{E}_{0\omega}$, hence $(\widehat{f \wedge g}) = \mathbf{E}_{1\omega+1}$; on the other hand $\hat{f} \wedge \hat{g} = \mathbf{E}_{0\omega} + \mathbf{E}_1$, hence $\hat{}$ does not preserve pull-backs.$\square$)

8.2.13. Definition.

Let $f, g \in I(x, y)$; then $f \leq g$ means that: $\forall z \in x \Big(f(z) \leq g(z)\Big)$.

8.2.14. Lemma.

If x, y are ordinals, if $f, g \in I(x, y)$, then $f \leq g$ iff there exist an ordinal z and functions $h \in I(y, z)$, $k \in I(z, z)$, such that $hg = khf$.

Proof. (i) If $hg = khf$, then, for all $t < x$, $h\Big(g(t)\Big) = k\Big(h\Big(f(t)\Big)\Big)$; but, since z is an ordinal, $k(u) \geq u$ for all u, hence $h\Big(g(t)\Big) \geq h\Big(f(t)\Big)$, and from this $g(t) \geq f(t)$.

(ii) Conversely, let $z = \omega^{2y}$, $h(t) = \omega^{2t+1}$; if $u < z$, write $u = u' + r$, with $u' = \mathsf{sup}\, \{\omega^{2f(t)+1}\,;\, \omega^{2f(t)+1}\}$, and let $v' = \mathsf{sup}\, \{\omega^{2g(t)+1}\,;\, \omega^{2f(t)+1} \leq u\}$, and let $k(u) = v' + r$; observe that $hg = khf$, so it remains to show that k is strictly increasing: assume that (with obvious notations) $u_1 + r_1 < u_2 + r_2$; then

- if $u_1 = u_2$, then $v_1 = v_2$, hence $v_1 + r_1 < v_2 + r_2$.
- if $u_1 < u_2$, then $v_1 < v_2$; choose t such that $r_1 < \omega^{2f(t)+1} \leq u_2$; then $r_1 < \omega^{2f(t)+1} \leq \omega^{2g(t)+1} \leq v_2$, hence $v_1 + r_1 < v_2 \leq v_2 + r_2$. □

8.2.15. Theorem.
Assume that F is a dilator; then $f \leq g \rightarrow F(f) \leq F(g)$.

Proof. If $f \leq g$, then write $hg = khf$, hence $F(h)F(g) = F(k)F(h)F(f)$, and so $F(f) \leq F(g)$. □

8.2.16. Definition.
A **predilator** is a functor from **OL** to **OL** preserving direct limits and pull-backs, and such that $\forall x \forall y \forall f \forall g \ \big(f, g \in I(x, y) \wedge f \leq g \rightarrow F(f) \leq F(g)\big)$.

8.2.17. Example.
The functor $\tilde{}$ is the typical example of a functor from **OL** to **OL** preserving direct limits and pull-backs, and which is not a predilator: if $f \leq g$, then $\tilde{g} \leq \tilde{f}$!

8.2.18. Proposition.
A functor F from **OL** to **OL** preserving direct limits and pull-backs is a predilator iff the denotations w.r.t. F are increasing in the coefficients, i.e. iff $x_0 \leq x'_0, ..., x_{n-1} \leq x_{n-1}$, and $x'_{n-1} < x \rightarrow (z_0\,; x_0, ..., x_{n-1}\,; x)_F \leq (z_0\,; x'_0, ..., x'_{n-1}\,; x)_F$.

Proof. If $f \leq g$, and F is a predilator, then $F(f)\big((z_0\,; 0, ..., n-1\,; n)_F\big) \leq F(g)\big((z_0\,; 0, ..., n-1\,; n)\big)$: define $f \in I(n, x)$ by $f(0) = x_0, ..., f(n-1) = x_{n-1}$, and $g \in I(n, x)$ by $g(0) = x_0, ..., g(n-1) = x_{n-1}$, then $f \leq g$ and so we get: $(z_0\,; x_0, ..., x_{n-1}\,; x)_F \leq (z_0\,; x'_0, ..., x'_{n-1}\,; x)_F$. Conversely, assume that $f \leq g$, and that the denotations w.r.t. F are increasing in the coefficients. Then if $z = (z_0\,; y_0, ..., y_{n-1}\,; y)$ and $f, g \in I(y, x)$, we get $F(f)(z) = (z_0\,; f(y_0), ..., f(y_{n-1})\,; x)$ and $F(g)(z) = (z_0\,; g(y_0), ..., g(y_{n-1})\,;$

x), and clearly $F(f)(z) \leq F(g)(z)$. □

8.2.19. Corollary.
A functor F from **OL** to **OL** preserving direct limits and pull-backs is a predilator iff its restriction to **ON** $< \omega$ enjoys the property $f \leq g \to F(f) \leq F(g)$.

Proof. In order to compare $(z_0\,;\, x_0, ..., x_{n-1}\,;\, x)_F$ and $(z_0\,;\, x'_0, ..., x'_{n-1}\,;\, x)_F$, when $x_0 \leq x'_0, ..., x_n \leq x'_{n-1} < x$, one can assume that x is an integer: this is a consequence of the:

8.2.20. Theorem.
Let F be a functor from **ON** (or **OL**, **ON** $\leq x$) to **OL**, and assume that $(z_0\,;\, x_0, ..., x_{n-1}\,;\, x)_F$ and $(z_1\,;\, y_0, ..., y_{m-1}\,;\, x)_F$ are denotations for elements $< F(x)$. Then the order relation between these two elements is completely determined by: (if one knows F) z_0, n, z_1, m and the order relations between the points x_i and y_j.

Proof. This means that given any two strictly increasing sequences $x'_0 < ... < x'_{n-1} < x'$ and $y'_0 < ... < y'_{m-1} < x'$ such that, for all i, j:

$$x_i < y_j \leftrightarrow \; x'_i < y'_j$$

and

$$y_j < x_i \leftrightarrow \; y'_j < x'_i$$

then

$$t < u \quad \leftrightarrow \; t' < u' \;,$$

where $t = (z_0\,;\, x_0, ..., x_{n-1}\,;\, x)_F$, $u = (z_1\,;\, y_0, ..., y_{m-1}\,;\, x)_F$, $t' = (z_0\,;\, x'_0, ..., x'_{n-1}\,;\, x')_F$, $u' = (z_1\,;\, y'_0, ..., y'_{m-1}\,;\, x)_F$. Define an integer p (resp. p') and a function $f \in I(p, x)$ (resp. $f' \in I(p', x')$) by: $rg(f) = \{x_0, ..., x_{n-1}\} \cup \{y_0, ..., y_{m-1}\}$ (resp. $rg(f') = \{x'_0, ..., x'_{n-1}\} \cup \{y'_0, ..., y'_{m-1}\}$). Then one can find $t_0, u_0 \in F(p)$, $t'_0, u'_0 \in F(p')$ such that $t = F(f)(t_0)$, $u = F(f)(u_0)$, $t' = F(f')(t'_0)$, $u' = F(f')(u'_0)$. Hence it suffices to show that $t_0 < u_0 \leftrightarrow t'_0 < u'_0$. Now observe that the hypothesis $x_i < y_j \leftrightarrow x'_i < y'_j$ and $y_j < x_i \leftrightarrow y'_j < x'_i$ implies that the sets $\{x_0, ..., x_{n-1}\} \cup \{y_0, ..., y_{m-1}\}$ and $\{x'_0, ..., x'_{m-1}\} \cup \{y'_0, ..., y'_{m-1}\}$ have the same order type, hence $p = p'$; for the same reason $f^{-1}(x_0) = f'^{-1}(x'_0), ..., f^{-1}(x_{n-1}) = f'^{-1}(x'_{n-1})$,

$f^{-1}(y_0) = f'^{-1}(y'_0), ..., f^{-1}(y_{m-1}) = f'^{-1}(y'_{m-1})$, hence $t_0 = t'_0$, $u_0 = u'_0$. □

end of the proof of 8.2.19.: one can find integers $y_0, ..., y_{n-1}, y'_0, ..., y'_{n-1}, y$ such that $x_i < x'_j \leftrightarrow y_i < y'_j$ and $x'_j < x_i \leftrightarrow y'_j < y_i$, hence $y_i \leq y'_i$ for all i, hence $(z_0\,;\, y_0, ..., y_{n-1}\,;\, y)_F \leq (z_0\,;\, y'_0, ..., y'_{n-1}\,;\, y)_F$, and by 8.2.20 $(z_0\,;\, x_0, ..., x_{n-1}\,;\, x)_F \leq (z_0\,;\, x'_0, ..., x'_{n-1}\,;\, x)_F$. □

8.2.21. Corollary.
If F is a dilator, then "the" extension of F into a functor from **OL** to **OL** is a predilator.

Proof. By 8.2.19 and 8.2.15. □

8.2.22. Remark.
Hence dilators can be viewed as special cases of predilators, just as well-orders are special cases of linear orders. The relation between the categories **DIL** and **PIL** of dilators and predilators is exactly the same as the relation between **ON** and **OL**: **OL** (resp. **PIL**) is "the closure" of **ON** (resp. **DIL**) w.r.t. direct limits. Direct limits in **DIL** and **PIL** will be investigated later on (see Sec. 8.3).

8.2.23. Definition.
A dilator is **weakly finite** iff $F(n)$ is finite for all n. A weakly finite dilator is **recursive** (resp. **primitive recursive**) iff the function $\ulcorner F \urcorner$ which associates to any sequence $\langle x_0, ..., x_{n-1}\,;\, m\rangle$ with $x_0 < ... < x_{n-1} < m$ (this sequence is the code for the function $f \in I(n, m) :\ f(0) = x_0, ..., f(n-1) = x_{n-1}$) the sequence encoding $F(f)$, is recursive (resp. prim. rec.). (The definition of $\ulcorner f \urcorner$ when $f \in I(n, m)$ is given in 8.1.22 (iii).

8.2.24. Proposition.
Assume that F is a recursive weakly finite dilator, and let $X = (|X|, \leq^1)$ be a recursive well-order; then $F(X)$ is (isomorphic to) a recursive well-order.

Proof. One can represent all elements of $F(X)$ by sequence $(z_0\,;\, x_0, ..., x_{n-1}\,;\, X)_F$ with:

(i) $(z_0\,;\, 0, ..., n-1\,;\, n)_F$ is a F-denotation.

(ii) $x_0 \;<^1\; ... \;<^1\; x_{n-1}$.

The condition (i) is perfectly recursive: it can be translated as: $lh\Big(\ulcorner F \urcorner(\langle 0, ..., n\rangle)\Big) > z_0 + 1 \wedge \Big(n = 0 \vee n \neq 0 \wedge \forall s\ (lh(s) \neq 0 \wedge \forall i < lh(s) - 1\ (x)_i < (x)_{i+1} \wedge (s)_{lh(s)-1} = n \wedge \exists p < lh(s) - 1\ z_0 \Big(\ulcorner F \urcorner(s)\Big)_p \rightarrow lh(s) = n+1\Big)$.

The comparison of two elements of $F(X)$ is given by

$$(z_0\,;\, x_0, ..., x_{n-1}\,;\, X) \;\leq^{F(X)}\; (z_1\,;\, x'_0, ..., x'_{m-1}\,;\, X)$$

iff

$$(z_0\,;\, p_0, ..., p_{n-1}\,;\, n+m) \;\leq^{F(n+m)}\; (z_1\,;\, p'_0, ..., p'_{m-1}\,;\, n+m)\ ,$$

where $p_0, ..., p_{m-1}, p'_0, ..., p'_{m-1}$ are integers such that:

$$p_i < p'_j \leftrightarrow x_i < x'_j$$

and

$$p'_j < p_i \leftrightarrow x'_j < x_i\ ;$$

the construction of the sequences $p_0, ..., p_{n-1}$ and $p'_0, ..., p'_{n-1}$ can be done from $x_0, ..., x_{n-1}$ and $x'_0, ..., x'_{n-1}$ in a recursive way, since the ordering of X is recursive. Finally $(z_0\,;\, p_0, ..., p_{n-1}\,;\, n{+}m)_F$ is exactly $\Big(\ulcorner F \urcorner(\langle p_0, ..., p_{n-1}, n+m\rangle)\Big)_{z_0}$, i.e. is obtained from $\langle p_0, ..., p_{n-1}\rangle, z_0$ in a recursive way, similarly for $(z_1\,;\, p'_0, ..., p'_{m-1}\,;\, n+m)$, hence the ordering is recursive. $\square$

8.2.25. Remarks.

(i) Obviously 8.2.24 holds when "primitive recursive" is substituted everywhere for "recursive".

(ii) One can imagine a more general meaning of "recursive dilator", namely when the dilator is not necessarily weakly finite, and the requirement is that the functions:

– to each integer n the code of the recursive well-order $F(n)$

– to each $\ulcorner f \urcorner$ $(f \in I(n, m))$ the code of the recursive function $F(f)$

are recursive. This concept still enjoys 8.2.24.

8.3. Trace of a dilator

8.3.1. Definition.
Assume that F is a dilator (resp. a predilator) then the **trace** of F, denoted by $\mathsf{Tr}(F)$, is the set of all pairs (z_0, n) such that $(z_0\,;\,0,\ldots,n-1\,;\,n)_F$ is a F-denotation.

8.3.2. Definition.
We shall consider in the sequel the following categories:

> **DIL**, the category of dilators: the objects are dilators.
> **PIL**, the category of predilators: the objects are predilators.

In both cases, the morphisms from F to G are given by the set $I^1(F,G)$ of all natural transformations from F to G. $I^1(F,G)$ is a set because $T \in I^1(F,G)$ is completely determined by $T(\omega)$. (We recall that if $T \in I^1(F,G)$, $U \in I^1(G,H)$, then $UT \in I^1(F,H)$ is defined by $(UT)(x) = U(x)T(x)$.)

8.3.3. Definition.
Assume that F, G are objects of **DIL** (resp. **PIL**), and that $T \in I^1(F,G)$; then one defines a function $\mathsf{Tr}(T)$ from $\mathsf{Tr}(F)$ to $\mathsf{Tr}(G)$ by:

$$\mathsf{Tr}(T)\Big((z_0,n)\Big) = (T(n)(z_0),n)\ .$$

8.3.4. Remarks.

(i) $\mathsf{Tr}(T)$ is well-defined because of 8.2.5 (ii).

(ii) tr is a functor from **DIL** (or **PIL**) to the category **SET** of sets; less pedantically this means that $\mathsf{Tr}(T)$ is the identity of $\mathsf{Tr}(F)$ when T is the identity of F, and that $\mathsf{Tr}(Tu) = \mathsf{Tr}(T)\mathsf{Tr}(U)$.

8.3.5. Examples.

(i) If F is the constant dilator $\underline{x}$, then $\mathsf{Tr}(\underline{x}) = \{(z,0)\,;\,z < x\}$; if $f \in I(x,y)$, define a natural transformation $\underline{f}$ from $\underline{x}$ to $\underline{y}$ by: $\underline{f}(a)(z) = f(z)$ for all $a \in 0n$. Then $\mathsf{Tr}(\underline{f})$ is defined by $\mathsf{Tr}(\underline{f})\Big((z,0)\Big) = f(z),0)$.

(ii) F is by no means determined by its trace; but if F and G are known, then any $T \in I^1(F,G)$ is completely determined by $\mathsf{Tr}(T)$, since

$$T(x)\Big((z_0\,;\,x_0,...,x_{n-1}\,;\,x)_F\Big) = (z_1\,;\,x_0,...,x_{n-1}\,;\,x)_G\ ,$$

with $\mathsf{Tr}(T)\Big((z_0,n)\Big) = (z_1,n)$.

8.3.6. Theorem.
Let G be a dilator (resp. a predilator); then given any subset X of $\mathsf{Tr}(G)$ one can find a unique dilator F (resp. a unique-up-to-isomorphism predilator F) and $T \in I^1(F,G)$ such that $rg\Big(\mathsf{Tr}(T)\Big) = X$.

Proof. Consider for all x the subsets $H(x)$ of $G(x)$, consisting of all denotations $(z_0\,;\,x_0,...,x_{n-1}\,;\,x)_G$, with $(z_0\,;\,n) \in X$, and let $H(f)$ be the function from $H(x)$ to $H(y)$ obtained by restriction of $G(f)$. It is immediate that H preserves direct limits, pull-backs and morphisms: if we start with a predilator G, H is therefore a predilator, and $rg(H) = X$; the transformation T is just the inclusion $T(x)$ from $H(x)$ into $G(x)$, and $\mathsf{Tr}(T)$ is just the inclusion map from $\mathsf{Tr}(H) = X$ into $\mathsf{Tr}(G)$. When G is a dilator, simply replace H and T defined as above by: $H'(x) = \|H(x)\|$, $H'(f) = \|H(f)\|$, $T'(x) = \|T(x)\|, \ldots$. H' is clearly a dilator, and $T' \in I^1(H',G')$ is such that $rg\Big(\mathsf{Tr}(T')\Big) = X$. Unicity conditions are left to the reader... . □

8.3.7. Theorem.
Let (F_i, T_{ij}) be a direct system in **DIL** (resp. in **PIL**) and let (F, T_i) be a family enjoying 8.1.11 (i)–(iii) w.r.t. (F_i, T_{ij}); then the following conditions are equivalent:

(i) $(F,T_i) = \varinjlim\ (F_i, T_{ij})$.

(ii) $\mathsf{Tr}(F) = \bigcup_i\ rg\Big(\mathsf{Tr}(T_i)\Big)$.

(iii) For all n $\Big(F(n), T_i(n)\Big) = \varinjlim\ \Big(F_i(n), T_{ij}(n)\Big)$.

(iv) For all direct systems (x_l, f_{lm}) indexed by L, with a direct limit (x, f_l) in **ON** (resp. in **OL**), one has

$$\Big(F(x), T_i(f_l)\Big) = \varinjlim_{I \times L} \Big(F_i(x_l), T_{ij}(f_{lm})\Big)\ .$$

As usual, one sets: $T(f) = G(f)T(x) = T(y)F(f)$ when $f \in I(x,y)$, and $T \in I^1(F,G)$.

<u>Proof</u>. (i) $\to$ (ii): Define $X \subset \mathsf{Tr}(F)$ by $X = \bigcup_i rg\Big(\mathsf{Tr}(T_i)\Big)$; define H and $U \in I^1(H,F)$ by $rg\Big(\mathsf{Tr}(U)\Big) = X$; one can surely write: $\mathsf{Tr}(T_i) = \mathsf{Tr}(U)u_i$ for some function u_i from $\mathsf{Tr}(F_i)$ to $\mathsf{Tr}(H)$; one easily constructs $U_i \in I^1(F_i, H)$ such that $\mathsf{Tr}(U_i) = u_i$, and we have $T_i = UU_i$. The family (H, U_i) enjoys 8.1.11 (i)–(iii), hence one can find $V \in I^1(F,H)$ such that $U_i = VT_i$ for all i: hence $T_i = UVT_i$, i.e. $\mathsf{Tr}(T_i) = \mathsf{Tr}(U)\mathsf{Tr}(V)\mathsf{Tr}(T_i)$ for all i: for all $a \in X$ $\mathsf{Tr}(U)\mathsf{Tr}(V)(a) = a$. We know that $rg\Big(\mathsf{Tr}(U)\Big) = X$: hence for $b \notin X$, $\mathsf{Tr}(U)\mathsf{Tr}(V) = a$ must belong to X: $\mathsf{Tr}(U)\mathsf{Tr}(V)(a) = \mathsf{Tr}(U)\mathsf{Tr}(V)(b)$, contradicting the obvious injectivity of $\mathsf{Tr}(U)$ and $\mathsf{Tr}(V)$. Hence $X = \mathsf{Tr}(F)$.

(ii) $\to$ (iv): The double direct system index by the product $I \times L$, and the family $\Big(F(x), T_i(f_l)\Big)$ obviously enjoys 8.1.11 (i)–(iii); we prove condition (iv)$'$: if $z \in F(x)$, write $z = (z_0\,; x_0, ..., x_{n-1}\,; x)$; choose an index i such that $(z_0\,; n) \in rg\Big(\mathsf{Tr}(T_i)\Big)$, and an index $l \in L$ such that $x_0, ..., x_{n-1} \in rg(f_l)$; then it is immediate (see Remark 8.1.8 (ii)) that $(z_0\,; x_0, ..., x_{n-1}\,; x) \in rg\Big(T_i(f_l)\Big)$.

(iv) $\to$ (iii): Immediate by considering $x_i = x = n$, $f_i = f_{ij} = \mathbf{E}_n$.

(iii) $\to$ (i): Assume that $\Big(F(n), T_i(n)\Big) = \varinjlim \Big(F_i(n), T_{ij}(n)\Big)$ for all n and let (G, U_i) be any family enjoying 8.1.11 (i)–(iii) w.r.t. (F_i, T_{ij}); if $f \in I(n,m)$, then $\Big(G(m), U_i(f)\Big)$ enjoys 8.1.11 (i)–(iii) w.r.t. $\Big(F_i(n), T_{ij}(n)\Big)$; hence there exists a unique function $V(f) \in I\Big(F(n), G(m)\Big)$ such that $U_i(f) = V(f)T_i(n)$. By unicity, one easily gets $G(g)V(f) = V(g)F(f)$, i.e. the family $V(n) = V(\mathbf{E}_n)$ defines a natural transformation V from $F\upharpoonright \mathbf{ON} < \omega$ to $G\upharpoonright \mathbf{ON} < \omega$. Such a natural transformation obviously extends to an element $V \in I^1(F,G)$, which is the unique solution of $U_i = VT_i$. □

8.3.8. <u>Remarks</u>.

(i) 8.3.7 clearly shows that the concept of direct limit of dilators, predilators, is very easy to handle: we have two ways of handling it:

- by means of the "pointwise" direct limits 8.3.7 (iii).
- by means of the traces (8.3.7 (ii)). Observe that this condition means that $\Big(\mathsf{Tr}(F), \mathsf{Tr}(T_i)\Big) = \varinjlim \Big(\mathsf{Tr}(F_i), \mathsf{Tr}(T_{ij})\Big)$ in the category **SET** of sets.

(ii) Given $rg\Big(\mathsf{Tr}(T)\Big)$ and $rg(f)$, it is easy to determine $rg\Big(T(f)\Big)$: since

$$T(f)\Big((z_0\,;\,x_0,...,x_{n-1}\,;\,x)_F\Big) \;=$$
$$(T(n)(z_0)\,;\,f(x_0),...,f(x_{n-1})\,;\,y)_G$$

one gets, provided $T \in I^1(F,G)$, $f \in I(x,y)$:

$$rg\Big(T(f)\Big) = \Big\{(z_1\,;\,y_0,...,y_{n-1}\,;\,y)\,;\,(z_1\,;\,n) \in rg\Big(\mathsf{Tr}(T)\Big) \wedge y_0,...,y_{n-1} \in rg(f)\Big\}\ .$$

8.3.9. <u>Theorem</u>.

(i) In **PIL**, every system has a direct limit.

(ii) In **DIL**, if (F_i, T_{ij}) and (F, T_i) enjoy 8.1.11 (i)–(iii), then (F_i, T_{ij}) has a direct limit.

<u>Proof</u>. (i) If (F_i, T_{ij}) is a direct system in **PIL**, let $\Big(F(n), T_i(n)\Big) = \varinjlim \Big(F_i(n), T_{ij}(n)\Big)$: $F(n)$ exists as a linear order in **OL**. If $f \in I(n,m)$, consider the system $F_i(f)$ indexed by the identity function from I to I and let $F(f) \in I\Big(F(n), F(m)\Big)$ be the direct limit of the $F_i(f)$'s. One easily checks that F is the restriction to $\mathbf{ON} < \omega$ of a predilator (still denoted by F) and that $T_i \in I^1(F_i, F)$. Conditions 8.1.11 (i)–(iii) are fulfilled, together with 8.3.7 (iii), hence $(F, T_i) = \varinjlim (F_i, T_{ij})$.

(ii) Define X, H, U, U_i exactly as in the proof of 8.3.7 (i) $\rightarrow$ (ii); then it is immediate that $\mathsf{Tr}(U) = \bigcup\limits_i rg\Big(\mathsf{Tr}(U_i)\Big)$, hence $(H, U_i) = \varinjlim (F_i, T_{ij})$.

□

8.3.10. Theorem.
Let $T_1 \in I^1(F_1, G)$, $T_2 \in I^1(F_2, G)$, $T_3 \in I^1(F_3, G)$; then the following conditions are equivalent:

(i) $T_3 = T_1 \wedge T_2$.

(ii) $rg\big(\mathsf{Tr}(T_3)\big) = rg\big(\mathsf{Tr}(T_1)\big) \cap rg\big(\mathsf{Tr}(T_2)\big)$.

(iii) $T_3(n) = T_1(n) \wedge T_2(n)$ for all integers n.

(iv) For all x_1, x_2, x_3, y and $f_1 \in I(x_1, y)$, $f_2 \in I(x_2, y)$ and $f_3 \in I(x_3, y)$ such that $f_3 = f_1 \wedge f_2$, one has:

$$T_3(f_3) = T_1(f_1) \wedge T_2(f_2)\ .$$

Proof. (i) → (ii): Assume that $T_3 = T_1 \wedge T_2$, hence $T_3 = T_1 T_{31} = T_2 T_{32}$, for some T_{31} and T_{32}, hence $\mathsf{Tr}(T_3) = \mathsf{Tr}(T_1)\mathsf{Tr}(T_{31}) = \mathsf{Tr}(T_2)\mathsf{Tr}(T_{32})$, and this implies $rg\big(\mathsf{Tr}(T_3)\big) \subset rg\big(\mathsf{Tr}(T_1)\big) \cap rg\big(\mathsf{Tr}(T_2)\big)$. Define F'_3 and $T'_3 \in I^1(F'_3, G)$ by $rg\big(\mathsf{Tr}(T'_3)\big) = rg\big(\mathsf{Tr}(T'_1)\big) \cap rg\big(\mathsf{Tr}(T'_2)\big)$. It is easy to find $T'_{31} \in I^1(F'_3, F_1)$ and $T'_{32} \in I^1(F'_3, F_2)$ such that $T'_3 = T_1 T'_{31} = T_2 T'_{32}$, and $U \in I^1(F_3, F'_3)$ such that $T_3 = T'_3 U$. (T'_{31}, T'_{32}, U can be easily obtained by means of their traces; see 8.3.11 (ii).) Since $T_3 = T_1 \wedge T_2$, there exists $V \in I^1(F'_3, F_3)$ such that $T'_{31} = T_{31} V$, $T'_{32} = T_{31} V$, hence $T'_3 = T_3 V$: from that we obtain $T_3 = T_3 V U$ and $T'_3 = T'_3 U V$. $\mathsf{Tr}(T_3) = \mathsf{Tr}(T_3)\mathsf{Tr}(V)\mathsf{Tr}(U)$ entails (since the function $\mathsf{Tr}(T_3)$ is injective) $\mathsf{Tr}(V)\mathsf{Tr}(U) =$ identity of $\mathsf{Tr}(F_3)$, hence VU is the identity of F_3, similarly UV is the identity of F'_3, and this forces $rg\big(\mathsf{Tr}(T'_3)\big)$ to be equal to $rg\big(\mathsf{Tr}(T_3)\big)$.

(ii) → (iv): Assume that $rg\big(\mathsf{Tr}(T_3)\big) = rg\big(\mathsf{Tr}(T_1)\big) \cap rg\big(\mathsf{Tr}(T_2)\big)$; and that $rg(f_3) = rg(f_1) \cap rg(f_2)$; then, by Remark 8.3.8 (ii): $rg\big(T_3(f_3)\big) = \big\{(z_0\,;\, x_0, ..., x_{n-1}\,;\, y)_G\,;\, (z_0\,;\, n) \in rg\big(\mathsf{Tr}(T_3)\big) \wedge x_0, ..., x_{n-1} \in rg(f_3)\big\} = rg\big(T_1(f_1)\big) \cap rg\big(T_2(f_2)\big)$.

(iv) → (iii): Take $f_3 = f_1 = f_2 = \mathbf{E}_n$.

(iii) → (i): Assume that $T_3(n) = T_1(n) \wedge T_2(n)$ for all n, and let $T'_3 \in I^1(F'_3, G)$, $T'_{31} \in I^1(F'_3, F_1)$, $T'_{32} \in I^1(F'_3, F_2)$ be such that 8.1.24

(i) holds, i.e. $T'_3 = T_1 T'_{31} = T_2 T'_{32}$; then we have when $f \in I(m,n)$ $T'_3(f) = T_1(n)T'_{31}(f) = T_2(n)T'_{32}(f)$, and by condition 8.1.24 (ii) of pull-backs: there exists a unique morphism $U(f) \in I\Big(F'_3(m), F_3(n)\Big)$ such that $T'_{31}(f) = T_{31}(n)U(f)$ and $T'_{32}(f) = T_{32}(n)U(f)$; unicity of $U(f)$ implies that

$$F_3(g)U(f) = U(g)F'_3(f)$$

when $g \in I(n,p)$, i.e. $U(n) = U(\mathbf{E}_n)$ is the restriction to $\mathbf{ON} < \omega$ of a natural transformation from F'_3 to F_3, still denoted by U. Hence we get $T'_{31} = T_{31}U$ and $T'_{32} = T_{32}U$, and U is clearly uniquely determined by these conditions, since U is determined by its restriction to $\mathbf{ON} < \omega$. $\square$

8.3.11. <u>Remarks</u>.

(i) 8.3.10 gives us two ways of computing pull-backs in the categories **DIL** and **PIL**:

  - by means of an intersection (8.3.10 (ii)); this means that $\mathsf{Tr}(T_1 \wedge T_2) = \mathsf{Tr}(T_1) \wedge \mathsf{Tr}(T_2)$ in the category **SET** of sets.
  - by means of the "pointwise" pull-backs $T_1(n) \wedge T_2(n)$.

(ii) We have several times made an implicit use of the following principle: assume that $T \in I^1(F,H)$, $U \in I^1(G,H)$ are such that $rg\Big(\mathsf{Tr}(T)\Big) \subset rg\Big(\mathsf{Tr}(U)\Big)$; then there exists a unique $V \in I^1(F,G)$ such that $T = UV$.

(<u>Proof</u>. If $u = \mathsf{Tr}(U)$, $t = \mathsf{Tr}(T)$, then one can find a unique v such that $t = uv$; the mapping v from $\mathsf{Tr}(F)$ to $\mathsf{Tr}(G)$ induces for all x a mapping $V(x)$ from $F(x)$ to $G(x)$, by

$$V(x)\Big((z_0\,;\,x_0,...,x_{n-1}\,;\,x)_F\Big) = (z_1\,;\,x_0,...,x_{n-1}\,;\,x)_G\ ,$$

with $(z_1,n) = v(z_0,n)$; one easily checks that $V(x) \in I\Big(F(x),G(x)\Big)$, i.e. that $V(x)$ is strictly increasing. The property $V(f)F(x) = G(y)V(f)$ is immediate as well. $\square$)

8.3.12. Theorem.

In **PIL** and **DIL**, pull-backs always exist. Moreover, in **DIL**, they are uniquely determined.

Proof. Immediate from 8.3.10 (ii) and 8.3.6. $\square$

8.4. Finite dimensional dilators

8.4.1. Definition.

Let F be a dilator (or a predilator); then the **dimension** of F, $\mathsf{dim}(F)$ is by definition the cardinal of $\mathsf{Tr}(F)$.

8.4.2. Remarks.

(i) If $T \in I^1(F, G)$, then $\mathsf{dim}(F) \le \mathsf{dim}(G)$.

(ii) A dilator (or a predilator) is **finite dimensional** iff its dimension is finite. Finite dimensional dilators form a full subcategory $\mathbf{DIL}_{fd}$ of **DIL**.

(iii) It is not possible to replace the cardinal $\mathsf{dim}(F)$ by an ordinal: there is no natural way of well-ordering $\mathsf{Tr}(F)$, even when F is a dilator. (However, we shall see that $\mathsf{Tr}(F)$ is naturally linearly ordered, see 8.4.22; but even when F is a dilator, this linear order need not to be a well-order.)

8.4.3. Theorem.

Let F be a finite dimensional predilator; then F is isomorphic to a dilator G. (This G is finite dimensional, and uniquely determined.)

Proof. We first prove that $F(x)$ is a well-order for all $x \in 0n$. Let (s_n) be a s.d.s. in $F(x)$, and let us write $s_n = (z_n\,;\, x_0^n, ..., x_{p_n-1}^n\,;\, x)_F$; the sequence (z_n, p_n) varies through the finite set $\mathsf{Tr}(F)$, hence one can find a subset $I \subset I\!N$, I infinite such that $(z_n, p_n) = (z, p) =$ constant for all $n \in I$; by renumbering the subsequence $(s_n)_{n\in I}$ one can find a s.d.s. (s'_n) of the form $(z\,;\, y_0^n, ..., y_{p-1}^n\,;\, x)_F$ in $F(x)$. Define a partition $I(2,\omega) = C_0 \cup ... \cup C_{p-1}$ by: $f \in C_i$ iff i is the smallest integer such that $y_i^{f(0)} > y_i^{f(1)}$. (Such an integer i exists, because if $y_i^{f(0)} \le y_i^{f(1)}$, this would entail $s'_{f(0)} \le s'_{f(1)}$ by 8.2.18.) By Ramsey's theorem, there exists an infinite set X and an integer $i_0 < p$ such that if $f(0), f(1) \in X$, then $f \in C_{i_0}$. Then the infinite sequence $(y_{i_0}^n)_{n\in X}$ is strictly decreasing in X, a contradiction. The theorem holds with $G(x) = \|F(x)\|$, $G(f) = \|F(f)\|$. □

8.4.4. Theorem.

$\mathbf{DIL}_{fd}$ is "dense" in **DIL** and **PIL** w.r.t. direct limits. In the case of **PIL**, $\mathbf{DIL}_{fd}$ is not, strictly speaking, a subcategory of **PIL**

(i) if F is any object in **DIL** or **PIL**, then one can find a direct system (F_i, T_{ij}), with all F_i's finite dimensional, together with morphisms T_i such that:

$$(F, T_i) = \varinjlim (F_i, T_{ij}) \ .$$

(ii) if F, G are objects of **DIL** (or **PIL**), and $(F, T_i) = \varinjlim (F_i, T_{ij})$, $(G, U_l) = \varinjlim (G_l, T_{lm})$ with F_i and G_l finite dimensional for all i and l, then given any $V \in I^1(F, G)$, one can find an increasing function φ together with a direct system of morphisms $(V_i)_{i\in\varphi}$, such that

$$V = \varinjlim (V_i) \ .$$

Proof. (i) Define $I = \{a\,;\, a \text{ finite}, \ a \subset \mathsf{Tr}(F)\}$. I (ordered by inclusion) is clearly directed. If $a \in I$ define F_a and $T_a \in I^1(F_a, F)$ by $rg\big(\mathsf{Tr}(T_a)\big) = a$; F_a is finite dimensional. Now, since when $a \subset b$, $rg\big(\mathsf{Tr}(T_a)\big) \subset rg\big(\mathsf{Tr}(T_b)\big)$, it is possible to define $T_{ab} \in I^1(F_a, F_b)$ by $T_a = T_b T_{ab}$ (Remark 8.3.11 (ii)); it is immediate that (F_i, T_{ij}) defines a direct system, and that (F, T_i) enjoys 8.1.11 (i)–(iii) w.r.t. (F_i, T_{ij}), hence it suffices to show 8.3.7 (ii); but if $(z_0, n) \in \mathsf{Tr}(F)$, clearly $(z_0, n) \in rg\big(\mathsf{Tr}(T_{\{(z_0,n)\}})\big)$.

(ii) is easily established on the model of 8.1.16 (ii). □

8.4.5. Remark.

We have therefore a strict analogy between:

$$\begin{cases} \mathbf{DIL}_{fd} & \mathbf{DIL} & \mathbf{PIL} \\ \mathbf{ON} < \omega & \mathbf{ON} & \mathbf{OL} \end{cases}$$

Similarly to $\mathbf{ON} < \omega$, $\mathbf{DIL}_{fd}$ has the property that the only systems having a direct limit *inside* the category are trivial.

8.4.6. Proposition.

Let F be a dilator; then F is finite dimensional iff the function $n \rightsquigarrow F(n)$ is a polynomial.

Proof. If F is finite dimensional, then $F(n) = a_0 + a_1 \cdot n + a_2 \cdot n(n-1)/2 + ... + a_k \cdot n(n-1)...(n-k+1)/k!$ with $k = \mathsf{sup}\,\{i\,;\, \exists z\ (z,i) \in \mathsf{Tr}(F)\}$, $a_i = \mathsf{card}\,\{z\,;\,(z,i) \in \mathsf{Tr}(F)\}$ (because the number of strictly increasing sequences $p_0 < ... < p_{i-1} < n$ is exactly $n(n-1)...(n-i+1)/i!$). Conversely, if the function $n \rightsquigarrow F(n)$ is a polynomial, it is still possible to write $F(n) = a_0 + a_1 n + ... + a_i \cdot n(n-1)...(n-i+1)/i! + ...$ with $a_i = \mathsf{card}\,\{z\,;\,(z,i) \in \mathsf{Tr}(F)\}$, and such an infinite sum of binomial polynomials is a polynomial iff almost all coefficients are equal to 0, and this forces the sum $\sum_i a_i = \mathsf{dim}(F)$ to be finite. □

8.4.7. Proposition.

Finite dimensional dilators are primitive recursive.

Proof. $F(k)$ can be viewed as the set of all denotations $(z_0\,;\, i_0,...,i_{n-1}\,;\,k)_F$, with $(z,n) \in \mathsf{Tr}(F)$ and $i_0 < ... < i_{n-1} < k$, and for $f \in I(k,k')$, $F(f)\big((z\,;\,i_0,...,i_{n-1}\,;\,k)_F\big) = (z\,;\,f(i_0),...,f(i_{n-1})\,;\,k')_F$, so all we need to show is that the ordering of $F(k)$ can be obtained as a prim. rec. function of k: but, in order to compare $(z\,;\,i_0,...,i_{n-1}\,;\,k)_F$ with $(z'\,;\,j_0,...,j_{m-1}\,;\,k)_F$, by 8.2.20 it is sufficient to compare $(z\,;\,i'_0,...,i_{n-1}\,;\,l)_F$ with $(z'\,;\,j'_0,...,j'_{m-1}\,;\,l)_F$ where the sequences i'_r and j'_s are such that $i'_r < j'_s \leftrightarrow i_r < j_s$; the problem is to find a uniform value for l; the only requirement on l is that $l \geq n+m$ (because if $l \geq n+m$, one can always find i'_r, j'_s as above), and since $\mathsf{Tr}(F)$ is finite, n and m vary through a finite set: it is therefore possible to find a uniform value l_0 for l, and the ordering of $F(l_0)$ determines the ordering of $F(k)$ for all k in a prim. rec. way. □

8.4.8. Remark.

It remains to answer the question: is it possible to generate all finite dimensional dilators *effectively*, for instance to enumerate them by a prim. rec. function. As we shall see below the answer is positive. For an alternative proof of this property, see [5], (4.3.9).

8.4.9. Proposition.
Let F be a dilator; then the following conditions are equivalent:

(i) $\mathsf{dim}(F) = 1$.

(ii) If G is dilator with $I^1(G, F) \neq \emptyset$, then $G = F$ or $G = \underline{0}$.

(iii) There exists an integer n such that all elements in $F(x)$ can be written $(0\,;\, x_0, ..., x_{n-1}\,;\, x)_F$ (for all $x \in 0n$).

(iv) The function $p \rightsquigarrow F(p)$ is equal to a binomial polynomial $p(p-1)...(p-n+1)/n!$ for some integer n.

When F satisfies these equivalent requirements, F is said to be **prime**.

Proof. Immediate, left to the reader. □

8.4.10. Definition.
Let F be a prime dilator, with $\mathsf{Tr}(F) = \{(0, n)\}$; then one defines a permutation σ^F of n as follows:
the points $a_i = (0\,;\, o, 2, ..., 2i-2, 2i+1, 2i+2, ..., 2n-2\,;\, 2n)_F$ are pairwise distinct, hence one can introduce σ^F by:

$$i < j \leftrightarrow a_{\sigma^F(i)} > a_{\sigma^F(j)} \ .$$

8.4.11. Theorem.
Assume that F is a prime dilator, with $\mathsf{Tr}(F) = \{(0, n)\}$; then in order to compare two distinct denotations: $t = (0\,;\, x_0, ..., x_{n-1}\,;\, x)_F$ and $t' = (0\,;\, x'_0, ..., x'_{n-1}\,;\, x)_F$, one can proceed as follows:
choose the smallest integer i_0 such that $x_{\sigma^F(i_0)} \neq x'_{\sigma^F(i_0)}$; then

$$t < t' \ \text{ if } \ x_{\sigma^F(i_0)} < x'_{\sigma^F(i_0)}$$

$$t > t' \ \text{ if } \ x_{\sigma^F(i_0)} > x'_{\sigma^F(i_0)} \ .$$

<u>Proof</u>. Let $q = \sigma^F(i_0)$; one will assume that $x_q < x'_q$, and prove that $t < t'$;

(i) We assume that $x_q + \omega \leq \mathsf{inf}\,(x_{q+1}, x'_q)$; then we prove that $t < t'$ by induction on the number k of indices i such that $x'_i < x_i$:

– if $k = 0$, then $x_i \leq x'_i$ for all i, hence by 8.2.18 $t \leq t'$, and since $t \neq t'$, $t < t'$.

– if $k \neq 0$, then construct t'', by replacing the coefficient x_r of t by x'_r and x_q by $x_q + 1$, where r is the smallest integer such that $x'_r < x_r$.

If $r < q$, then the parameters in t and t'' can be listed as follows: $x_0, ..., x_{r-1}, x'_r, x_r, ..., x_q, x_q + 1, x_{q+1}, ...$, this list is to be compared with $0, ..., 2r - 2, 2r, 2r + 1, ..., 2q, 2q + 1, 2q + 2, ...$, i.e. the order relations between t and t'' is the same as between: $a_r = (0\,; 0, 2, ..., 2r - 2, 2r + 1, 2r + 2, ..., 2q, 2q + 2, ..., 2n - 2\,; 2n)_F$ and $a_q = (0\,; 0, 2, ..., 2r - 2, 2r, 2r + 2, ..., 2q + 1, 2q + 2, ..., 2n - 2\,; 2n)_F$; now, if one looks back at the way the integer q was chosen, it is immediate that $a_r < a_q$, hence $t < t''$. Similarly, one concludes that $t < t''$ in the case $q < r$. Now we can apply the induction hypothesis to t'' and t', hence $t'' < t'$, so $t < t'$.

(ii) In the general case, we only know that $x_q + 1 \leq \mathsf{inf}\,(x_{q+1}, x'_q)$; consider the function $f \in I(x, \omega \cdot x)$ defined by $f\ :\ \mathbf{E}_{1\omega}\mathbf{E}_x$; the image under f of the coefficients x_i, x'_j and such that $f(x_q) + \omega \leq \mathsf{inf}\,\big(f(x_{q+1}), f(x'_q)\big)$, and by (i) above $F(f)(t) < F(f)(t')$, hence $t < t'$. □

8.4.12. <u>Theorem</u>.
If σ is a permutation of n, there is a unique prime dilator F such that $\sigma = \sigma^F$.

<u>Proof</u>. Once the permutation σ^F of n is known, we are able to compare any two F-denotations $(0\,; x_0, ..., x_{n-1}\,; x)_F$ and $(0\,; x'_0, ..., x'_{n-1}\,; x)_F$, and this determines F completely. Hence the solution of the theorem is unique. Consider now the dilator $G = \mathsf{Id}^n$ $(= \mathsf{Id} \cdot id \cdot ... \cdot \mathsf{Id})$; in $G(x) = x^n$, we consider all points $x^{n-1} \cdot x_{\sigma(0)} + ... + x^0 \cdot x_{\sigma(n-1)}$, where $x_0, ..., x_{n-1}$ is any strictly increasing sequence of ordinals $< x$. Clearly such a point can be expressed as $(z\,; x_0, ..., x_{n-1}\,; x)_G$, with $z = n^{n-1} \cdot \sigma(0) + ... + n^0 \cdot \sigma(n-1)$, so there is a unique prime dilator F together with $T \in I^1(F, G)$ such that

$rg\Big(\mathsf{Tr}(T)\Big) = \{(z,n)\}$; in order to compare σ^F, we need to compare the points: $b_i = 2n^{n-1} \cdot a_{i,0} + ... + 2n^0 \cdot a_{i,n-1}$, with $a_{i,j} = 2\sigma(j)$ when $i \neq j$, $a_{i,i} = 2\sigma(i) + 1$. Everybody knows how to compare integers written in number base $2n$: we have $b_{\sigma(n-1)} < ... < b_{\sigma(0)}$, hence $\sigma = \sigma^F$. □

8.4.13. Corollary.
There are exactly $n!$ prime dilators such that $\mathsf{Tr}(F) = \{(0,n)\}$.

Proof. Immediate from 8.4.12. □

8.4.14. Definition.
Assume that F is a dilator (or a predilator); then one defines, when $(z_0, n) \in \mathsf{Tr}(F)$ a permutation $\sigma^F_{z_0,n}$ of n as follows: let $a_i = (z_0\,;\, 0, ..., 2i-2, 2i+1, 2i+2, ..., 2n-2\,;\, 2n)_F$ then $i < j \leftrightarrow a_{\sigma^F_{z_0,n}(i)} > a_{\sigma^F_{z_0,n}(j)}$.

8.4.15. Remarks.

(i) We already know, by 8.2.20, that $(z_0\,;\, x_0, ..., x_{n-1}\,;\, x)_F$ and $(z_0\,;\, x'_0, ..., x'_{n-1}\,;\, x)_F$ can be compared by means of finitary data. The permutation $\sigma^F_{z_0,n}$ gives some kind of uniform answer to the question: "what is the particular process of comparing $(z_0\,;\, x_0, ..., x_{n-1}\,;\, x)_F$ and $(z_0\,;\, x'_0, ..., x'_{n-1}\,;\, x)_F$?" The answer is: compare the coefficients $x_{\sigma(0)}$ and $x'_{\sigma(0)}, ..., x_{\sigma(i)}$ and $x'_{\sigma(i)}, ...$ until we find some integer i such that $x_{\sigma(i)} \neq x'_{\sigma(i)}$, then the two denotations compare exactly as their coefficients $x_{\sigma(i)}$.

(Proof. We already know that this is true when F is a prime dilator, hence it suffices to consider G prime and $T \in I^1(F,G)$ such that $rg\Big(\mathsf{Tr}(T)\Big) = \{(z_0\,;\, n)\}$: if $x_{\sigma(i)} < x'_{\sigma(i)}$, then $(0\,;\, x_0, ..., x_{n-1}\,;\, x)_G < (0\,;\, x'_0, ..., x'_{n-1}\,;\, x)_G$, hence if one takes the images under $T(f)$, one gets $(z_0\,;\, x_0, ..., x_{n-1}\,;\, x)_F < (z_0\,;\, x'_0, ..., x'_{n-1}\,;\, x)_F$. □)

(ii) In the proof just made, we have implicitly used the fact that $\sigma^G_{0,n}$ $(= \sigma^G)$ is equal to $\sigma^F_{z_0,n}$: this is a general property, immediate to check: if $U \in I^1(H,H')$ and $a \in \mathsf{Tr}(H)$, then $\sigma^{H'}_{\mathsf{Tr}(U)(a)} = \sigma^H_a$. The permutation associated with a point in the trace of a dilator is therefore an invariant.

(iii) The idea of listing the coefficients in "order of importance" in order to be able to compare two denotations, is familiar from the practice of decimal (or more generally: Cantor normal form) numeration. Here we have established that this principle is in some sense universal. However, we still don't know *how* to compare $(z_0\,;\,x_0,...,x_{n-1}\,;\,x)_F$ and $(z_1\,;\,x'_0,...,x'_{m-1}\,;\,x)_F$ when $(z_0,n) \neq (z_1,m)$.

8.4.16. Proposition.
Assume that F is a dilator of dimension 2, and let (a,n), (b,m) be the elements of its trace. Define p to be the greatest integer such that:

$$(P)\ :\ \forall i \forall j\ \Big(i < p \wedge j < p \rightarrow \big(\sigma^F_{a,n}(i) < \sigma^F_{a,n}(j) \leftrightarrow \sigma^F_{b,m}(i) < \sigma^F_{b,m}(j)\big)\Big)$$

assume that $x_0 < ... < x_{n-1} < x$, $y_0 < ... < y_{m-1} < x$, and that $x_{\sigma(0)} = y_{\tau(0)}, ..., x_{\sigma(p-1)} = y_{\tau(p-1)}$, with $\sigma = \sigma^F_{a,n}$, $\tau = \sigma^F_{b,m}$. Then the order between $(a\,;\,x_0,...,x_{n-1}\,;\,x)_F$ and $(b\,;\,y_0,...,y_{m-1}\,;\,x)_F$ deos not depend on the ordinals $x_{\sigma(p)},...,x_{\sigma(n-1)}$, $y_{\tau(p)},...,y_{\tau(m-1)}$.

Proof. Assume for instance that $(a\,;\,x_0,...,x_{n-1}\,;\,x)_F < (b\,;\,y_0,...,y_{m-1}\,;\,x)_F$ and let $x'_0 < ... < x'_{n-1} < x$, $y'_0 < ... < y'_{m-1} < x$ be such that $x'_{\sigma(0)} = x_{\sigma(0)},...,x'_{\sigma(p-1)} = x_{\sigma(p-1)}$, $y'_{\tau(0)} = y_{\tau(0)},...,y'_{\tau(p-1)} = y_{\tau(p-1)}$; we prove that

$$(a\,;\,x'_0,...,x'_{n-1}\,;\,x)_F < (b\,;\,y'_0,...,y'_{m-1}\,;\,x)_F$$

(i) if $n = p$, then the mutual orders of the x_i's and the y_j's are the same as the mutual orders between the x_i's and the y_j's. By 8.2.20 one gets $(a\,;\,x'_0,...,x'_{n-1}\,;\,x)_F < (b\,;\,y'_0,...,y'_{m-1}\,;\,x)_F$.

(ii) if $m = p$: similar to (i).

(iii) if $p < \mathsf{inf}\,(n,m)$, let $k = \sigma(p)$, $k' = \tau(p)$; since p is maximum such that (P) holds, it follows that, for some $i < p$:

either: $x_k < x_{\sigma(i)} = y_{\tau(i)} < y_{k'}$ (subcase a)

or : $y_{k'} < y_{\tau(i)} = x_{\sigma(i)} < x_k$ (subcase b)

Subcase a: Since obviously $x'_k < x'_{\sigma(i)} = y'_{\tau(i)} < y'_{k'}$, we get

1. if $x'_k < x_k$, $y_{k'} < y'_{k'}$, then, by 8.4.15 (i): $(a\,;\,x'_0,...,x'_{n-1}\,;\,x)_F < (a\,;\,x_0,...,x_{n-1}\,;\,x)_F < (b\,;\,y_0,...,y_{m-1}\,;\,x)_F < (b\,;\,y'_0,...,y'_{m-1}\,;\,x)_F$.
2. in general, it is possible to construct an ordinal x', functions $f.g \in I(x,x')$ such that $f(x_{\sigma(i)}) = g(x_{\sigma(i)})$ for all $i < p$ and $f(x'_k) < g(x_k)$, $g(y_{k'}) < f(y'_{k'})$. (The existence of x', f, g is immediate from $x'_k, x_k < y'_{k'}, y_{k'}$.) We get $(a\,;\,f(x'_0),...,f(x'_{n-1})\,;\,x')_F < (b\,;\,f(y'_0),...,f(y'_{m-1})\,;\,x')_F$, by applying 1 to $(a\,;\,g(x_0),...,g(x_{n-1})\,;\,x')_F < (b\,;\,g(y_0),...,g(y_{m-1})\,;\,x')_F$. Hence $(a\,;\,x'_0,...,x'_{n-1}\,;\,x)_F < (b\,;\,y'_0,...,y'_{m-1}\,;\,x)_F$.

Subcase b: Symmetric to subcase a. □

8.4.17. <u>Proposition</u>.
Let F be a dilator of dimension 2, with $\mathsf{Tr}(F) = \{(a,n),(b,m)\}$, and let p be an integer enjoying (P) (see 8.4.16). Let $k < p$, and assume that, for some sequences $x_0 < ... < x_{n-1} < x$, $y_0 < ... < y_{m-1} < x$ enjoying $x_{\sigma(0)} = y_{\tau(0)},...,x_{\sigma(k-1)} = y_{\tau(k-1)}$, we have: $(a\,;\,x_0,...,x_{n-1}\,;\,x)_F < (b\,;\,y_0,...,y_{m-1}\,;\,x)_F$; then given any sequences $x'_0 < ... < x'_{n-1} < x$, $y'_0 < ... < y'_{m-1} < y$, with $x'_{\sigma(o)} = y'_{\tau(0)},...,x'_{\sigma(k-1)} = y'_{\tau(k-1)}$, $x'_{\sigma(k)} < y'_{\tau(k)}$ we have:

$$(a\,;\,x'_0,...,x'_{n-1}\,;\,x)_F < (b\,;\,y'_0,...,y'_{m-1}\,;\,x)_F\ .$$

<u>Proof</u>. (i) Iff $x_{\sigma(0)} = x'_{\sigma(0)},...,x_{\sigma(k-1)} = x'_{\sigma(k-1)}$, and $x'_{\sigma(k)} < x_{\sigma(k)}$, $y_{\tau(k)} < y'_{\tau(k)}$, then by 8.4.15 (i): $(a\,;\,x'_0,...,x'_{n-1}\,;\,x)_F < (a\,;\,x_0,...,x_{n-1}\,;\,x)_F < (b\,;\,y_0,...,y_{m-1}\,;\,x)_F < (b\,;\,y'_0,...,y'_{m-1}\,;\,x)_F$.

(ii) In general, one easily constructs an ordinal x' together with functions $f,g \in I(x,x')$, such that $f(x_{\sigma(0)}) = g(x'_{\sigma(0)}),...,f(x_{\sigma(k-1)}) = g(x_{\sigma(k-1)})$, and $g(x'_{\sigma(k)}) < f(x_{\sigma(k)})$, $f(y_{\tau(k)}) < g(y'_{\tau(k)})$. (The existence of x', f, g is immediate from the hypothesis $x'_{\sigma(k)} < y'_{\tau(k)}$.) The case (i) yields $(a\,;\,g(x'_0),...,g(x'_{n-1})\,;\,x')_F < (b\,;\,g(y'_0),...,g(y'_{m-1})\,;\,x')_F$, which in turn implies $(a\,;\,x'_0,...,x'_{n-1}\,;\,x)_F < (b\,;\,y'_0,...,y'_{m-1}\,;\,x)_F$. □

8.4.18. <u>Definition</u>.
Assume that F is a dilator (or a predilator) and let (a,n), (b,m) be two elements of $\mathsf{Tr}(F)$, with $(a,n) \neq (b,m)$; one defines $\S^F(a(a,n\,;\,b,m)$ as follows: $\S(a,n\,;\,b,m)$ is a pair (p,ε), where:

(i) p is the smallest integer enjoying property (P) of 8.4.16, and such that for all sequences $x_0 < ... < x_{n-1} < x$ and $y_0 < ... < y_{m-1} < x$ such that $x_{\sigma(i)} = y_{\tau(i)}$ for $i = 0, ..., p-1$, the order between $t = (a\,;\,x_0, ..., x_{n-1}\,;\,x)_F$ and $u = (b\,;\,y_0, ..., y_{m-1}\,;\,x)_F$ does not depend on $x_{\sigma(p)}, ..., x_{\sigma(n-1)}, y_{\tau(p)}, ..., y_{\tau(n-1)}$.

(ii) $\varepsilon = +1$ when, under the conditions of (i), one has $t < u$.

(iii) $\varepsilon = -1$ when, under the conditions of (i), one has $t > u$.

8.4.19. <u>Remarks</u>.

(i) The existence of an integer p such that 8.4.18 (i) holds is an immediate consequence of 8.4.16 applied to G such that there is a $T \in I^1(F,G)$, with $rg\big(\mathsf{Tr}(T)\big) = \{(a,n),(b,m)\}$.

(ii) If $\S^F(a,n\,;\,b,m) = (p,\varepsilon)$, then $\S^F(b,m\,;\,a,n) = (p,-\varepsilon)$; it will be convenient to set $\S^F(a,n\,;\,a,n) = (n,0)$. We shall use the abbreviation $|\S^F(a,n\,;\,b,m)|$ for the first component p of the pair.

(iii) The definition of $\S^F(a,n\,;\,b,m)$ is perfectly finitistic and effective: if $\S^F(a,n\,;\,b,m) = (p,\varepsilon)$, then it will be possible to compute p as follows: by 8.2.20, $(a\,;\,x_0, ..., x_{n-1}\,;\,x)_F$ and $(b\,;\,y_0, ..., y_{m-1}\,;\,x)_F$ are ordered in the same wau as $(a\,;\,p_0, ..., p_{n-1}\,;\,n{+}m)_F$ and $(b\,;\,q_0, ..., q_{m-1}\,;\,n{+}m)_F$, where the sequences p_i and q_j are such that $p_i < q_j \leftrightarrow x_i < y_j$ and $p_i > q_j \leftrightarrow x_i > y_j$: so, in order to compute p, it will suffice to look at the value $x = n + m$: the same process yields ε.

8.4.20. <u>Theorem</u>.
Assume that (a,n) and (b,m) are distinct elements of $\mathsf{Tr}(F)$, with $\S^F(a,n\,;\,b,m) = (p,\varepsilon)$; then $t = (a\,;\,x_0, ..., x_{n-1}\,;\,x)_F < (b\,;\,y_0, ..., y_{m-1}\,;\,x)_F = u$ iff one of the following conditions holds:

(i) either $\varepsilon = +$‘, and $x_{\sigma(0)} = y_{\tau(0)}, ..., x_{\sigma(p-1)} = y_{\tau(p-1)}$.

(ii) or for some $q < p$, $x_{\sigma(0)} = y_{\tau(0)}, ..., x_{\sigma(q-1)} = y_{\tau(q-1)}$, $x_{\sigma(q)} < y_{\tau(q)}$.

Proof. If (i) holds, then this means that for any sequences $x_0 < ... < x_{n-1} < x$ and $y_0 < ... < y_{m-1} < x$ such that $x_{\sigma(0)} = y_{\tau(0)}, ..., x_{\sigma(p-1)} = y_{\tau(p-1)}$, we have always $(a\,;\,x_0, ..., x_{n-1}\,;\,x)_F < (b\,;\,y_0, ..., y_{m-1}\,;\,x)_F$.

If (ii) holds, then it is possible (since the order between $(a\,;\,x'_0, ..., x'_{n-1}\,;\,x')_F$ and $(b\,;\,y'_0, ..., y'_{m-1}\,;\,x')_F$ is $<$ or $>$ according to the choice of the sequences $x'_0 < ... < x'_{n-1} < x'$ and $y'_0 < ... < y'_{m-1} < x'$ such that $x'_{\sigma(0)} = y'_{\tau(0)}, ..., x'_{\sigma(q-1)} = y'_{\tau(q-1)}$) to find sequences $x'_0 < ... < x'_{n-1} < x'$, $y'_0 < ... < y'_{m-1} < x'$ such that $x'_{\sigma(0)} = y'_{\tau(0)}, ..., x'_{\sigma(q-1)} = y'_{\tau(q-1)}$, and $(a\,;\,x'_0, ..., x'_{n-1}\,;\,x')_F < (b\,;\,y'_0, ..., y'_{m-1}\,;\,x')_F$, hence, with $x'' = \mathsf{sup}(x, x')$, $(a\,;\,x_0, ..., x_{n-1}\,;\,x'')_F < (b\,;\,y'_0, ..., y'_{m-1}\,;\,x'')_F$; by 8.4.16 $(a\,;\,x_0, ..., x_{n-1}\,;\,x'')_F < (b\,;\,y_0, ..., y_{m-1}\,;\,x'')_F$, hence $(a\,;\,x_0, ..., x_{n-1}\,;\,x)_F < (b\,;\,y_0, ..., y_{m-1}\,;\,x)_F$.

Assume now that (i) and (ii) are false; then either (i)$'$: $\varepsilon = -1$, and $x_{\sigma(0)} = y_{\tau(0)}, ..., x_{\sigma(p-1)} = y_{\tau(p-1)}$ or (ii)$'$: there exists $q < p$, such that $x_{\sigma(0)} = y_{\tau(0)}, ..., x_{\sigma(q-1)} = y_{\tau(q-1)}$ and $x_{\sigma(q)} > y_{\tau(q)}$.

Clearly (i)$'$ and (ii)$'$ imply $(a\,;\,x_0, ..., x_{n-1}\,;\,x)_F > (b\,;\,y_0, ..., y_{m-1}\,;\,x)_F$. □

8.4.21. Theorem.
Assume that $\S^F(a, n\,;\,b, m) = (p, +1)$ and $\S^F(b, m\,;\,c, l) = (q, +1)$; then $\S^F(a, n\,;\,c, l) = (\mathsf{inf}\,(p, q), +1)$.

Proof. Assume that $\S^F(a, n\,;\,c, l) = (r, \varepsilon)$, then

(i) $r \geq \mathsf{inf}\,(p, q)$: let $\sigma = \sigma^F_{a,n}$, $\tau = \sigma^F_{b,m}$, $\rho = \sigma^F_{c,l}$, let $s = \mathsf{inf}\,(q, p)$ and assume $s \neq 0$; choose sequences $x_0 < ... < x_{n-1}$, $y_0 < ... < y_{m-1}$, $z_0 < ... < z_{l-1}$ bounded by $x = n + m + l$ such that: $x_{\sigma(0)} = y_{\tau(0)} = z_{\rho(0)}, ..., x_{\sigma(s-2)} = y_{\tau(s-2)} = z_{\rho(s-2)}$ and:

1. $x_{\sigma(s-1)} < y_{\tau(s-1)} < z_{\rho(s-1)}$: by 8.4.20 (variant (ii)) one gets $(a\,;\,x_0, ..., x_{n-1}\,;\,x)_F < (b\,;\,y_0, ..., y_{m-1}\,;\,x)_F < (x\,;\,z_0, ..., z_{l-1}\,;\,x)_F$.

2. $x_{\sigma(s-1)} > y_{\tau(s-1)} > z_{\rho(s-1)}$: by 8.4.20 one gets $(a\,;\,x_0, ..., x_{n-1}\,;\,x)_F > (b\,;\,y_0, ..., y_{m-1}\,;\,x)_F > (c\,;\,z_0, ..., z_{l-1}\,;\,x)_F$.

This shows that the order between $(a\,;\,x_0,...,x_{n-1}\,;\,x)_F$ and $(c\,;\,z_0, ..., z_{l-1}\,;\,x)_F$ depends on the order between $x_{\sigma(s-1)}$ and $z_{\tau(s-1)}$ and from that $r \geq s$.

(ii) $r \leq \mathsf{inf}\,(p,q)$: assume that $x_0,...,x_{n-1}, z_0,...,z_{l-1}$ are such that $x_{\sigma(0)} = z_{\rho(0)},...,x_{\sigma(s-1)} = z_{\rho(s-1)}$; we shall prove that $(a\,;\,x_0,...,x_{n-1}\,;\,x)_F < (c\,;\,z_0,...,z_{l-1}\,;\,x)_F$; necessarily $\varepsilon = +1$, and $r \leq s$. By 8.2.20, there is no loss of generality in supposing that $x_0,...,x_{n-1}, z_0,...,z_{l-1}$, x are limit ordinals. This makes the construction of the sequence $y_0,...,y_{m-1}$ possible;

- if $p = q$, choose a sequence $y_0,...,y_{m-1}$ with $x_{\sigma(0)} = y_{\tau(0)}$, $x_{\sigma(p-1)} = y_{\tau(p-1)}$; by 8.4.20 (variant (i)) one gets: $(a\,;\,x_0,...,x_{n-1}\,;\,x)_F < (b\,;\,y_0,...,y_{m-1}\,;\,x)_F < (c\,;\,z_0,...,z_{l-1}\,;\,x)_F$.
- if $p < q$, choose a sequence $y_0,...,y_{m-1}$ with $x_{\sigma(0)} = y_{\tau(0)},...,x_{\sigma(p-1)} = y_{\tau(p-1)}$, $y_{\tau(p)} < z_{\rho(p)}$; again by 8.2.20: $(a\,;\,x_0,...,x_{n-1}\,;\,x)_F < (b\,;\,y_0, ..., y_{m-1}\,;\,x)_F < (c\,;\,z_0,...,z_{l-1}\,;\,x)_F$.
- if $q < p$, choose a sequence $y_0,...,y_{m-1}$ with $x_{\sigma(0)} = y_{\tau(0)},...,x_{\sigma(q-1)} = y_{\tau(q-1)}$, $x_{\sigma(q)} < y_{\tau(q)}$: once again, by 8.2.20: $(a\,;\,x_0,...,x_{n-1}\,;\,x)_F < (b\,;\,y_0,...,y_{m-1}\,;\,x)_F < (c\,;\,z_0,...,z_{l-1}\,;\,x)_F$. $\square$

8.4.22. Remarks.

(i) If one defines $\leq^F$, a binary relation on $\mathsf{Tr}(F)$, by $(a,n) \leq^F (b,m)$ iff the second component ε of $\S^F(a,n\,;\,b,m)$ is $\neq\ -1$,then $\leq^F$ is clearly an order relation, (and better: a linear order) on $\mathsf{Tr}(F)$. However, there are dilators F such that $\leq^F$ is not a well-order.

(ii) It is possible to define a metric d^F on $\mathsf{Tr}(F)$, by $d^F(a,n\,;\,b,m) = 2^{-|\S^F(a,n\,;\,b,m)|}$ when $(a,n) \neq (b,m)$, $= 0$ otherwise. This metric enjoys the ultrametric inequality:

$$d^F(a,n\,;\,c,l) \leq \mathsf{sup}\left(d^F(a,n\,;\,b,m), d^F(b,m\,;\,c,l)\right)\ .$$

(iii) $\S^F$ is the other invariant of dilators: if $T \in I^1(F,G)$, then $\S^G\Big(\mathsf{Tr}(T)(a), \mathsf{Tr}(T)(b)\Big) = \S^F(a,b)$. A description of dilators by means of their invariants is possible, see 8.4.23.

(iv) 8.4.20 gives us the effective way of comparing $t = (z_0\,;\,x_0,...,x_{n-1}\,;\,x)_F$ and $u = (z_1\,;\,y_0,...,y_{m-1}\,;\,x)_F$ when $(z_0, n) \neq (z_1, m)$: one looks for the smallest i_0 such that $x_{\sigma(i)} \neq y_{\tau(i)}$ ($i_0 = \mathsf{inf}\,(n, m)$, if there is no such i); then, if $i_0 < |\S^F(z_0, n\,;\,z_1, m)|$, $t < u$ iff $x_{\sigma(i)} < y_{\tau(i)}$; if $i_0 \geq |\S^F(z_0, n\,;\,z_1, m)|$, then $t < u$ iff the second component of $\S^F(z_0, n\,;\,z_1, m)$ is $+1$. Hence, we have found a universal algorithm corresponding to 8.2.20: this completes 8.4.15 (i).

8.4.23. Theorem.
Assume that X is a set, together with the following data:

(i) a linear order $\leq$ on X.

(ii) a function f from X^2 to $I\!N$ such that:

– $f(x, y) = f(y, x)$ for all $x, y \in X$.

– $f(x, z) = \mathsf{inf}\,\big(f(x, y), f(y, z)\big)$ for all $x, y, z \in X$ such that $x \leq y \leq z$.

(iii) for all $x \in X$ a permutation Σ_x of $f(x, x)$; the permutations Σ_x have the property that:

$$\forall i < f(x, y)\ \forall j < f(x, y)\ \Big(\Sigma_x(i) < \Sigma_x(j) \leftrightarrow \Sigma_y(i) < \Sigma_y(j)\Big)\ .$$

Then there is a predilator F (unique up to isomorphism) together with a bijective function φ from $\mathsf{Tr}(F)$ to X, such that:

1. if $\S^F(a, n\,;\,b, m) = (p, +1)$, then $\varphi(a, n) \leq \varphi(b, m)$ and $f\big(\varphi(a, n), \varphi(b, m)\big) = p$.

2. $\sigma^F_{a,n} = \Sigma_{\varphi(a,n)}$.

Conversely, every predilator can be obtained in that way.

Proof. Let u be a linear order; then $F(u)$ consists by definition of all formal expressions: $(z\,;\,u_0,...,u_{n-1}\,;\,u)$ with $z \in X$, $u_0,...,u_{n-1}$ a strictly increasing sequence of length $n = f(z, z)$. By definition, $(z\,;\,u_0,...,u_{n-1}\,;\,u) \leq^u (z'\,;\,u'_0,...,u'_{m-1}\,;\,u)$ iff either (i): $z \leq z'$ and $u_{\Sigma_z(i)} = u'_{\Sigma_{z'}(i)}$ for $i =$

$0, \ldots, f(z,z') - 1$, or (ii) there exists an integer $i < f(z,z')$ such that $u_{\Sigma_z(j)} = u'_{\Sigma_{z'}(j)}$ for all $j < i$, and $u_{\Sigma_z(i)} < u'_{\Sigma_{z'}(i)}$.

It is immediate that $\leq^u$ is a linear ordering of $F(u)$; if $f \in I(u,v)$, then it is immediate that the function $F(f)$ defined by:

$$F(f)\Big((z_0\,;\, u_0, \ldots, u_{n-1}\,;\, u)\Big) = (z\,;\, f(u_0), \ldots, f(u_{n-1})\,;\, v)$$

is strictly increasing. One easily checks that F is a predilator. (By 8.2.23 F preserves direct limits and pull-backs; $F \leq g \rightarrow F(f) \leq F(g)$ is immediate $\ldots$.) It is immediate that $\mathsf{Tr}(F) = \Big\{\Big(z, f(z,z)\Big)\,;\, z \in X\Big\}$, that $\sigma^F_{z,f(z,z)} = \Sigma_z$, and that $\S^F\Big(z, f(z,z)\,;\, z', f(z',z')\Big) = (f(z,z'), \varepsilon)$ with $\varepsilon = +1$ if $z \leq z'$, $\varepsilon = -1$ if $z' \leq z$, $\varepsilon = 0$ if $z = z'$: hence the theorem holds with $\varphi\Big(\Big(z, f(z,z)\Big)\Big) = z$.

Conversely, every dilator F can be written in this form, with $X = \mathsf{Tr}(F)$, $\leq = \leq^F$, $\Sigma_{a,n} = \sigma^F_{a,n}$, $f(a,n\,;\, a',m) = |\S^F(a,n\,;\, a',m)|$. $\square$

8.4.24. <u>Corollary</u>.
There is a prim. rec. function which enumerates all finite dimensional dilators.

<u>Proof</u>. It suffices to enumerate all 4-uples $(X, \leq, f, \Sigma)$ of Theorem 8.4.23, with X equals some integer. The unique finite dimensional dilator defined by the data $(X, \leq, f, \Sigma)$ is obtained from these data in a prim. rec. way. $\square$

8.4.25. <u>Remarks</u>.

(i) Another equivalent formulation is possible: instead of considering the permutations Σ_x of $f(x,x)$, one could introduce a linear order R_x of $f(x,x)$ by

$$i\, R_x\, j \leftrightarrow \Sigma_x(i) \leq \Sigma_x(j)\ .$$

8.4.23 (iii) shows that R_x and R_y coincide when restricted to $f(x,y)$: let $R_{xy} = R_x \upharpoonright f(x,y) = R_y \upharpoonright f(x,y)$.

We obtain the following alternative formulation of predilators: a set X, together with

1. a linear order $\leq$ on X.

2. for all $x, y \in X$ s.t. $x \leq y$, a linear order $R(x, y)$ of some integer $f(x, y)$.

   These data must satisfy:

$$\forall x \forall y \forall z \; x \leq y \leq z \to R(x, z) = R(x, y) \cap R(y, z) \; .$$

   This formulation is easily shown to be equivalent to 8.4.20: this is the *abstract construction of predilators*.

(ii) Let us see how the abstract construction (i) enables us to determine all predilators.

1. We can construct all dilators of finite dimension n, as follows: Let $X = n$, ordered as usual, and choose, for all $i < n$ a linear order $R(i, i)$ of some integer $f(i, i)$; then choose for all i s.t. $i + 1 < n$ a linear order $R(i, i + a)$ such that $R(i, i + 1)$ is the restriction of $R(i, i) \cap R(i + 1, i + 1)$ to some integer $f(i, i + 1)$. Then we can define

$$R(i, j) = R(i, i + 1) \cap ... \cap R(j - 1, j) \quad \text{for} \quad i < j < n \; ,$$

   and clearly $R(i, j) \cap R(j, k) = R(i, k)$.

2. A morphism of predilators is clearly represented by a function $g \in I(X, \leq ; X', \leq')$ such that

$$R'\big(g(x), g(y)\big) = R(x, y) \;\; \text{for all} \;\; x, y \;\; \text{s.t.} \;\; x \leq y \; .$$

   Since we know how to construct finite dimensional dilators and their morphisms, the direct limits enable us to construct all predilators.

(iii) The abstract construction of predilators is a way of obtaining them from a set of "independent data", just like the matricial representation of endomorphisms of a vector space enables us to describe them by "independent" data.

(iv) It is quite remarkable that this abstract construction was obtained by piecing together:

1. the characterization of dilators of dimension 1 (8.4.11).
2. the characterization of dilators of dimension 2 (8.4.12).
3. the characterization of dilators of dimension 3 by means of 8.4.24, which expresses in which way the three 2-dimensional subdilators of a 3-dimensional dilator behave w.r.t. one another.

(v) the interest of the abstract construction is that it enables us to see rather subtle aspects of the theory. For instance one can imagine a notion of morphism of predilator not satisfying

$$R'\Big(g(x), g(y)\Big) = R(x, y) \ ,$$

for instance we are given $h(x, y) \in I\Big(R(x, y), R\Big(g(x), g(y)\Big)\Big)$, together with compatibility conditions... . These morphisms have no obvious interpretation in terms of the associated functors, but are rather natural objects... . The direct limits along these morphisms could produce objects which are not any longer dilators or predilators, but which must play a role at a certain stage... .

(vi) We have succeeded in characterizing in a complete way the concept of predilator; all this was done in the bulk of elementary mathematics. Now, no such elementary (or algebraic) approach is possible for dilators, since the concept of dilator is Π^1_2-complete. There is another part of the theory, where the algebraic features are less prominent, and in which we turn our attention towards well-foundedness properties of dilators: this will be done in Chapter 9.

Annex 8.A. A theory of ordinal denotation

This is not the first attempt to analyze the act of denoting ordinals by means of expressions involving other (in general strictly smaller) ordinals. However, compared to other approaches, e.g. Feferman [77], this theory seems to be more general, and mathematically of greater interest, since it enables us to identify systems of ordinal denotations with dilators, and the mathematical structure of dilators is rather subtle. (Whereas former authors were rather concerned with *normal functions*, i.e. strictly increasing and continuous function from $0n$ to $0n$; normal functions are of very limited theoretical interest: if $f(x)$ is any function from $0n$ to $0n$, then $x \rightsquigarrow \sum\limits_{y<x} (f(y)+1)$ is a normal function. Normal functions were thought to be of interest because of their continuity property, but the example given above shows that, except at limit points, the behaviour of a normal function is completely inpredictible. Dilators are another conception of "continuity", which is mathematically interesting, since the behaviour of the functor is completely determined by its behaviour on finite objects and morphisms, i.e. on $\mathbf{ON} < \omega$. When F is a dilator, the function $x \rightsquigarrow F(x)$ is not necessarily continuous at limits (but this is the case when F is a *flower*), but the "inner" continuity of a dilator is anyhow more real than the superficial continuity of normal functions... .)

8.A.1. Examples.
Practice gives us example of ordinal denotation:

(i) The Cantor Normal Form of base, say, 10:
any ordinal can be uniquely written

$$z = 10^{x_0} \cdot a_0 + ... + 1 -^{x_{n-1}} \cdot a_{n-1} \ ,$$

where $0 < a_0, ..., a_{n-1} < 10$, and $x_0 >, , , > x_{n-1}$.

(ii) any ordinal $< x^2$ can be uniquely written

$$z = x \cdot x_0 + x_1 \ ,$$

with $x_0, x_1 < x$.

8.A.2. Discussion.

We must now find out in these examples what are the inherent and the contingent features: of course one cannot pay too much attention to the two-dimensional way of representing the denotation; on the other hand, the list of the "ordinal parameters" is obviously crucial. As to the ordinal parameters, there is an important difference between (i) and (ii): the denotation (ii) depends on the choice of an x such that $z < x^2$, and this x is present in the denotation. In Example (i), the denotation is universal: it does not depend on the choice of an x such that $z < 10^x$. We shall unify the two situations by deciding that (ii) is the natural case, whereas (i) is exceptional, and that this is only an abuse of notations which allowed us to get rid of an x such that $z < 10^x$. (The distinction between (ii) and (i) is exactly the distinction between general dilators and *flowers*.)

Given a denotation d, then the basic distinction is to divide it into two parts:

– a dynamic part: the ordinal parameters.

– a static part: what is "independent" of the ordinal parameters, we shall call it *skeleton* or *configuration*.

For example, in 8.A.1 (i), the ordinals $x_0, ..., x_{n-1}$ are ordinal parameters, whereas $a_0, ..., a_{n-1}$ are part of the skeleton. ($x_0, ..., x_{n-1}$ can be changed arbitrarily, provided $x_0 > ... > x_{n-1}$, whereas the coefficients $a_0, ..., a_{n-1}$ are bound to stay in the interval $]0, 10[$.) In 8.A.1 (ii) all ordinals x, x_0, x_1 are parameters.

Up to now, this distinction between skeletons and parameters is not very precise; we shall ask conditions later on.

If we have a denotation d, we shall represent it in an abstract form as $d = (S\,;\, y_0, ..., y_{k-1}\,;\, x)$, where

(i) $y_0, ..., y_{k-1}, x$ are the ordinal parameters of d, listed in strictly increasing order (recall that x may be absent from the "actual" d).

(ii) S is the skeleton of the denotation.

This gives the following representation for 8.A.1 (i)–(ii):

(i) $z = (S\,;\,x_{n-1},...,x_0\,;\,x)$, where S is the skeleton: $S = 10^{P_{n-1}} \cdot a_0 + ... + 10^{P_0} \cdot a_{n-1}$ ($P_0,...,P_{n-1}$ are just extra symbols: P_i stands for "the i^{th} parameter").

(ii) Here we have three cases:

1. If $x_0 < x_1$, then $z = (S\,;\,x_0, x_1\,;\,x)$ with $S = P \cdot P_0 + P_1$, P being an extra symbol for "the parameter x".
2. If $x_0 = x_1$, then $z = (S\,;\,x_0\,;\,x)$ with $S = P \cdot P_0 + P_0$.
3. If $x_1 < x_0$, then $z = (S\,;\,x_1, x_0\,;\,x)$ with $S = P \cdot P_1 + P_0$.

We are now in a position to give a general definition:

8.A.3. Definition.
A **system of ordinal denotation** D consists of the following data:

(i) a set of **configurations** (**skeletons**): a configuration is a pair (X_0, n): n is the number of parameters $\neq x$.

(ii) a function $|\ \ |^D$ which associates to any D-**denotation**, i.e. any formal expression $(C_0\,;\,x_0,...,x_{n-1}\,;\,x)$ with (C_0, n) skeleton in D and $x_0 < ... < x_{n-1} < x$ ($x \in 0n$) an ordinal $z = |(C_0\,;\,x_0,...,x_{n-1}\,;\,x)|^D$; one will say that $(C_0\,;\,x_0,...,x_{n-1}\,;\,x)$ **denotes** z in D, and we shall use the notation $z = (C_0\,;\,x_0,...,x_{n-1}\,;\,x)_D$.

We require a certain number of properties to hold:

(iii) two distinct denotations with the same x denote distinct ordinals.

(iv) the set of ordinals of the form $(C_0\,;\,x_0,...,x_{n-1}\,;\,x)$, x being fixed, forms an initial segment of $0n$, i.e. an ordinal $D(x)$.

(v) the order between $|(C_0\,;\,x_0,...,x_{n-1}\,;\,x)|^D$ and $|(C_1\,;\,y_0,...,y_{m-1}\,;\,x)|^D$ depends only on the relative orders of the x_i's and the y_j's.

Properties (iii) and (iv) state that any point $< D(x)$ is uniquely denotable by means of a D-denotation ending with x, whereas (v) stresses the *local* character of the order between denotations.

8.A.4. Examples.
It is immediate that our Examples 8.A.1 (i) and (ii) are systems of ordinal denotation in the sense of 8.A.3. This is completely immediate.

8.A.5. Theorem.
Assume that D is a system of ordinal denotation, and define D^* by:

– $D^*(x) = D(x)$ when $x \in 0n$; if $f \in I(x, y)$, then

– $D^*(f)\Big((C_0\,;\, x_0, ..., x_{n-1}\,;\, x)_D\Big) = (C_0\,;\, f(x_0), ..., f(x_{n-1})\,;\, y)_D$.

Then D^* is a dilator.

Proof. We first show that D^* is a functor from **ON** to **ON**; the crucial condition is that $D^*(f) \in I\Big(D^*(x), D^*(y)\Big)$, i.e. $D^*(f)$ is strictly increasing. But, if $(C_0\,;\, x_0, ..., x_{n-1}\,;\, x)_D < (C_1\,;\, y_0, ..., y_{m-1}\,;\, x)_D$, the order between the x_i's and the y_j's is the same as the order between the $f(x_i)$'s and the $f(y_j)$'s, hence, by Condition 8.A.3 (v): $(C_0\,;\, f(x_0), ..., f(x_{n-1})\,;\, y)_D < (C_1\,;\, f(y_0), ..., f(y_{m-1})\,;\, y)_D$. In order to prove that D^* is a dilator, it suffices, by Theorem 8.2.3, to show the existence of uniquely determined z_0, n, f such that 8.2.3 (i)–(ii) hold; but if $z = (C_0\,;\, x_0, ..., x_{p-1}\,;\, x)_D$, let $n = p$, $f \in I(n, p)$ be defined by $f(0) = x_0, ..., f(p-1) = x_{p-1}$, and $z_0 = (C_0\,;\, 0, ..., p-1\,;\, p)_D$. Then by 8.A.3 (iii), z_0, n, f are uniquely determined. □

8.A.6. Examples.

(i) If D is the denotation system 8.A.1 (i), then

$$D^* \text{ is the dilator } 10^{\mathsf{Id}} \; (= (1 + \underline{g})^{\mathsf{Id}})$$

$$10^{\mathsf{Id}}(x) = 10^x$$

$$10^{\mathsf{Id}}(f)(10^{x_0} \cdot a_0 + ... + 10^{x_{n-1}} \cdot a_{n-1}) =$$

$$10^{f(x_0)} \cdot a_0 + ... + 10^{f(x_{n-1})} \cdot a_{n-1} \,.$$

(ii) If D is the denotation system 8.A.1 (ii), then

$$D^* \text{ is the dilator } \mathsf{Id}^2$$

$$\mathsf{Id}^2(x) = x^2$$

$$\mathsf{Id}^2(f)(x \cdot x_0 + x_1) = y \cdot f(x_0) + f(x_1) \qquad (f \in I(x,y)) \ .$$

8.A.7. Theorem.
Assume that F is a dilator; then one can construct (by 8.2.4) a denotation system F^+.

Proof. 8.2.4 enables us to construct a denotation system F^+, with:
(i) the skeletons are the pairs (z_0, n) with $(z_0, n) \in \mathsf{Tr}(F)$.
(ii) the meaning of $(z_0\,;\, x_0, ..., x_{n-1}\,;\, x)_{F^+}$ is by definition $(z_0\,;\, x_0, ..., x_{n-1}\,;\, x)_F$.

Properties (iii) and (iv) are fulfilled because of the Normal Form Theorem 8.2.3, whereas Property (v) follows from 8.2.20. □

8.A.8. Theorem.
The operations * and $^+$ are inverse.

Proof. This is practically immediate: if F is a dilator, then $F^{+*} = F$; if D is a denotation, then D^{*+} is a denotation system isomorphic to D. (The notion of isomorphism of denotation systems is clear.) □

8.A.9. Discussion.

(i) Finally, the concept of dilator is perfectly equivalent to the concept of system of ordinal denotation. Hence the idea of "representing ordinals by means of expressions involving ordinals" is universally solved by the concept of dilator.

(ii) However, this is a typical situation where one must be cautious: the danger is to find an *ad hoc* solution. For instance in that case, we found the concept of dilator *before* thinking about ordinal denotations. The conditions on systems of ordinal denotations are very simple and general (8.A.3) and correspond to practice, but it is still possible to have a doubt as to the adequacy of the mathematical solution we have found w.r.t. the general idea of ordinal denotation, although in that case we must confess that we are rather convinced by this solution.

(iii) Ordinal notations, i.e. representation of ordinals by means of notations in a given language, do not lead to any interesting theory. The natural concept corresponding to the concept of system of ordinal notations is that of a recursive well-order, whose mathematical structure is limited. In practice people have been mainly interested in constructing many *notation* systems. A closer look would show that these systems are indeed *denotation* systems (in which the ordinals parameters have been replaced by integers $< \omega$). This remark gives us a way of associating dilators with various constructions connected with the subject labelled as "ordinal notations", and gives a general framework for a part of mathematical logic where theoretical considerations were particularly missing!

(iv) One will find another exposition of the results of this section in [83].

Annex 8.B. Categories of sequences

Most of the results of this section are without proof; a certain number of functors considered here are introduced in Chapter 9.

8.B.1. Definition.
Let $\mathcal{C}$ be a category; then one defines the category $\mathcal{C}^{\mathbf{ON}}$ as follows:

(i) The objects of $\mathcal{C}^{\mathbf{ON}}$ are families $(a_i)_{i<x}$ of objects of $\mathcal{C}$, indexed by all $i < x$, where x is an ordinal.

(ii) The set of morphisms from $(a_i)_{i<x}$ to $(b_i)_{i<y}$ consists of all families $(t_i)_{i<f}$ such that: $f \in I(x,y)$ and for all $i < x$, t_i is a $\mathcal{C}$-morphism from a_i to $b_{f(i)}$. This set is denoted by $I\big((a_i)_{i<x}, (b_i)_{i<y}\big)$; the composition of $(t_i)_{i<f}$ with $(u_i)_{i<g}$ is by definition $(t_{g(i)}u_i)_{i<fg}$.

8.B.2. Theorem.
Assume that $\big((a_i^l)_{i<x_l}, (t_i^{lm})_{i<f_{lm}}\big)$ is a direct system in $\mathcal{C}^{\mathbf{ON}}$ and that $\big((a_i^l)_{i<x}, (t_i^l)_{i<f_l}\big)$ enjoys 8.1.11 (i)–(iii) w.r.t. this direct system; then 8.1.11 (iv) holds (i.e. this is a direct limit) iff:

(i) $(x, f_l) = \varinjlim_{L} (x_l, f_{lm})$.

(ii) Given $z \in x$, define a subset L_z of L by: $l \in L_z \leftrightarrow z \in rg(f_l)$, and a direct system (b_z^l, g_z^{lm}) indexed by L_z, by: $b_z^l = a_z^l$ (if $z = f + l(z_l)$) $g_z^{lm} = t_{z_l}^{lm}$. Define a family (b, g^l) by $b = a_z$ and $g^l = t_{z_l}^l$. Then $(b, g^l) = \varinjlim_{L_z} (b^l, g^{lm})$.

Proof. This is a good exercise.... □

8.B.3. Theorem.
Assume that $(t_i^j)_{i<f_j}$ $(j = 1,2,3)$ are morphisms in $\mathcal{C}^{\mathbf{ON}}$ with a common target $(a_i)_{i<y}$; then $(t_i^3)_{i<f_3} = (t_i^1)_{i<f_1} \wedge (t_i^2)_{i<f_2}$ iff:

(i) $f_3 = f_1 \wedge f_2$.

(ii) given $z \in y$, define i_1, i_2, i_3 by $z = f_1(i_1) = f_2(i_2) = f_3(i_3)$; then $t^3_{i_3} = t^1_{i_1} \wedge t^2_{i_2}$.

Proof. Another exercise. □

8.B.4. Definition.
We define the functor $\sum$ from $\mathbf{ON}^{\mathbf{ON}}$ to $\mathbf{ON}$: $\sum\limits_{i<x} y_i$ is the usual sum of the family y_i,

$$\Big(\sum_{i<f} g_i\Big)\Big(\sum_{i<i_0} y_i + z\Big) = \sum_{i<f(i_0)} y'_i + g_{i_0}(z)$$

when $(g_i)_{i<f} \in I^{\mathsf{set}}\Big((y_i)_{i<x}, (y'_i)_{i<x'}\Big)$ and $i_0 < x$, $z < y_{i_0}$.

8.B.5. Theorem.
The functor of 8.B.4 preserves direct limits and pull-backs.

Proof. Easy consequence of 8.B.2 and 8.B.3… . □

8.B.6. Remark.
The sum of dilators, as defined in 9.1.1, appears as a functor from $\mathbf{DIL}^{\mathbf{ON}}$ to $\mathbf{ON}$. Obviously, this functor preserves direct limits and pull-backs. The two functors $\sum$ are related by:

$$\Big(\sum_{i<x} F_i\Big)(y) = \sum_{i<x} F_i(y)$$
$$\Big(\sum_{i<x} F_i\Big)(f) = \sum_{i<\mathbf{E}_x} F_i(f)$$
$$\Big(\sum_{i<g} T_i\Big)(y) = \sum_{i<g} T_i(y)\ .$$

8.B.7. Remark.
Another example of functor from a category of sequences to $\mathbf{ON}$ is given by Π (see 9.5).

Annex 8.C. Dendroids

Dendroids are an alternative description of dilators in terms of well-trees (in Greek $\tau\grave{o}\ \delta\acute{\varepsilon}\nu\delta\rho o\nu$ means "true"). This concept is reminiscent of a former attempt, made by Jervell to handle the concepts of Π^1_2-logic by means of *homogeneous trees* ([78]). Unfortunately, homogeneous trees are not sufficiently general, and it was necessary to consider this more complex notion. We follow our exposition in [5], Chapter 6.

8.C.1. Definition.

A **dendroid of type** x is a pair (x, D), where x is an ordinal and D is a set of sequences $(a_0, ..., a_p)$ of ordinals, such that:

(i) if $s = (a_0, ..., a_p) \in D$, then p is even.

(ii) if $s = (a_0, ..., a_p) \in D$ and $q < p$, then $(a_0, ..., a_q) \notin D$.

In order to state the remaining properties, we introduce D^* : $s \in D^*$ iff $s = (\,)$ or $s * s' \in D$ for some s' ($*$ denotes as usual concatenation).

(iii) if $s = (a_0, ..., a_{2p}) \in D^*$, if $s * (t)$, $s * (u) \in D^*$ and $t \leq u$, then $a_{2i+1} < t \leq u$ or $t \leq u < a_{2i+1}$ for all $i < p$. (As a consequence, $a_1, a_3, ..., a_{2p-1}$ are pairwise distinct.) Moreover, for all $i < p$, $a_{2i+1} < x$.

(iv) if $s = (a_0, ..., a_{2p-1}) \in D^*$, then the set $\{a\,;\, s * (a) \in D^*\}$ is an ordinal, i.e. $(a_0, ..., a_{2p-1}, b) \in D^*$ and $a < b \rightarrow (a_0, ..., a_{2p-1}, a) \in D^*$.

(v) there is no sequence (a_n) such that, for all n, $(a_0, ..., a_{n-1}) \in D^*$.

8.C.2. Remarks.

(i) Usually, when the context is clear, we shall identify a dendroid (x, D) with the second component D.

(ii) Condition (v) is a well-foundedness condition for D^*.

(iii) Observe that D is not an *abstract tree*; D^* is of course a well-founded abstract tree.

8.C.3. <u>Definition</u>.
Assume that (x, D) and (x', D') are dendroids; then $I(x, D\,;\, x', D')$ is the set of all pairs (f, g) such that:

(i) $f \in I(x, x')$.

(ii) g is a function from D to D' such that $g(s)$ has the same length as s.

(iii) if $s = (a_0, ..., a_p) \in D^*$ and $s * t$, $s * u \in D$, then one can find $s' \in D'^*$, $s' = (a'_0, ..., a'_p)$ such that $g(s * t) = s' * t'$, $g(s * u) = s' * u'$ for some t' and u'; s' will be denoted by $g^*(s)$; we define $g^*\big(()\big) = ()$: g^* maps D^* into D'^*.

(iv) $g^*\big(s * (a)\big) = g^*(s) * \big(f(a)\big)$ when $s = (a_0, ..., a_{2p}) \in D^*$.

(v) assume that $s = (a_0, ..., a_{2p-1}) \in D^*$ and that $g^*\big(s * (a)\big) = g^*(s) * (b)$ and $g^*\big(s * (a')\big) = g^*(s) * (b')$, and that $a < b$ (and $s * (a) \in D^*$, $s * (b) \in D^*$); then $a' < b'$.

8.C.4. <u>Remark</u>.
If $(f, g) \in I(x, D\,;\, x', D')$ and $(f', g') \in I(x', D'\,;\, x'', D'')$, then $(f'f, g'g) \in I(x, D\,;\, x'', D'')$.

8.C.5. <u>Example</u>.
Given a dendroid (y, D') and $f \in I(x, y)$, we shall define a new dendroid $(x, D) = f^{-1}(y, D')$ (we shall also write $D = f^{-1}(D')$) by:

(i) In D', remove all sequences $(a_0, ..., a_{2p})$ such that $a_{2i+1} \notin rg(f)$ for some $i < p$; one obtains a set D'' of sequences enjoying all properties of dendroids, except perhaps (iv); we define D''^* as in 8.C.1; the process of construction of D'' and D''^* is **mutilation** (w.r.t. f).

(ii) There exists one and only one dendroid D and one and only one function f from D to D' such that $(f, g) \in I(x, D\,;\, x', D')$, and

$rg(g) = D''$. In fact we shall define the associated function g^*. We define the members $s = (a_0, ..., a_{n-1})$ of D^* and $g^*(s)$ by induction on n. The case $n = 0$ is trivial: obviously $(\,) \in D^*$ and $g^*\big((\,)\big) = (\,)$; if $n \neq 0$, two subcases; let us write $s = t * (a)$:

1. If n is even, then $s \in D^*$ iff $t \in D^*$ and $g^*(t) * \big(f(a)\big) \in D''^*$; in that case, $g^*(s) = g^*(t) \times \big(f(a)\big)$.
2. If n is odd, then $s \in D^*$ iff $t \in D^*$ and the order type of the set $X = \{u\,;\, g^*(t) * (u) \in D''^*\}$ is $> a$; if φ is the isomorphism between $\|X\|$ and X, let $g^*(s) = g^*(t) * \big(\varphi(a)\big)$.

(iii) The function g just defined is the **mutilation function**, and is denoted by $m_f^{D'}$ (or simply m_f). $(f, m_f) \in I(x, D\,;\, x', D')$.

8.C.6. Definition.
The category **DEN** of dendroids is defined by

objects: dendroids (x, D)

morphisms from (x, D) to (x', D'): the elements of $I(x, D\,;\, x', D')$.

8.C.7. Remark.
In **DEN**, there are only trivial isomorphisms: if $(f, f') \in I(x, D\,;\, x', D')$, $(g, g') \in I(x', D'\,;\, x, D)$ are such that $(fg, f'g') = (\mathbf{E}_{x'}, \mathsf{id}_{D'})$ and $(gf, g'f') = (\mathbf{E}_x, \mathsf{id}_D)$, then $x = x'$ and $f = g = \mathbf{E}_x$. Using condition (iv) of dendroids, one easily proves that $D = D'$ and $f' = g' = \mathsf{id}_D$.

8.C.8. Definition.
The functor **type** from **DEN** to **ON** is defined by: $t(x, D) = x$, $t(f, g) = f$.

8.C.9. Proposition.
In **DEN** $(x, D\,;\, f_i, g_i) = \varinjlim (x_i, D_i\,;\, f_{ij}, g_{ij})$ iff 8.1.11 (i)–(iii) hold and

(i) $x = \bigcup_i rg(f_i)$ (i.e. $(x, f_i) = \varinjlim (x_i, f_{ij})$).

(ii) $D = \bigcup_i rg(g_i)$ (equivalently: $D^* = \bigcup_i rg(g_i^*)$).

Proof. Left to the reader (see also [5], 6.2.2). $\square$

8.C.10. Corollary.

(i) The functor t preserves direct limits.

(ii) In **DEN**, $(x, D\,;\, f_i, m_{f_i}^D) = \varinjlim\, (x_i, D_i\,;\, f_{ij}, m_{f_{ij}}^{D_j})$ iff 8.1.11 (i)–(iii) hold and $(x, f_i) = \varinjlim\, (x_i, f_{ij})$.

Proof. (i) is immediate from 8.C.9.

(ii): If 8.1.11 (i)–(iii) hold and $(x, f_i) = \varinjlim\, (x_i, f_{ij})$, take $s \in D$, $s = (a_0, ..., a_{2k}, a_{2k+1}, ...)$, and choose i such that $a_1, a_3, a_5, ... \in rg(f_i)$. Then $s \in rg(m_{f_i}^D)$, hence conditions 8.C.9 (i)–(iii) hold, hence $(x, D\,;\, f_i, m_{f_i}^D) = \varinjlim\, (x_i, D_i\,;\, f_{ij}, m_{f_{ij}}^{D_j})$. The converse implication is exactly (i). $\square$

8.C.11. Proposition.
In **DEN**, given $(f_i, g_i) \in I(x_i, D_i\,;\, x, D)$ $(i = 1, 2, 3)$; then $(f_1, g_1) \wedge (f_2, g_3) = (f_3, g_3)$ iff

(i) $rg(f_1) \cap rg(f_2) = rg(f_3)$ (i.e. $f_1 \wedge f_2 = f_3$).

(ii) $rg(g_1) \cap rg(g_2) = rg(g_3)$ (equivalently $rg(g_1^*) \cap rg(g_2^*) = rg(g_3^*)$).

Proof. Left to the reader, see also [5], 6.2.4. $\square$

8.C.12. Corollary.

(i) The functor t preserves pull-backs.

(ii) If $(x_i, D_i) = f_i^{-1}(x, D)$ $(i = 1, 2, 3)$, then $(f_1, m_{f_1}^D) \wedge (f_2, m_{f_2}^D) = (f_3, m_{f_3}^D)$ iff $f_1 \wedge f_2 = f_3$.

Proof. (i) is a trivial consequence of 8.C.11.

(ii): If $f_1 \wedge f_2 = f_3$, then $rg(f_1) \cap rg(f_2) = rg(f_3)$; furthermore it is immediate that $rg(m_{f_1}^D) \cap rg(m_{f_2}^D) = rg(m_{f_3}^D)$, hence by 8.C.11 $(f_1, m_{f_1}^D) \wedge$

$(f_2, m^D_{f_2}) = (f_3, m^D_{f_3})$. The converse implication is just (i). □

8.C.13. Definition.

(i) If (x, D) is a dendroid, then the **height** of (x, D) is the ordinal isomorphic with the well-ordering of D defined by: if $t \neq u$, then

$$s * (t) * s' <^+ s * (u) * s'' \leftrightarrow t < u \ .$$

(ii) The functor **height** from **DEN** to **ON** is defined by:

$$h(x, D) = \|(D, \leq^+)\|$$
$$h(f, g) = \|g\| \ .$$

8.C.14. Remark.
$\leq^+$ is the restriction to D of (the analogue of) the *Brouwer-König ordering* of finite sequences of ordinals. The (analogue of) the *linearization principle* (5.4.17) for ordinal trees, proves that D^* is well-ordered by $\leq^+$, and so is its subset D.

8.C.15. Proposition.
The functor h preserves direct limits and pull-backs.

Proof. Immediate consequence of 8.C.9 and 8.C.11. □

8.C.16. Definition.
A dendroid (x, D) is **homogeneous** iff:

$*$ for all $x' \leq x$, $f, g \in I(x', x)$, $f^{-1}(D) = g^{-1}(D)$.

$**$ for all $x' \leq x$, $f, g \in I(x', x)$, and $s = (x_0, ..., x_{2k}) \in f^{-1}(D)^*$, if $f(x_1) = g(x_1)$, $f(x_3) = g(x_3), ..., f(x_{2k-1}) = g(x_{2k-1})$, then $m^{D*}_f(s) = m^{D*}_g(s)$.

8.C.17. Examples.

(i) The following dendroid of type 10 is not homogeneous:

```
0       0       0     1       0       0     1     2
|       |                     |             |
3       4          5          6             7

                         0                                1

                                        .
```

This denotes the dendroid $D = \{(0,3,0),(0,4,0),(0,5,0),(0,5,1),$ $(0,6,0),(0,7,0),(0,7,1),(0,7,2),(1)\}$. Take $f \in I(3\,\mathit{cdot}\,10)$ defined by $f(0) = 1,\ f(1) = 2,\ f(2) = 9$; then $rg(m_f^D)$ is the set $\{(1)\}$, hence $f^{-1}(D) = \{(0)\}$. If one considers now $f' \in I(3,10)$ defined by $f'(0) = 4,\ f'(1) = 5,\ f'(2) = 6$, then $rg(m_{f'}^D)$ is

```
0       0     1       0
|                     |
4          5          6

           0                1

                      .
```

and $f'^{-1}(D)$ is therefore

```
0       0     1       0
|                     |
0          1          2

           0                1

                      .
```

Clearly $f^{-1}(D) \neq f'^{-1}(D)$: Condition $(*)$ is violated.

(ii) The following dendroid of type 5 is not homogeneous:

$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ | & | & | & | & | \\ 0 & 1 & 2 & 3 & 4 \\ | & | & | & | & | \\ 0 & 1 & 2 & 3 & 4 \end{array}$$

$$.$$

Condition $(*)$ of homogeneity is fulfilled, but not the subtler condition $(**)$: define for instance $f, f' \in I(1,5)$ by $f(0) = 2$, $f'(0) = 4$; then $f^{-1}(D)$ and $f'^{-1}(D)$ are both equal to:

$$\begin{array}{c} 0 \\ | \\ 0 \\ | \\ 0 \\ | \\ . \end{array}$$

but $m_f^{D*}\big((0)\big) = (2) \neq m_{f'}^{D*}\big((0)\big) = (4)$, and this contradicts condition $(**)$.

(iii) The following dendroid of type 5 is homogeneous:

```
0 1 0 1 0 1 0 1   0  0 1 0 1 0 1  0 0 0 1 0 1   0 0 0 0 1  0 0 0 0
                  |               | |           | | |      | | | |
 1   2   3   4    0   2   3   4   0 1  3   4    0 1 2  4   0 1 2 3
                          |                       |    |
       0            0     1        0     1        0    1      0
       |                                                      |
       0              1              2               3        4
                                     |
                                     0
                                     |
                                     .
```

For instance, when $f \in I(3,5)$, $f^{-1}(D)$ is always equal to

```
 0      1       0       1          0       0       1          0       0
                                   |                          |       |
    1               2              0           2              0       1
                                   |           |
            0                      0           1                  0
            |                                                     |
            0                          1                          2
                                       |
                                       0
                                       |
                                       .
```

8.C.18. <u>Proposition</u>.
In Definition 8.C.16, $(*)$ and $(**)$ can be weakened in $(*)'$ and $(**)'$ by replacing "$\forall x' \leq x$" by "$\forall x'$ $\mathsf{finite} \leq x$".

<u>Proof</u>. Left to the reader (see [5], 6.3.3). $\square$

8.C.19. <u>Proposition</u>.
Assume that (x, D) is homogeneous; then there is a functor F from $\mathbf{ON} \leq x$ to $\mathbf{DEN}$ such that:

(i) for all $x' \leq x$, $F(x')$ is a dendroid of type x'.

(ii) for all $x', x'' \leq x$ and $f \in I(x', x'')$, $F(f) = (f, m_f^{F(x'')})$.

(iii) $F(x) = (x, D)$.

Proof. Assume that (x, D) is homogeneous; then define $F(x') = (\mathbf{E}_{x'x})^{-1} \cdot (x, D)$; it is immediate that, when $f \in I(x', x'')$, then $(f, m_f^{F(x'')}) \in I\big(f^{-1} \cdot \big(F(x'')\big)\,;\, F(x'')\big)$; but $f^{-1}\big(F(x'')\big) = f^{-1}\mathbf{E}_{x''x}^{-1}\big(F(x)\big) = (\mathbf{E}_{x''x}f)^{-1}\big(F(x)\big)$; by homogeneity $(\mathbf{E}_{x''x}f)^{-1}\big(F(x)\big) = \mathbf{E}_{x'x}^{-1}\big(F(x)\big) = F(x')$. The fact that F is a functor follows from the equality

$$m_{fg}^D = m_f^D m_g^{f^{-1}(D)}\ . \qquad \square$$

8.C.20. Proposition.
Let (x, D) be a homogeneous dendroid and let F be the functor associated with (x, D) by 8.C.19; then F preserves direct limits and pull-backs.

Proof. F preserves $\varinjlim$: if $(x', f_i) = \varinjlim (x_i, f_{ij})$, then by 8.C.10 (ii), $\big(F(x'), F(f_i)\big) = \varinjlim \big(F(x_i), F(f_{ij})\big)$.

F preserves $\wedge$: if $f_1 \wedge f_2 = f_3$, then, by 8.C.12 (ii) $F(f_1) \wedge F(f_2) = F(f_3)$. $\square$

8.C.21. Definition.
A dendroid D of type ω is **strongly homogeneous** (in short: D is a **sh. dendroid**) iff for all $x \geq \omega$ there is a homogeneous dendroid (x, D') such that $D = \mathbf{E}_{\omega x}^{-1}(D')$.

8.C.22. Proposition.
D is strongly homogeneous, iff there is a functor D^0 from **ON** to **DEN** such that:

(i) for all x, $D^0(x)$ is a homogenous dendroid of type x.

(ii) for all x, y and $f \in I(x, y)$, $D^0(f) = (f, m_f^{D^0(y)})$.

(iii) $D^0(\omega) = (\omega, D)$.

<u>Proof</u>. (1) If such a functor exists, then $\mathbf{E}_x^{-1}\Big(D^0(x)\Big) = D^0(\omega) = D$, hence D is strongly homogeneous.

(2) Conversely, assume that D is strongly homogeneous; then $(x, D') = D^0(x)$ is uniquely determined by the condition $\mathbf{E}_{\omega x}^{-1}(D') = D$: since D' is homogeneous, there is a functor F from $\mathbf{ON} \leq x$ to $\mathbf{DEN}$ enjoying 8.C.19 (with D' instead of D). If $(x, f_i) = \varinjlim (x_i, f_{ij})$, with all x_i's finite, then $\Big(F(x), F(f_i)\Big) = \varinjlim \Big(F(x_i), F(f_{ij})\Big)$. But $F(x_i)$ and $F(f_{ij})$ are uniquely determined by: $F(x_i) = \mathbf{E}_{x_i\omega}^{-1}(D)$ and $F(f_{ij}) = m_{f_{ij}}^{F(x_j)}$. In $\mathbf{DEN}$, since we have no non-trivial isomorphisms, then the direct limit is unique, when it exists: hence $F(x)$ is uniquely determined. We define D^0 to be the union of all functors F defined as above, when x varies through $0n$. By Property 8.C.19 of these functors, we get (i), (ii) and (iii). $\square$

8.C.23. <u>Remark</u>.
We have seen in the proof of 8.C.22, that 8.C.22 determines D^0 uniquely. Also, observe that D^0 preserves $\varinjlim$ and $\wedge$, by 8.C.20.

8.C.24. <u>Definition</u>.
The following data define a category $\mathbf{SHD}$:

> objects: strongly homogeneous dendroids.
>
> morphisms from D to D': the set $I_{\mathsf{sh}}(D, D')$ of functions g from D to D' such that $(\mathbf{E}_{\omega'} g) \in I(\omega, D\,;\, \omega, D')$ and $m_h^{D'} g = g m_h^D$ for all $h \in I(\omega, \omega)$.

8.C.25. <u>Proposition</u>.
Assume that D and D' are sh. dendroids; then

(i) If T is a natural transformation from D^0 to D'^0, then for all x, $T(x) = (\mathbf{E}_x, T_x)$ for some T_x. (Hence $T_\omega \in I_{\mathsf{sh}}(D, D')$.)

(ii) Conversely, given $g \in I_{\mathsf{sh}}(D, D')$, there is a unique natural transformation $g^0 \in I^1(D^0, D'^0)$ such that $g^0(\omega) = (\mathbf{E}_{\omega'} g)$.

Proof. (i) $t \circ T$ is a natural transformation from $t \circ D^0$ to $t \circ D'^0$; but $t \circ D^0 = t \circ D'^0 = \mathsf{Id}$, hence $t \circ T \in I^1(\mathsf{Id}, \mathsf{Id})$. Hence $t \circ T = \mathbf{E}^1_{\mathsf{Id}}$. (For instance observe that $\mathsf{Tr}(f)\Big((0,0)\Big) = (0,0)$, if $f = t \circ T$.) Hence $t\Big(T(x)\Big) = \mathbf{E}_x$ for all x. The condition $m^{D'}_n T(\omega) = T(\omega) m^D_n$ is immediate.

(ii) Let $\mu_n = m^D_{\mathbf{E}_{n\omega}}$, let $\mu'_n = m^{D'}_{\mathbf{E}_{n\omega}}$. Let $X_n \subset D$ and $X'_n \subset D'$ be the respective ranges of μ_n and μ'_n: X_n (resp. X'_n) is the set of all sequences $(a_0, ..., a_{2p}) \in D$ (resp. D') and such that $a_1, a_3, a_5, ... < n$. It is immediate that g maps X_n into X'_n, hence there is a unique function T_n from $D^0(n)$ to $D'^0(n)$ such that $\mu'_n T_n = g\mu_n$, and it is immediate that $(\mathbf{E}_n, T_n) \in I\Big(D^0(n), D'^0(n)\Big)$; if $f \in I(n,m)$, choose $h \in I(\omega,\omega)$ such that $h\mathbf{E}_{n\omega} = \mathbf{E}_{m\omega} \cdot f$; then

$$\begin{aligned}&\mu'_m m_f^{D'^0(m)} T_n = m_h^{D'} \mu'_n T_n = m_h^{D'} g\mu_n = g m_h^D \mu_n \ = \\ &g\mu_m m_f^{D^0(m)} = \mu'_m T_m m_f^{D^0(m)} \ ,\end{aligned}$$

hence $m_f^{D'^0(m)} T_n = T_m^{D^0(m)}$. From that, it follows that $T(n) = (\mathbf{E}_n, T_n)$ defines a natural transformation from $D^0 \restriction \mathbf{ON} < \omega$ to $D'^0 \restriction \mathbf{ON} < \omega$. By general category-theoretic results (for instance see [5], 2.1.5 (ii)) T can be uniquely extended into a natural transformation from D^0 to D'^0. $\square$

8.C.26. Remark.

The monstruous condition $m_h^{D'} g = g m_h^D$ for all $h \in I(\omega,\omega)$ can obviously be replaced by the condition: $m_f^{D'^0(m)} T_n = T_m m^{D^0(m)}$ for all n, m and $f \in I(n,m)$; this enables us to replace the quantification over the non-denumerable set $I(\omega,\omega)$ by a denumerable quantifier.

8.C.27. Theorem.

One can define a functor **LIN** (**linearization**) from **SHD** to **DIL** by:

(i) If D is a sh. dendroid, then $\mathbf{LIN}(D)(x) = h\Big(D^0(x)\Big)$ and $\mathbf{LIN}(D)(f) = h\Big(D^0(f)\Big)$.

(ii) If $g \in I_{\mathsf{sh}}(D, D')$, then $\mathbf{LIN}(g)(x) = h\Big(g^0(x)\Big)$.

Proof. D^0 preserves direct limits and pull-backs, as well as h (8.C.22 and 8.C.15), hence $\mathbf{LIN}(D)$ is a dilator… . $\square$

8.C.28. <u>Definition</u>.

Assume that F is a dilator; then one defines for x an ordinal and p an integer, a binary relation $\sim_p^{F,x}$ on the ordinal $F(x)$, as follows:

$$z = (a\,;\, x_0, ..., x_{n-1}\,;\, x)_F \;\sim_p^{F,x}\; (b\,;\, y_0, ..., y_{m-1}\,;\, x) = z'$$

iff $z = z'$ or

(i) $|\S^F(a, n\,;\, b, m)| > p/2$ and

(ii) if $\sigma = \sigma^F_{a,n}$, $\tau = \sigma^F_{b,m}$, then, for all $i < p/2$, $x_{\sigma(i)} = y_{\tau(i)}$.

8.C.29. <u>Proposition</u>.

$\sim_p^{F,x}$ is an equivalence relation.

<u>Proof</u>. By Remark 8.4.22 (ii). $\square$

8.C.30. <u>Theorem</u>.

For all p, the equivalence classes modulo $\sim_p^{F,x}$ are intervals.

<u>Proof</u>. We prove the result by induction on p: assume that $A = (a\,;\, x_0, ..., x_{n-1}\,;\, x)_F$, $B = (b\,;\, y_0, ..., y_{m-1}\,;\, x)_F$, $C = (c\,;\, z_0, ..., z_{l-1}\,;\, x)$, and that $A \sim_p^{F,x} C$ and $A < B < C$. We prove that $A \sim_p^{F,x} B$: assume for contradiction that $A \not\sim_p^{F,x} B$:

(i) If $p = 0$, then $\S^F(a, n\,;\, b, m) = (0, \varepsilon)$, $\S^F(b, m\,;\, c, l) = (0, \varepsilon')$ (8.C.28 (i)); since $A < B < C$, we must have $\varepsilon = \varepsilon' = +1$, hence by 8.4.21 $\S^F(a, n\,;\, c, l) = (0, +1)$, so $A \not\sim_0^{F,x} C$.

(ii) If $p = 2q + 1$, the induction hypothesis yields $A \sim_{2q}^{F,x} B$; since we assume that $A \not\sim_{[}^{F,x} B \not\sim_p^{F,x} C$, we get $x_{\sigma(q)} \neq y_{\tau(q)} \neq z_{\rho(q)}$. If $y_{\tau(q)} < x_{\sigma(q)}$, then since $x_{\sigma(0)} = y_{\tau(0)}, ..., x_{\sigma(q-1)} = y_{\tau(q-1)}$ and $|\S^F(a, n\,;\, b, m)| > q$, it would follow that $B < A$; hence $x_{\sigma(q)} < y_{\tau(q)}$ and similarly $y_{\tau(q)} < z_{\rho(q)}$, hence we get $x_{\sigma(q)} < z_{\rho(q)}$, contradiction with $A \sim_p^{F,x} C$.

(iii) If $p = 2q+2$, the induction hypothesis yields $A \sim_{2q+1}^{F,x} B$, and so the hypothesis $A \not\sim_p^{F,x} B \not\sim_p^{F,x} C$ implies $\S^F(b, m\,;\, c, l) = (q + 1, \varepsilon)$ and $\S^F(b, m\,;\, c, l) = (q+1, \varepsilon')$; since $x_{\sigma(0)} = y_{\tau(0)}, ..., x_{\sigma(q)} = y_{\tau(q)}$ it follows that $\varepsilon = +1$, similarly $\varepsilon' = +1$. By 8.4.21, we get $\S^F(a, n\,;\, c, l) = (q+1, +1)$, a contradiction with $A \sim_p^{F,x} C$. $\square$

8.C.31. Proposition.

(i) If $f \in I(x, y)$ and $z \sim_p^{F,x} z'$, then $F(f)(z) \sim_p^{F,y} F(f)(z')$.

(ii) If $T \in I^1(F, G)$ and $z \sim_p^{F,x} z'$, then $T(x)(z) \sim_p^{G,x} T(x)(z')$.

(iii) The equivalence class of $(a\,;\, x_0, ..., x_{n-1}\,;\, x)_F$ modulo $\sim_{2n}^{F,x}$ consists of exactly one point.

Proof. (i) and (ii) are immediate. For (iii) observe that $|\S^F(a, n\,;\, b, m)| \leq \mathsf{inf}\,(n, m) \leq 2n/2$, hence the only possibility left in Definition 8.C.28, when $z = (a\,;\, x_0, ..., x_{n-1}\,;\, x)_F \sim_{2n}^{F,x} z'$ is $z = z'$. $\square$

8.C.32. Notation.

We shall use the notation I_k to denote the pair (I_k, k) where I_k is an equivalence class modulo $\sim_k^{F,x}$. (F and x are assume to be clear from the context; if not, we shall say "I_k in $F(x)$".) I_k is an interval, and observe that there is a unique sequence $I_0, ..., I_{k-1}$ such that $I_0 \supset ... \supset I_{k-1} \supset I_k$ (for $i \leq k$, I_i is the equivalence class modulo $\sim_i^{F,x}$ containing I_k). Define $I_k \lhd J_k$ by $\forall z \in I_k\ \forall z' \in J_k\ z < z'$, and $I_k \unlhd J_k$: $I_k = J_k$ or $I_k \lhd J_k$. Define the ordinals $|I_k|$ as follows:

(i) $|I_0|$ is the order type of the set of predecessors of I_0 for $\lhd$.

(ii) $|I_{2k+1}|$ = the ordinal b defined by: if $(a\,;\, x_0, ..., x_{n-1}\,;\, x)_F \in I_{2k+1}$ then $b = x_{\sigma_{a,n}^F(k)}$ if $k < n$, $b = 0$ otherwise.

(iii) $|I_{2k+2}|$ = the order type of the set of predecessors of I_{2k+2} for $\lhd$ which are included in I_{2k+1}.

Assume that I_k is an interval in $F(x)$, $f \in I(x, y)$; we shall denote by $\bar{F}(f)(I_k)$ the interval J_k in $F(y)$ containing the image of I_k under $F(f)$.

Similarly, if I_k in $F(x)$, $T \in I^1(F, G)$, we shall denote by $T(x)(I_k)$ the interval J_k in $G(x)$ containing the image of I_k under $T(x)$.

8.C.33. Definition.

(i) If F is a dilator, one defines the function $\boldsymbol{\varphi}_{F,x}$ from $F(x)$ to the class of finite sequences of ordinals by:

$$\boldsymbol{\varphi}_{F,x}\Big((a\,;\,x_0,...,x_{n-1}\,;\,x)_F\Big) = (|I_0|,...,|I_{2n}|)$$

with $I_{2n} = \{(a\,;\,x_0,...,x_{n-1}\,;\,x)_F\}$ and $I_0 \supset ... \supset I_{2n}$.

(ii) If F is a dilator, one defines the function $\boldsymbol{\psi}_{F,f}$, when $f \in I(x,y)$, from $rg(\boldsymbol{\varphi}_{F,x})$ to $rg(\boldsymbol{\varphi}_{F,y})$:

$$\boldsymbol{\psi}_{F,f}(|I_0|,...,|I_{2n}|) = (|\bar{F}(f)(I_0)|,...,|\bar{F}(f)(I_{2n})|)\ .$$

(iii) If $T \in I^1(F,G)$, one defines a function $\boldsymbol{\psi}_{T,x}$ from $rg(\boldsymbol{\varphi}_{F,x})$ to $rg(\boldsymbol{\varphi}_{G,x})$ by:

$$\boldsymbol{\psi}_{T,x}(|I_0|,...,|I_{2n}|) = (|\bar{T}(f)(I_0),...,|\bar{T}(f)(I_{2n})|)\ .$$

8.C.34. <u>Theorem</u>.

(i) $rg(\boldsymbol{\varphi}_{F,x})$ is a dendroid of type x.

(ii) $(f,\boldsymbol{\psi}_{F,f}) \in I\Big(x, rg(\boldsymbol{\varphi}_{F,x})\,;\,y, rg(\boldsymbol{\varphi}_{F,y})\Big)$; furthermore $\boldsymbol{\psi}_{F,f}\boldsymbol{\varphi}_{F,x} = \boldsymbol{\varphi}_{F,y}F(f)$.

(iii) $(E_x,\boldsymbol{\psi}_{T,x}) \in I\Big(x, rg(\phi_{F,x})\,;\,x, rg(\boldsymbol{\varphi}_{G,x})\Big)$; furthermore $\boldsymbol{\psi}_{T,x}\boldsymbol{\varphi}_{F,x} = \boldsymbol{\varphi}_{G,x}T(x)$.

<u>Proof</u>. First of all, remark that the sequence $(|I_0|,...,|I_p|)$ determines I_p in a unique way.

(<u>Proof</u>. By induction on p: $|I_0|$ determines I_0, since I_0 is the $|I_0|^{\text{th}}$ equivalence class modulo $\sim_0^{F,x}$; if $p = 2q+1$, then by the induction hypothesis, I_{2q} is uniquely determined; the ordinal $|I_{2q+1}|$ gives the common value $x_{\sigma(q)}$ of all points in I_{2q+1}: $(a\,;\,x_0,...,x_{n-1}\,;\,x)_F \in I_{2q+1}$ iff it belongs to I_{2q} and $x_{\sigma(q)} = |I_{2q+1}|$. Finally, if $p = 2q+2$, and if I_{2q+1} is well-determined (induction hypothesis), then I_{2q+2} is the $|I_{2q+2}|^{\text{th}}$ equivalence class modulo $\sim_{2q+2}^{F,x}$ included in I_{2q+1}. $\square$)

(i): We check properties 8.C.1 (i)–(v): (i)–(iv) are immediate, for instance (ii): if $z \neq z'$, then the associated sequences $(I_0, ..., I_{2n})$ and $(I'_0, ..., I'_{2n'})$ are such that $I_{2n} = \{z\} \neq \{z'\} = I_{2n'}$, hence by our preliminary remark one can find $p \leq \inf(n, n')$ such that $|I_p| \neq |I'_p|$.... Property (v) is much more delicate: by our preliminary remark a s.d.s. in $rg(\boldsymbol{\varphi}_{F,x})^*$ for $\leq^*$ is the same thing as a decreasing sequence

$$I_0 \supset ... \supset I_p \supset ... .$$

Define $f \in I(x, \omega \cdot x)$ by $f(z) = \omega \cdot z$, and let $J_p = \bar{F}(f)(I_p)$; then

$$J_0 \supset ... \supset J_p \supset ... .$$

Write $J_p = [a_p, b_p[$; the ordinals b_p form a decreasing sequence, hence $b_p = b = \mathsf{constant}$ for all $p \geq N$. But, if $c = (z\,; p_0, ..., p_{n-1}\,; \omega \cdot x)_F \in J_{2N+1}$, define $q_0, ..., q_{n-1}$: $q_{\sigma(0)} = p_{\sigma(0)}, ..., q_{\sigma(N)-1} = p_{\sigma(N)-1}$, $q_{\sigma(N)} = p_{\sigma(N)+1}$, $q_{\sigma(n-1)} = p_{\sigma(n-1)} + 1$; with $\sigma = \sigma^F_{z,n}$. Such a definition is possible because $p_{\sigma(0)} = |J_1|, ..., p_{\sigma(N-1)} = |J_{2N-1}|$ are in $rg(f)$, i.e. are limit. If $d = (z\,; q_0, ..., q_{n-1}\,; \omega \cdot x)$, then $x \sim_{2N} d$, hence $d < b_{2N}$; but $c \not\sim_{2N+1} d$ and $c < d$, hence $b_{2N+1} \leq d$; a contradiction with $b_{2N} = b_{2N+1}$.

(ii): $\boldsymbol{\psi}_{F,f}\boldsymbol{\varphi}_{F,x} = \boldsymbol{\varphi}_{F,y}F(f)$ is exactly the definition; so we need only show that $(f, \boldsymbol{\psi}_{F,f})$ is a morphism of dendroids. This is completely evident; observe that

$$\boldsymbol{\psi}^*_{F,f}(|I_0|, ..., |I_{p-1}|) = (|\bar{F}(f)(I_0)|, ..., |\bar{F}(f)(I_{p-1})|) .$$

(iii): Similar to (ii), left to the reader.... □

8.C.35. <u>Theorem</u>.
It is possible to define a functor **BCH** (**branching**) from **DIL** to **SHD** by:

$$\mathbf{BCH}(F) = rg(\boldsymbol{\varphi}_{F,\omega})$$

$$\mathbf{BCH}(T) = \boldsymbol{\psi}_{T,\omega} .$$

BCH enjoys the following properties:

(i) $\mathbf{BCH}(F)^\circ(x) = \big(x, rg(\boldsymbol{\varphi}_{F,x})\big)$.

(ii) $\mathbf{BCH}(F)^\circ(f) = (f, \boldsymbol{\psi}_{F,f})\ (=\ (f, m_f^{rg(\boldsymbol{\varphi}_{F,y})}))$.

(iii) $\mathbf{BCH}(T)^\circ(x) = (E_x, \boldsymbol{\psi}_{T,x})$.

Proof. Essentially we need to prove that $\mathbf{BCH}(F)$ is a sh. dendroid for all dilators F, and that $\mathbf{BCH}(T) \in I_{\mathsf{sh}}\big(\mathbf{BCH}(F), \mathbf{BCH}(G)\big)$. It will suffice to show that (i)–(ii) define a functor $\mathbf{BCH}(F^)\circ$ from **ON** to **DEN** enjoying Conditions 8.C.22 (i)–(iii), and (by 8.C.25 (i)) that (iii) defines a natural transformation from $\mathbf{BCH}(F)^\circ$ to $\mathbf{BCH}(G)^\circ$.

1. We claim that $(|I_0|, ..., |I_{2n}|) \in rg(\boldsymbol{\psi}_{F,f})$ iff $(|I_0|, ..., |I_{2n}|) \in rg(\boldsymbol{\varphi}_{F,y})$ and $I_1, I_3, I_5, ... \in rg(f)$.

(Proof. Assume that $I_{2n} = \{(z\,;\, x_0, ..., x_{n-1}\,;\, x)_F\}$; then $(|I_0|, ..., |I_{2n}|) \in rg(\boldsymbol{\psi}_{F,f})$ iff $I_{2n} = \bar{F}(f)(I_{2n})$ for some J_{2n} $(=\ \{(z\,;\, f^{-1}(x_0), ..., f^{-1}(x_{n-1})\,;\, x)_F\}$, this is equivalent to $x_0, ..., x_{n-1} \in rg(f)$, in other terms $|I_1|, ..., |I_{2n-1}| \in\ rg(f)$. $\square$)

Hence $rg(\boldsymbol{\psi}_{F,f})$ is equal to the range of the mutilation function $m_f^{rg(\boldsymbol{\varphi}_{F,y})}$, hence $\boldsymbol{\psi}_{F,f} = m_f^{rg(\boldsymbol{\varphi}_{F,y})}$: this establishes 8.C.22 (ii). The homogeneity of $rg(\boldsymbol{\varphi}_{F,x})$ is left to the reader.

2. By 8.C.33 (ii), (iii) we get:

$$\boldsymbol{\psi}_{G,f}\boldsymbol{\psi}_{T,x}\boldsymbol{\varphi}_{F,x} = \boldsymbol{\psi}_{G,f}\boldsymbol{\varphi}_{G,x}T(x) = \boldsymbol{\varphi}_{G,y}G(f)T(x)\ =$$
$$\boldsymbol{\varphi}_{G,y}T(y)F(f) = \boldsymbol{\psi}_{T,y}\boldsymbol{\varphi}_{F,y}F(f) = \boldsymbol{\psi}_{T,y}\boldsymbol{\psi}_{F,f}\boldsymbol{\varphi}_{F,x}\ .$$

This clearly proves that $\boldsymbol{\psi}_{G,f}\boldsymbol{\psi}_{T,x} = \boldsymbol{\psi}_{T,y}\boldsymbol{\psi}_{F,f}$ hence $\mathbf{BCH}(T)^\circ$ is a natural transformation... . $\square$

8.C.36. Theorem.

(i) $\mathbf{LIN} \circ \mathbf{BCH} = \mathbf{ID}_{\mathbf{DIL}}$.

(ii) $\mathbf{BCH} \circ \mathbf{LIN} = \mathbf{ID}_{\mathbf{SHD}}$.

Proof. (i): The function $\boldsymbol{\varphi}_{F,x}$ is strictly increasing: if $z < z'$, and $\boldsymbol{\varphi}_{F,x}(z) = (|I_0|, ..., |I_{2n}|)$, $\boldsymbol{\varphi}_{F,x}(z') = (|I'_0|, ..., |I'_{2n'}|)$, choose i minimum such that $I_i \neq I'_i$; then $I_i \lhd I'_i$ (since $z \in I_i$, $z' \in I'_i$), hence $|I_0| = |I'_0|, ..., |I_{i-1}| = |I'_{i-1}|$,

$|I_i| < |I'_i|$, and $\boldsymbol{\varphi}_{F,x}(z) <^+ \boldsymbol{\varphi}_{F,x}(z')$; hence the order type of $bfBCH(F)^\circ(x)$ is equal to $F(x)$: $\mathbf{LIN}\Big(\mathbf{BCH}(F)\Big)(x) = h\Big(\mathbf{BCH}(F)^\circ(x)\Big) = F(x)$. Now, if $f \in I(x,y)$, the formula

$$\boldsymbol{\psi}_{F,f}\boldsymbol{\varphi}_{F,x} = \boldsymbol{\varphi}_{F,y}F(f)$$

means exactly that $h(f, \boldsymbol{\psi}_{F,f}) = F(f)$, so

$$\mathbf{LIN}\Big(\mathbf{BCH}(F)\Big)(f) = h\Big(\mathbf{BCH}(F)^\circ(f)\Big) = F(f)\ .$$

If $T \in I^1(F,G)$, the formula

$$\boldsymbol{\psi}_{T,x}\boldsymbol{\varphi}_{F,x} = \boldsymbol{\varphi}_{G,x}T(x)$$

means that $(\mathbf{E}_x, \boldsymbol{\psi}_{T,x}) = T(x)$, hence

$$\mathbf{LIN}\Big(\mathbf{BCH}(T)\Big)(x) = h\Big(\mathbf{BCH}(T)^\circ(x)\Big) = T(x)\ .$$

We have proved that $\mathbf{LIN}\Big(\mathbf{BCH}(F)\Big) = F$ and $\mathbf{LIN}\Big(\mathbf{BCH}(T)\Big) = T$.

(ii): If D is a sh. dendroid, then we shall first study the dilator $F = \mathbf{LIN}(D)$:

1. If $s \in D$, then let z be the order type of the set of predecessors of s w.r.t. $<^+$, and write $z = (a\,;\, x_0, ..., x_{n-1}\,;\, \omega)_F$, $s = (u_0, ..., u_{2m})$; from the equality $F(f) = h\Big(D^\circ(f)\Big)$ it follows that $z \in rg\Big(F(f)\Big)$ if $s \in rg(m^D_f)$, for all $f \in I(x,\omega)$, and all $x \le \omega$: from this it follows that $m = n$, and that $\{x_0, ..., x_{n-1}\} = \{u_1, ..., u_{2n-1}\}$. Define a permutation τ of n by: $\tau(i) = j$ iff $u_{2i+1} = x_j$; we claim that $\tau = \sigma^F_{a,n}$; we shall compare the points $z_f = F(f)(z)$ when $f \in I(\omega,\omega)$; obviously $z_f < z_g$ iff $s_f <^+ s_g$, with $s_f = m^D_f(s)$. Here we use property $(**)$ of homogeneous dendroids: assume that $f(u_1) = g(u_1), ..., f(u_{2k-1}) = g(u_{2k-1})$, then $s_f\restriction 2k+1 = s_g\restriction 2k+1$; now, if $f(u_{2k+1}) < g(u_{2k+1})$, it follows that $s_f <^+ s_g$: $z_f < z_g$ iff $f(u_1) = g(u_1), ..., f(u_{2k-1}) = g(u_{2k-1})$, $f)u_{2k+1}) < g(u_{2k+1})$ for some $k < n$, equivalently $f(x_{\tau(0)}) = g(x_{\tau(0)}), ..., f(x_{\tau(k-1)}) = g(x_{\tau(k-1)})$, $f(x_{\tau(k)}) < g(x_{\tau(k)})$. Hence $\tau = \sigma^F_{a,n}$.
2. We shall now compute the relations $\sim^{F,\omega}_k$: assume that $s' = (v_0, ..., v_{2n'}) \in D$, and that $z' = (b\,;\, y_0, ..., y_{n'-1}\,;\, \omega)_F$ is the order type of the set of predecessors of s' w.r.t. $<^+$; then we claim that (we assume that $z \neq z'$)

$$z \sim_k^{F,\omega} z' \quad \text{iff } u_0 = v_0, ..., u_k = v_k \ .$$

Assume that $z \sim_k^{F,\omega} z'$: hence $x_{\sigma(i)} = y_{\tau(i)}$ for all $i < k/2$, i.e. $u_{2i+1} = v_{2i+1}$ for all i such that $2i + 1 \leq k$. Let $r = [k/2]$ (the greatest integer $\leq k/2$); then $\sigma(i) < \sigma(r)$ iff $\tau(i) < \tau(r)$ for all $i < r$ (with $\sigma = \sigma^F_{a,n}$, $\tau = \tau^F_{b,n}$), hence one can find functions $f, g, h \in I(\omega, \omega)$ such that: $f(x_{\sigma(i)}) = g(y_{\tau(i)}) = h(x_{\sigma(i)})$ for all $i < r$ and $f(x_{\sigma(r)}) < g(y_{\tau(r)}) < h(x_{\sigma(r)})$. Now, observe that $|\S^F(a, n\,; b, n')| > k/2 \geq r$ (easy exercise for the reader; this can be obtained on the model of Part 1. above), hence, if $z'_f = F(f)(z')$, one gets:

$$z_f < z'_g < z_h \ ,$$

and write $s'_f = m^D_f(s')$: $s_f <^+ s'_g <^+ s_h$. Now we apply once more property $(**)$ of homogeneous dendroids, since $f(2i+1) = g(2i+1) = h(2i+1)$ for all i s.t. $2i + 1 < 2r$:

$$m^{D*}_f\Big((u_0, ..., u_{2r})\Big) = m^{D*}_g\Big((u_0, ..., u_{2r})\Big) = m^{D*}_h\Big((u_0, ..., u_{2r})\Big) \qquad (= A)$$
$$m^{D*}_f\Big((v_0, ..., v_{2r})\Big) = m^{D*}_g\Big((v_0, ..., v_{2r})\Big) = m^{D*}_h\Big((v_0, ..., v_{2r})\Big) \qquad (= B)$$

and the inequalities $s_f <^+ s_f <^+ s_h$ will be rewritten as:

$$A * t_f <^+ B * t'_g <^+ A * t_h \ ,$$

and so $A = B$. This forces $u_i = v_i$ for all $i < 2r$, and this establishes $u_0 = v_0, ..., u_k = v_k$ when k is even; but when k is odd, we know that $u_k = v_k \ldots$.

Conversely, assume that $u_0 = v_0, ..., u_k = v_k$; observe that $u_{2i+1} < u_{2r+1}$ iff $v_{2i+1} < v_{2r+1}$ for $i < r$: if k is odd, then $k = 2r + 1$, and this is simply trivial; if k is even, then by Property 8.C.1 (iii) of dendroids, the point $u_{2i+1} = v_{2i+1}$ cannot belong to the interval $[u_{2r+1}, v_{2r+1}]$ (or $[v_{2r+1}, u_{2r+1}]$). Hence it will be possible to define functions f, g, h exactly as above, and it is immediate that $s_f <^+ s'_g <^+ s_h$, hence $z_f < z'_g < z_h$; the definition of $\S^F(a, n\,; b, n')$ shows that $|\S^F(a, n\,; b, n')| > r$ hence $|\S^F(a, n\,; b, n')| > k/2$. The hypothesis yields

$$x_{\sigma(i)} = y_{\tau(i)} \quad \text{for all } i < k/2 \ ,$$

so $z \sim_k^{F,\omega} z'$.

3. Let us compute the ordinals $|I_k|$: assume that $z \in I_k$, and choose s as above; we claim that $|I_k| = u_k$:

   – If k is odd, then $|I_k| = x_{\sigma(k-1/2)} = u_k$.

   – If $k = 0$, then $z \sim_0^{F,\omega} z$ iff $u_0 = v_0$: I_0 is the u_0^{th} class, hence $|I_0| = u_0$.

   – If $k = 2p+2$, then $z \sim_{2p+1}^{F,\omega} z'$ iff $u_0 = v_0, ..., u_{2p+1} = v_{2p+1}$, hence u_k enables us to distinguish between the various classes modulo $\sim_{2p+2}^{F,\omega}$ which are included in I_{2p+1}, and from this, again $|I_k| = u_k$.

4. From 3, it becomes possible to compute $\mathbf{BCH}\Big(\mathbf{LIN}(D)\Big)$: this dendroid is the range of the function which associates to $s \in D$, the sequence $(|I_0|, ..., |I_{2n}|)$, with $I_{2n} = \{z\}$. Clearly $\mathbf{BCH}\Big(\mathbf{LIN}(D)\Big) = D$.

5. Finally, we compute $\mathbf{BCH}\Big(\mathbf{LIN}(g)\Big)$ when $g \in I_{\mathsf{sh}}(D, D')$: if $g(s) = s''$ and if z'' is the order type of the set of predecessors of s'' w.r.t. $<^+$ in D', then

$$\mathbf{BCH}\Big(\mathbf{LIN}(g)\Big)(s) = (|\bar{T}(\omega)(I_0)|, ..., |\bar{T}(\omega)(I_{2n})|) \ ,$$

   with $T = \mathbf{LIN}(g)$; but, by definition, $T(\omega)(z) = z''$, and we know by 3 that (if $z'' \in J_k$ in D') $|J_k| = w_k$, (if $s'' = (w_0, ..., w_{2n})$), hence

$$\mathbf{BCH}\Big(\mathbf{LIN}(g)\Big)(s) = s'' = g(s) \ . \qquad \square$$

8.C.37. Corollary.
The functors **BCH** and **LIN** preserve direct limits and pull-backs.

Proof. Trivial. □

Annex 8.D. Quasi-dendroids

The concept of quasi-dendroid is akin to the concept of dendroid; its main interest lies in its greater flexibility and its direct relation to $\boldsymbol{\beta}$-proofs (Chapter 10). But the category of strongly homogeneous quasi-dendroids is no longer isomorphic to **DIL**. A certain number of results will be listed without proof, especially those who duplicate results of 8.C.

8.D.1. Definition.
A **quasi-dendroid** of type x is a pair (x, D) where x is an ordinal and:

(i) D is a set of finite sequences $s = (x_0, ..., x_n)$ such that, for all $i < n$:

- either x_i is an ordinal,
- or x_i is a pair $(a_i, -)$, with $a_i < x$; such a pair will be denoted by $\underline{a}_i$.

(For instance $s = (0, \omega, \underline{5}, \omega + 1, \underline{\omega^2 + \omega})$.)

(ii) If $s = (x_0, ..., x_n) \in D$ and $m < n$, then $(x_0, ..., x_m) \notin D$.

(iii) Let $D^* = \{(\,)\} \cup \{s\,;\, s * s' \in D \text{ for some } s'\}$, then:if x, x' are ordinals and $s * (x) \in D^*$, then $s * (\underline{x}') \notin D^*$.

(iv) There is no sequence x_n such that, for all n:

$$(x_0, ..., x_{n-1}) \in D^* \ .$$

8.D.2. Remarks.

(i) We shall abbreviate "quasi-dendroid" into "qd".

(ii) Dendroids will be identified with particular qd's: if D is a dendroid, define D':

$$D' = \{(x_0, \underline{x}_1, x_2, ..., \underline{x}_{2n-1}, x_{2n})\,;\, (x_0, x_1, x_2, ..., x_{2n-1}, x_{2n}) \in D\} \ .$$

Then we identify D with the qd D'.

(iii) The **height** $h(D)$ is its order type for the order $<^+$, defined as usual. (Of course $s * (\underline{a}) <^+ s * (\underline{b})$ iff $a < b$.) There is a bijective function φ_D from D to $h(D)$, such that $s <^+ s' \leftrightarrow \varphi_D(s) < \varphi_D(s')$.

8.D.3. Definition.
Assume that (x, D) and (x', D') are qd's; then $I_q(x, D\,;\, x', D')$ is the set of all pairs (f, g) such that:

(i) $f \in I(x, x')$.

(ii) g is a function from D to D' such that $g(s)$ has the same length as s.

(iii) If $s = (a_0, ..., a_p) \in D^*$ and $s * t$, $s * u \in D$, then one can find $s' \in D'^*$, $s' = (a'_0, ..., a'_p)$ such that $g(s * t) = s' * t'$, $g(s * u) = s' * u'$ for some t' and u'; s' will be denoted by $g^*(s)$; we define $g^*\big(()\big) = ()$: g^* maps D^* into D'^*.

(iv) $g^*\big(s * (\underline{a})\big) = g^*(s) * \big(\underline{f(a)}\big)$ if $s * (\underline{a}) \in D^*$.

(v) If a is an ordinal, then $g^*\big(s * (a)\big) = g^*(s) * (b)$ for some ordinal b, depending on s and a; moreover, if $a < a'$ and $g^*\big(s * (a')\big) = g^*(s) * (b')$, then $b < b'$.

8.D.4. Definition.
The following data define a category **QDN**

objects: quasi-dendroids (x, D)
morphisms from (x, D) to (x', D'): $I_q(x, D\,;\, x', D')$.

8.D.5. Remark.
It is possible to define functors *type* and *height* from **QDN** to **ON**. It is also possible to establish for **QDN** the exact analogue of 8.C.8–15. (But the mutilation functions m_f^D are replaced by μ_f^D..., see 8.D.6.)

8.D.5. Definition.

Assume that D' is a qd of type y and let $f \in I(x, y)$; we shall define a new qd $(x,^f D)$ by:

(i) In D' remove all sequences $(a_0, ..., a_p)$ such that $a_i = \underline{z}$ and $z \notin rg(f)$ for some $i < p$; one obtains a set D'' of sequences which is still a qd of type y. The process of construction of D'' is called **mutilation** w.r.t. f.

(ii) Define a function μ_f as follows

$$\begin{aligned} &\mu_f\Big(()\Big) = () \\ &\mu_f\Big(s * (a)\Big) = \mu_f(s) * (a) && \text{if } a \in 0n \\ &\mu_f\Big(s * (\underline{a})\Big) = \mu_f(s) * \Big(\underline{f(a)}\Big) && \text{if } a < x \ . \end{aligned}$$

We define ${}^f D' = \{s \, ; \, \mu_f(s) \in D'\}$, i.e.

$${}^f D' = \mu_f^{-1}(D') = \mu_f^{-1}(D'') \ .$$

(iii) Define a function $\mu_f^{D'}$ from ${}^f D'$ to D' by:

$$\mu_f^{D'}(s) = \mu_f(s) \quad \textsf{for all } s \in {}^f D' \ .$$

Clearly $(f, \mu_f^{D'}) \in I_q(x,^f D' \, ; \, y, D')$.

8.D.7. Remark.
The mutilations μ_f^D are simpler than the mutilations m_f^D; when D is a dendroid, then ${}^f D \neq f^{-1}(D)$ in general: ${}^f D$ is not necessarily a dendroid: for instance, in the Example 8.C.17 (iii) ${}^f D$ is (if $f(0) = 1$, $f(1) = 2$, $f(2) = 4$)

$$
\begin{array}{ccccccccccccccc}
0 & & 1 & & 0 & & 1 & 0 & & 0 & & 1 & 0 & & 0 \\
 & & & & & & & | & & & & & | & & | \\
 & \underline{1} & & & & \underline{2} & & \underline{0} & & & \underline{2} & & \underline{0} & & \underline{1} \\
 & & & & & & & | & & & | & & & & \\
 & & & 1 & & & & 0 & & & 1 & & & 0 & \\
 & & & | & & & & & & & & & & | & \\
 & & & \underline{0} & & & & & \underline{1} & & & & & \underline{2} & \\
 & & & & & & & & | & & & & & & \\
 & & & & & & & & 0 & & & & & & \\
 & & & & & & & & | & & & & & & \\
 & & & & & & & & . & & & & & &
\end{array}
$$

and ${}^fD$ differs from $f^{-1}(D)$ by the choice of the non-underlined ordinals.

8.D.8. Definition.
A qd D of type x is **homogeneous** iff the following property holds

$(*)$ for all $x' \le x$ and $f \in I(x', x)$, $f' \in I(x', x)$, then ${}^fD ={}^{f'} D$.

8.D.9. Remarks.

(i) A good surprise is that the technical condition $(**)$ has disappeared.

(ii) The analogues of 8.C.18–20 hold.

(iii) One defines in a similar way **strongly homogeneous** qd; they form a category **SHQ** whose morphisms are denoted by $I_{\mathsf{qsh}}(D, D')\ldots$. The analogues of 8.C.22–26 hold for **shqd**'s (strongly homogeneous quasi-dendroids).

(iv) One defines the functor $\mathbf{LIN}_q$ from **SHQ** to **DIL** by $\mathbf{LIN}_q(D)(x) = h\big(D^\circ(x)\big)$, $\mathbf{LIN}_q(D)(f) = h\big(D^\circ(f)\big)$ and $\mathbf{LIN}_q(T)(x) = h\big(T^\circ(x)\big)$. The main problem is to define a functor $\mathbf{BCH}_q$ from **DIL** to **SHQ**; one cannot expect that $\mathbf{BCH}_q\big(\mathbf{LIN}_q(D)\big) = D$ (for instance, if $D = \{(0)\}$ and $D' = \{(0)\}$, then $\mathbf{LIN}_q(D) = \mathbf{LIN}_q(D') = \underline{1}$).

8.D.10. Definition.

(i) Assume that F is a dilator and let $D = \mathbf{BCH}(F)$; let $D' = D^\circ(\omega^2)$, and let $f \in I(\omega, \omega^2)$: $f(n) = \omega \cdot (1+n)$. We define $\mathbf{BCH}_q(F) =^f D'$.

(ii) If $T \in I^1(F, G)$, let $g = \mathbf{BCH}(T)$ and $g' = g^\circ(\omega^2)$; we define $\mathbf{BCH}_q(T)$ by:

$$\mathbf{BCH}_q(T)\mu_f^{D'} = \mu_f^{E'} g' \quad (\text{if } E' = \mathbf{BCH}(G)^\circ(\omega^2)) .$$

8.D.11. Theorem.

(i) $\mathbf{BCH}_q$ is a functor from **DIL** to **SHQ**.

(ii) $\mathbf{LIN}_q \circ \mathbf{BCH}_q = \mathbf{ID}_{\mathbf{DIL}}$.

Proof. (i): Let $D'_\beta = D^\circ\Big(\omega(1+\beta)\Big)$ and $D''_\beta =^{f_\beta} D'_\beta$, with $f_\beta \in I\Big(\beta, \omega(1+\beta)\Big)$: $f_\beta(z) = \omega(1+z)$. We show that if $g \in I(\beta, \beta')$, then ${}^g D''_{\beta'} = D''_\beta$; since $D''_\omega = \mathbf{BCH}_q(F)$, this will establish that $\mathbf{BCH}_q(F)$ is strongly homogeneous.
We establish the following property: let $g' = \mathbf{E}_\omega \cdot (\mathbf{E}_1 + g)$; then if $s = (x_0, \underline{x}_1, x_2, ..., \underline{x}_{2n-1}, x_{2n}) \in D'_\beta$ and $x_1, x_2, ..., x_{2n-1}$ are limit, then $m_{g'}^{D'_{\beta'}}(s) = (x_0, \underline{g'(x_1)}, x_2, ..., \underline{g'(x_{2n-1})}, x_{2n})$ $(= \mu_{g'}(s))$.

(Proof. Looking back at the definition of $m_{g'}^{D'_{\beta'}}$ we see that it suffices to show that (with $X = rg(m_{g'}^{D'_{\beta'}})$): if $t = (y_0, \underline{y}_1, ..., ...) \in {D'_{\beta'}}^*$, with $y_1, y_3, y_5, ...$ limit, then $t \in X^* \leftrightarrow y_1, y_3, y_5, ... \in rg(g')$. Clearly, if $t \in X^*$ then $y_1, y_3, y_5, ...$ are in $rg(g')$; conversely, if $y_1, y_3, y_5, ... \in rg(g')$, choose t' such that $t * t' \in D'_{\beta'}$; if $t * t' = (y_0, \underline{y}_1, ..., y_{2n})$, choose a function $g'' \in I\Big(\omega \cdot (1+\beta), \omega \cdot (1+\beta')\Big)$ such that: $y_1, y_3, y_5, ..., y_{2n-1} \in rg(g'')$, and consider $m_{g'}^{D'}\Big((m_{g''}^{D'_{\beta'}})(t * t')\Big)$. Using $(**)$ this point is of the form $t * t''$ and all its odd coefficients $y'_1, y'_3, y'_5, ..., y'_{2n-1}$ are in $rg(g')$: this proves that $t * t'' \in X$ and so $t \in X^*$. □)

Using the property, one easily shows that ${}^g D''_{\beta'} = D''_\beta$.

(ii) is left as an exercise to the reader... □

8.D.12. Definition.
A dilator is **finitistic** iff the associated shqd $\mathbf{BCH}_q(F) = D$ has the following property:

$$\forall n\ \exists m\ \forall s \in D^*\ \Big(lh(s)\Big) = 2n \to \{x\,;\ s * (x) \in D^*\} \subset m\ .$$

8.D.13. Examples.

(i) Finite dimensional dilators are finitistic. (In 8.D.12 take $m = \mathsf{dim}(F)$).

(ii) Finitistic dilators are weakly finite: since $D(n)$ has finite branchings, $F(n)$ is finite.

(iii) But there are weakly finite dilators which are not finitary, for instance $F = \sum\limits_{n<\omega} D_n$, with D_n a prime dilator with $\mathsf{Tr}(D_n) = \{(0, n)\}$.

8.D.14. Comments.
Finitistic dilators are closer to our intuition of a dilator than other concepts so far introduced: a dilator is a way to describe ordinals by means of "finitary data". Of course the dilator must itself be "finitary" in some sense. Finite dimensional dilators form a very small class which cannot fit our intuition, whereas weakly finite dilators contain objects that cannot be accepted as "finitary", for instance the dilator of Example 8.D.13 (iii) which is of kind ω! Observe that it was necessary to make a rather complex definition (leading to $\mathbf{BCH}_q(F)$) in order to find the concept of a finitistic dilator; dendroids and quasi-dendroids are useful because they help us to visualize the "algebraic" structure of dilators... . Another use of qd's can be found in the

8.D.15. Corollary.
The ordering $\leq^F$ (8.4.22) has the following property: let $\mathsf{Tr}_n(F) = \{(z_0, n)\,;\ (z_0, n) \in \mathsf{Tr}(F)\}$; then $\leq^F \restriction \mathsf{Tr}_n(F)$ is a well-order when F is a dilator.

Proof. If $(z_0, n) \in \mathsf{Tr}(F)$, then we associate to this point a sequence $(x_0, ..., x_n) = g(z_0, n)$ as follows: If φ is the order-preserving isomorphic between $F(\omega)$ and $(D, <^+)$ (with $D = \mathbf{BCH}_q(F)$), consider $\varphi\Big((z_0\,;\ y_0, ...,$

$y_{n-1}\,;\,\omega)_F\big)$: this is a point of D of the form $(x_0, \underline{y}_0, x_1, ..., \underline{y}_{n-1}, x_n)$; we observe now that $x_0, ..., x_n$ are independent of the choice of the integers $y_0 < ... < y_{n-1}$.

(<u>Proof</u>. If $\varphi\big((z_0\,;\,y'_0, ..., y'_{n-1}\,;\,\omega)_F\big)$ is equal to $(x'_0, \underline{y}'_0, x'_1, ..., \underline{y}'_{n-1}, x'_n)$, then since the diagram

$$\begin{tikzcd} F(\omega) \arrow[r, "F(f)"] \arrow[d, "\varphi"'] & F(\omega) \arrow[d, "\varphi"] \\ D \arrow[r, "\mu_f^D"'] & D \end{tikzcd}$$

is commutative for all $f \in I(\omega, \omega)$, we get, if we choose f, f' such that $f(y_0) = f'(y'_0), ..., f(y_{n-1}) = f'(y'_{n-1})$:

$$(x_0, f(y_0), x_1, ..., f(y_{n-1}), x_n) = \mu_f^D \varphi\big((z_0\,;\,y_0, ..., y_{n-1}\,;\,\omega)_F\big) =$$
$$\varphi F(f)\big((z_0\,;\,y_0, ..., y_{n-1}\,;\,\omega)_F\big) = \varphi\big((z_0\,;\,f(y_0), ..., f(y_{n-1})\,;\,\omega)_F\big)$$

and

$$(x'_0, \underline{f'(y'_0)}, x'_1, ..., \underline{f'(y_{n-1})}, x_n) = \big((z_0\,;\,f'(y'_0), ..., f'(y'_{n-1})\,;\,\omega)_F\big)$$

by symmetry. Hence $z_0 = x'_0, ..., x_n = x'_n$. □)

Now observe that the function g is strictly increasing: if $(z_0, n) <^F (z_1, n)$ let $p = |\S^F(z_0, n\,;\,z_1, n)|$; choose integers $y_0 < ... < y_{n-1}$ and $y'_0 < ... < y'_{m-1}$ such that

$$y_{\sigma(0)} = y'_{\sigma(0)}, ..., y_{\tau(p-1)} = y'_{\tau(p-1)}\ ,$$

with $\sigma = \sigma^F_{z_0,n}$, $\tau = \sigma^F_{z_1,n}$. Then $(z_0\,;\,y_0, ..., y_{n-1}\,;\,\omega)_F < (z_1\,;\,y'_0, ..., y'_{n-1}\,;\,\omega)_F$; let $s = \varphi\big((z_0\,;\,y_0, ..., y_{n-1}\,;\,\omega)_F\big)$, $s' = \varphi\big((z_0\,;\,y'_0, ..., y'_{n-1}\,;\,\omega)_F\big)$; then $s <^+ s'$. If $s = (x_0, \underline{y}_0, ..., \underline{y}_{n-1}, x_n)$, $s' = (x'_0, \underline{y}'_0, ..., \underline{y}'_{n-1}, x'_n)$, then either $(x_0, ..., x_p) <^+ (x'_0, ..., x'_p)$ hence $g(z_0, n) <^+ g(z_1, n)$, or $(x_0, ..., x_p) = (x'_0, ..., x'_p)$. But this case is impossible: since D obviously enjoys Condition 8.C.1 (iii) of dendroids, then y_p and y'_p lay in the same of the intervals determined by $y_0, ..., y_{p-1}$, and we can assume that $y'_p < y_p$: but this would entail $(x'_0, ..., x'_p) <^+ (x_0, ..., x_p)$.

The existence of a strictly increasing function from $\mathsf{Tr}_n(F)$ to $0n$ implies that $\mathsf{Tr}_n(F)$ is a well-order. □

8.D.16. Remarks.

(i) The set $\mathsf{Tr}_{\leq n}(F) = \{(z_0, p) \in \mathsf{Tr}(F)\,;\, p \leq n\}$ is well-ordered by $<^F$: this is an easy consequence of 8.D.15.

(ii) But $\mathsf{Tr}(F)$ is not necessarily well-ordered by $<^F$. For instance the points (2,1),(12.2),(112,3),(1112,4),... form a s.d.s. in $\mathsf{Tr}(10^{\mathsf{Id}})$.

One can define an equivalence relation $\asymp$ between quasi-dendroids of the same type. This relation has the property that all equivalence classes contain one and only one dendroid: to each quasi-dendroid one can associate the unique dendroid $N(D)$ such that $D \asymp N(D)$. The definition of the equivalence relation $\asymp$ and its immediate properties are rather long to establish, and the reader will find all necessary information in [5], Chapter 7.

The equivalence relation $\asymp$ is very useful because the conditions defining quasi-dendroids are not as drastic as the conditions defining dendroids. It is therefore possible to define operations on dendroids as follows:

+ define the operation on quasi-dendroids.

+ then "normalize", i.e. apply $N(\cdot)$.

Of course the operations one defines, must be compatible with the equivalence relations $\asymp$ ..., see [5], 7.2 for a list of such operations..., see also 8.G.4.

Annex 8.E. Π^1_2-completeness of dilators

This is one of the very first results of Π^1_2-logic, that I proved a few months before the $\boldsymbol{\beta}$-completeness theorem. The original proof is too archaic to be presented here; moreover, the $\boldsymbol{\beta}$-completeness theorem yields Π^1_2-completeness of $\boldsymbol{\beta}$-proofs, and by linearization, of dilators, see for instance 10.1.27; hence the only interest of this annex is to present a simple and direct argument. The proof which follows is due to Normann.

8.E.1. Theorem (Girard, 1978).
The set of all indices of prim. rec. dilators is Π^1_2-complete; more precisely

(i) the formula expressing that "e is the index of a prim. rec. dilator" is Π^1_2.

(ii) if A is a closed Π^1_2 formula, then one can construct (primitive recursively in A) a prim. rec. predilator P such that:

$$A \leftrightarrow \text{“}P \text{ is a dilator”} \ .$$

Furthermore this equivalence is provable in $PRA^2 + \Sigma^0_1 - CA^*$.

Proof. (i) is left to the reader (for a proof, see 10.1.26).

(ii): We start with a Π^1_2 formula A, and we assume that A is

$$\forall f \ \exists g \ \forall n \ R\Big(f^*(n), g^*(n)\Big)$$

with R prim. rec.; this is made possible by the general result of 5.1. If f is a function, we define T_f by:

$$s \in T_f \leftrightarrow Seq(S) \wedge \forall n < lh(s) \quad R(f^*(n), s{\restriction}n) \ .$$

Then clearly: $A \leftrightarrow \forall f \ \neg WTR(T_f)$.

If α is an ordinal, we define an abstract tree $D(\alpha)$ as consisting of all sequences

$$s = (a_0, ..., a_{n-1})$$

such that:

(i) a_{2i} is an integer, for all i s.t. $2i < n$; let $t = \langle a_0, ..., a_{2i}, ...\rangle$ $(2i < n)$.

(ii) a_{2i+1} is an ordinal $< \alpha$, for all i s.t. $2i + 1 < n$.

(iii) For all s_1 and s_2 such that $2s_1 + 1, 2s_2 + 1 < n$, $s_1 <^* s_2$ and $\forall p < lh(s_i)$, $R(t\restriction p, s_i\restriction p)$ $(i = 1, 2)$, we have:

$$a_{2s_1+1} < a_{2s_2+1} \ .$$

8.E.2. Lemma.
$A \leftrightarrow \forall\alpha \in 0n\ D(\alpha)$ is well-founded w.r.t. the familiar ordering of sequences.

Furthermore this result is provable in $PRA^2 + \Sigma^0_1 - CA^*$.

Proof. (i): Assume A, and for contradiction that $s_n = (a_0, ..., a_{n-1})$ is a s.d.s. in some $D(\alpha)$; then let $f(n) = a_{2n}$, and, when $s \in T_f$, $h(s) = a_{2s+1}$; clearly h is a strictly increasing function from T_f to α, and this forces T_f to be a wf-tree, contradiction.

(ii): Conversely, assume $\neg A$, hence there is an f such that $WTR(T_f)$. Now there is a strictly increasing function h from T_f to some $\alpha \neq 0$, and we define $h(n) = 0$ when $n \notin T_f$; the sequence $s_n = (a_0, ..., a_{n-1})$, with $a_{2i} = f(i)$, $a_{2i+1} = h(i)$ is a s.d.s. in $D(\alpha)$, contradiction.

As to the formalization, observe that ordinals must be interpreted by well-orders. Everything is immediate, except in Part (ii), the existence of a strictly increasing function from T_f to a well-order, which follows from the linearization principle, *a consequence of $\Sigma^0_1 - CA^*$ (see 5.4).* □

We define a predilator Q by:
$Q(\alpha)$ = the set $D(\alpha$, equipped with the Brouwer-Kleene ordering of sequences; $Q(f)$ is by definition equal to $D(f)$: $D(f)\Big((..., x_{2i}, ..., x_{2j+1}, ...)\Big) = (..., x_{2i}, ..., f(x_{2j+1}), ..)$.

Hence we obtain

8.E.3. Lemma.
$A \leftrightarrow Q$ is a dilator.

This result is provable in $PRA^2 + \Sigma^0_1 - CA^*$.

Proof. Immediate use of the *linearization principle*. □

However, Q is not necessarily weakly finite; we introduce P as follows: $P(\alpha)$ is the set of all pairs $(z, 2^{u_0} + ... + 2^{u_{n-1}})$ such that:

(i) $z = (z_0, ..., a_{k-1}) \in D(\alpha)$; let $t = \langle a_0, ..., a_{2i}, ...\rangle$ $(2i < k)$.

(ii) $u_{n-1} < ... < u_0 < \alpha$ and $n = t$.

The pairs are lexicographically ordered, and

$$P(f)\Big((z, 2^{u_0} + ... + 2^{u_{n-1}})\Big) = (D(f)(z), 2^{f(u_0)} + ... + 2^{f(u_{n-1})}) \ .$$

8.E.4. Lemma.
The predilator P is weakly finite and prim. rec.

Proof. In $P(n)$, an element $(z, 2^{u_0} + ... + 2^{u_{m-1}})$ is such that $m \leq n$, hence if $z = (a_0, ..., a_{k-1})$, the value of $t = \langle a_0, ..., a_{2i}, ...\rangle$ is bounded by n. We have only finitely many choices for the ordinal parameters: hence $F(n)$ is finite. Looking back at the definition of Q and P, we immediately see that P is prim. rec. □

8.E.5. Lemma.
P is a dilator $\leftrightarrow$ Q is a dilator.

This result is provable in $PRA^2 + \Sigma_1^0 - CA^*$.

Proof. (i): If (z_n) is a s.d.s. in $Q(\alpha)$, we may assume that α is infinite; write $z_n = (a_0^n, ..., a_i^n, ...)$ and let $t_n = \langle a_0^n, ..., a_{2i}^n, ...\rangle$; then $(z_n, 2^{t_n - 1} + ... + 2^0)$ is a s.d.s. in $P(\alpha)$.

(ii): $P(\alpha)$ appears as a subset of the product $2^\alpha \cdot Q(\alpha)$, with induced order; if $Q(\alpha)$ is a well-order so is $P(\alpha)$.

$\Sigma_1^0 - CA^*$ *is clearly needed in Part (ii).* □

The lemma concludes the proof of 8.E.1. □

8.E.6. Remarks.

(i) Of course, if A contains second order parameters, then the construction is still valid and all we have to do is to remark that P is then prim. rec. in these parameters.

The proof of the equivalence can still be carried out in PRA^2.

(ii) A typical application of 8.E.1 can be found in next annex (8.F.4).

Annex 8.F. The basis theorems

Basis theorems are results of the kind: "if the class $\mathcal{C}$ of functions is non-void, then it contains a function of $\mathcal{D}$", i.e.: $\mathcal{C} \neq \emptyset \to \mathcal{C} \cap \mathcal{D} \neq \emptyset$.

There are three main basis theorems, which are connected to the logical complexities Σ^0_1, Π^1_1, Π^1_2.

8.F.1. Theorem (Kreisel basis theorem, [79]).
Let $A(X)$ be of Π^0_1 formula of L^2 depending on a variable X of type s; then, if $\exists X\ A(X)$, X can be chosen Δ^0_2, i.e. there is a Δ^0_2 set X s.t. $A(X)$.

Proof. Replace X by its characteristic function f; then $A(X)$ can be rewritten $A'(f)$, and the hypothesis is that $\exists f\Big(\forall x(f(x) = \bar{0} \vee f(x) = \bar{1}) \wedge A'(f)\Big)$. In other terms, there is an infinite branch in a prim. rec. tree T associated with A', and consisting of 0's and 1's:

$$s \in T \leftrightarrow Seq(s)\ \wedge\ \forall i < lh(s)\Big((s)_i = \bar{0} \vee (s)_i = \bar{1}\Big)\ \wedge$$
$$\forall n \leq lh(s) R(s\restriction n)$$

where the prim. rec. predicate R is such that:

$$A'(f) \leftrightarrow \forall n\ R\Big(f^*(n)\Big)$$

(see 5.2.4 for more details).

If T is not a wf-tree, then we explicitly define a s.d.s. $\langle f(0), ..., f(n-1)\rangle = s_n$ in T as follows: observe that T cannot be finite, and let $s_0 = \langle\,\rangle$; if s_n has been defined and $\in T$, and T_{s_n} is infinite, then

– either $T_{s_n * \langle 0\rangle}$ is infinite: let $s_{n+1} = s_n\ *\ \langle 0\rangle$.

– or $T_{s_n * \langle 0\rangle}$ is finite: let $s_{n+1} = s_n\ *\ \langle 1\rangle$.

Now, observe that "T_s is infinite" can be expressed by a Π^0_1 formula: let φ be the prim. rec. function defined by:

$$\varphi(n) = \mathsf{sup}\,\{s\,;\, lh(s) = n \wedge \forall i < n (s)_i \leq 1\}\ .$$

Then T_s infinite $\leftrightarrow \forall n\ \exists t \leq \varphi(n)(Seq(t) \wedge lh(t) = n \wedge s\ *\ t \in T)$. Now, there is a formula $B(s, n)$ which holds iff s is s_n:

$$B(s,n) \leftrightarrow Seq(s) \wedge lh(s) = n \wedge \forall i < n\Big((s)_i = \bar{0} \leftrightarrow T_{s\restriction i * \langle 0 \rangle} \text{ infinite}\Big) .$$

Observe that B can be put in both of Σ^0_2 and Π^0_0 forms; hence the function f such that $s_n = f^*(n)$ can be expressed by:

$$f(n) = m \leftrightarrow \forall s(B(s,n+1) \to (s)_n = m) \qquad (\Pi^0_2)$$

$$\leftrightarrow \exists s(B(s,n+1) \to (s)_n = m) \qquad (\Sigma^0_2)$$

Hence the graph of f is Σ^0_2 and Π^0_2, i.e. f is a Δ^0_2 function, i.e. if $f = X_X$, X is Δ^0_2. □

8.F.2. <u>Theorem</u> (Kleene basis theorem, [80]).
Let $A(f)$ be a Π^0_1 formula of L^2_{pr}, and assume that $\exists f\ A(f)$l then such a f can be chosen among the set of functions which are recursive in some Π^1_1 predicate.

<u>Proof</u>. Write $\exists f\ A(f)$ as "R is not a well-order", for some recursive linear order R; then, by 5.6.7, if $\exists f\ A(f)$, one can find a s.d.s. in R which is recursive in the Π^1_1 set $|Acc(R)|$. □

8.F.3. <u>Theorem</u> (Novikoff-Kondo-Addison, [82]).
Let $A(f)$ be a Π^1_1 formula of L^2_{pr}, and assume that $\exists f\ A(f)$; then such a f can be chosen among Δ^1_2 functions.

<u>Proof</u>. This basis theorem is a standard corollary to the Novikoff-Kondo-Addison uniformization lemma; the reader can find a proof in [7], pp. 188–189; dilators enable us to give a rather elegant proof of a result which is slightly weaker than the uniformization lemma, but which contains 8.F.3:

8.F.4. <u>Theorem</u>.
Let P be a prim. rec. predilator; there is a Π^1_1 predicate $B(h)$, with the following properties:

(i) $B(h) \wedge B(h') \to h = h'$.

(ii) $\exists h\ B(h) \leftrightarrow P$ is not a dilator.

(iii) If P is not a dilator, then the unique h s.t. $B(h)$ encodes a pair $\langle f_1, f\rangle$ such that: f_1 is the characteristic function of a denumerable well-order $\leq$, f is a s.d.s. in $P(\preceq)$.

<u>Proof</u>. A priori, there is some trouble in defining $\preceq$ uniquely (because $\preceq$ is an order between integers); but the ordinal $\alpha = \| \preceq \|$ can be chosen unambiguously: $\alpha = \mathsf{inf}\,\{x \in 0n\,;\, P(x)$ is not a well-order$\}$. We shall not bother about the representation of α as an order between integers, and we turn to the problem of choosing a specific s.d.s. in $P(\alpha)$. Let I be the *inaccessible part* of $P(\alpha)$, i.e. the set $\{y \in P(\alpha)\,;\, P(\alpha)\restriction y$ is not a well-order$\}$; we shall construct a s.d.s.

$$g(n) = (i_n\,;\, x_1^n, ..., x_{r_n}^n\,;\, \alpha)_P$$

in $P(\alpha)$: $g(n+1) < g(n)$ ($\leq$ is the order of $P(\alpha)$). Assume that $g(0) > ... > g(n-1)$ have already been constructed, and let $I_n = \{z \in I\,;\, z < g(i)$ for all $i < n\}$; then I_n is non void, because I has no smallest element. Consider the set

$$Sp(I_n) = \Big\{(z_0\,;\, p) \in \mathsf{Tr}(P)\,;\, \exists \alpha_0, ..., \alpha_{p-1}(\alpha_0 < ... < \alpha_{p-1} < \alpha \,\wedge\, (z_0\,;\, \alpha_0, ..., \alpha_{p-1}\,;\, \alpha)_P \in I_n)\Big\}\ .$$

Define $\langle i_n, r_n\rangle = \mathsf{inf}\,\{\langle z_0, n\rangle\,;\, (z_0\,;\, n) \in Sp(I_n)\}$; now we consider the subset J_n of $P(\alpha)$ defined by:

$$J_n = \Big\{(i_n\,;\, \alpha_1, ..., \alpha_{r_n}\,;\, \alpha)_P\,;\, \alpha_1 < ... < \alpha_{r_n} < \alpha\Big\}\ ,$$

i.e. J_n is the set of all points of $P(\alpha)$ whose denotation w.r.t. P has the "skeleton" $(i_n\,;\, r_n)$. Observe that

8.F.5. <u>Lemma</u>.
J_n is a well-ordered subset of $P(\alpha)$.

<u>Proof</u>. Define Q and $T \in I^1(Q, P)$ by $rg\big(\mathsf{Tr}(T)\big) = \{(i_n\,;\, r_n)\}$; then Q is a finite dimensional predilator, and is therefore (isomorphic to) a dilator (8.4.3), hence J_n, which is isomorphic to $Q(\alpha)$ is a well-order. $\square$

Hence $J_n \cap I_n$, which is non void, has a smallest element: $(i_n\,;\, x_1^n, ..., x_{r_n}^n\,;\, \alpha)_P$; this element is by definition $g(n)$.

First observe that:

8.F.6. Lemma.

$$\alpha = \{x_i^n\,;\, n \in I\!N \wedge 1 \leq i \leq r_n\}\ .$$

Proof. Let $Y = \{x_i^n\,;\, n \in I\!N \wedge 1 \leq i \leq r_n\}$; define β and $h \in I(\beta, \alpha)$ by $rg(h) = Y$. Then the sequence $g'(n) = (i_n\,;\, h^{-1}(x_1^n), ..., h^{-1}(x_{r_n}^n)\,;\, \alpha)_P$ is a s.d.s. in $P(\beta)$. By the choice of α, it follows that $\beta = \alpha$; since $h^{-1}(x_i^n) \leq x_i^n$, it follows that $g'(n) \leq g(n)$ for all n; $g'(n)$ is a point in $J_n \cap I_n$ which is $\leq g(n)$, hence $g'(n) = g(n)$ for all n: this forces $h(y) = y$ for all $y \in Y$; but since $rg(h) = Y$, this is only possible if $Y = \alpha$. $\square$

8.F.6 enables us to define a surjection from the set $X = \{\langle n, i\rangle\,;\, n \in I\!N \wedge 1 \leq i \leq r_n\}$ onto α:

$$\varphi(\langle n, i\rangle) = x_i^n\ .$$

We define $\langle n, i\rangle \preceq \langle m, j\rangle$ by $x_i^n \leq x_j^m$. $(X, \preceq)$ is a preorder, and the associated order is a well-order isomorphic to α; let $\asymp$ be the associated equivalence, and $N(\langle n, i\rangle) = \mathsf{inf}\,(\langle m, j\rangle\,;\, \langle n, i\rangle \asymp \langle m, j\rangle)$; then $Y = \{\langle n, i\rangle \in X\,;\, N(\langle n, i\rangle) = \langle n, i\rangle\}$ is well ordered by $\preceq$.

We define the s.d.s. f in $P\Big((Y, \leq)\Big)$:

$$f(n) = \langle i_n, N(\langle n, 1\rangle), ..., N(\langle n, r_n\rangle)\rangle$$

(the last component is not indicated...).

Now we show that Y, $\preceq\restriction Y$, and f are the only solution of a Π_1^1 property; the property is a conjunction $(\mathsf{i}) \wedge ... \wedge (\mathsf{vi})$:

(i) For all n, $f(n)$ is a sequence $\langle i_n, p_1^n, ..., p_{r_n}^n\rangle$, with:

- $(i_n, t_n) \in \mathsf{Tr}(P)$.
- $p_i^j \in Y$.
- $p_1^n \prec ... \prec p_{r_n}^n$.

(ii) $(Y, \preceq)$ is a well order; moreover

$$Y = \Big\{\langle i,j\rangle\,;\, 1 \leq j \leq r_i \wedge p^i_j = \langle i,j\rangle \wedge \\ \forall i' < i\ \forall j'(1 \leq j' \leq r_{i'} \to p^i_j \neq p^{i'}_{j'})\Big\}\ .$$

In order to state the remaining properties, let us order the set S of all formal sequences $\langle i, a_0, ..., a_{n-1}\rangle$ such that $(i\,;\,n) \in \mathsf{Tr}(P)$ and $a_0, ..., a_{n-1} \in Y$ and $a_0 \prec ... \prec a_{n-1}$:

$$\langle i, a_0, ..., a_{n-1}\rangle < \langle j, b_0, ..., b_{m-1}\rangle \ \text{ iff}$$
$$(i\,;\,c_0, ..., c_{n-1}\,;\,n+m\rangle_P < (j\,;\,d_0, ..., d_{m-1}\,;\,n+m)_P\ ,$$

where $c_0 < ... < c_{n-1} < n+m$, and $d_0 < ... < d_{m-1} < n+m$ are such that: $c_i < d_j \leftrightarrow a_i < b_j$ (resp. $c_i > d_j \leftrightarrow a_i > b_j$).

(iii) If $z \in Y$, then the set of all points $\langle i, a_0, ..., a_{n-1}\rangle$ in S such that $a_{n-1} < z$, is well ordered by $<$.

(iv) For all n, $f(n+1) < f(n)$.

(v) If $z = \langle i, a_0, ..., a_{p-1}\rangle < f(n)$, then

– either $\langle i, p\rangle > \langle i_n, r_n\rangle$.

– or the set of the predessors of Y w.r.t. $\preceq$ is a well-order.

(vi) Let $z = \langle i, a_0, ..., a_{p-1}\rangle \in S$; then

– $z \geq f(0) \to \langle i, p\rangle \geq \langle i_0, r_0\rangle$.

– $f(n+1) \leq z < f(n) \to \langle i_{n+1}, r_{n+1}\rangle \leq \langle i, p\rangle$.

Y, $\preceq\restriction Y$, and f are solutions of this Π^1_1 property; if Y', $\preceq'$, f' is another solution, observe that:

– $\|(Y', \preceq')\| = \alpha$: this follows from (ii) and (iii); let ψ be the order-preserving isomorphism from $(Y, \preceq')$ to α; define $g'(n) = (i_n\,;\,\psi(p^n_1), ..., \psi(p^n_{r_n})\,;\,\alpha)_P$; then, by (iv), (v), (vi), we obtain

– $g' = g$.

Finally observe that (ii) forces $Y' = Y$, $\preceq' = \preceq\lceil Y$, and $f' = f$.

The final solution of 8.F.4 is obtained by expressing $(\mathsf{i}) \wedge \ldots \wedge (\mathsf{vi})$ as a Π^1_1 predicate depending on a code $h = \langle f_1, f\rangle$ for 3-uple $(y, \preceq\lceil Y, f)\ldots$. □

Proof of 8.F.3. If $A(f)$ is Π^1_1, and $\exists f\ A(f)$, then, by Π^1_2 completeness of dilators, we can construct a prim. rec. predilator P such that P is not a dilator. Using 8.F.4, we obtain $\exists! f\ B(f)$; this unique f_0 such that $B(f_0)$ is Δ^1_2, i.e. its graph is both Σ^1_2 and Π^1_2:

$$\begin{aligned} f_0(n) = m \leftrightarrow\ & \exists f (B(f) \wedge f(n) = m) \\ \leftrightarrow\ & \forall f (B(f) \rightarrow f(n) = m)\ . \end{aligned}$$

Now, an inspection of the proof of Π^1_2-completeness of dilators shows the existence of an arithmetical $g = T(f)$ (g is defined arithmetically in f) such that $B(f) \rightarrow A(g)$; hence if $g_0 = T(f_0)$ it is clear that $A(g_0)$; finally observe that g_0, which is arithmetical in a Δ^1_2 function, is Δ^1_2 as well. □

8.F.7. Remarks.

(i) The three basis theorems can be restated as:

- the class of all Δ^0_2 sets is a basis for Π^0_1 classes of sets.
- the class of all functions which are recursive in a Π^1_1 set, is a basis for Π^0_1 classes of functions.
- the class of all Δ^1_2 functions is a basis for Π^1_1 classes of functions.

(ii) The three basis theorems are still valid when A is modified into $A' = \exists Y\ A$ (in 8.F.1) and $A' = \exists g\ A'$ (in 8.F.2–3); in particular:

- the class of all functions which are recursive in a Π^1_1 set, is a basis for Δ^1_1 classes of functions.
- the class of all Δ^1_2 functions is a basis for Σ^1_2 classes of functions.

Hence 8.A.1, and 8.A.2, 8.A.3 modified as above, give us the logical complexity of models of consistent theories in:

Σ^0_1-logic: the model can be chosen Δ^0_2.

Π^1_1-logic: the model can be chose recursive in Π^1_1.
Π^1_2-logic: the model can be chosen Δ^1_2.

(iii) In the proof of 8.F.4, one fact was absolutely essential: ordinals are linearly ordered; hence it is possible to make a unique choice of $\|Y\|$. If we try to adopt 8.F.4 to Π^1_3-logic (i.e., if we start with a preptyx P of type $(\mathbf{O} \to \mathbf{O}) \to \mathbf{O}$, which is not a ptyx), there will be no way of choosing a particular dilator F such that $P(F)$ is not an ordinal, because dilators are not "naturally" linearly ordered. But, as soon as this F is given (it may be part of the data), the descending sequence $g(n)$ can be uniquely constructed, exactly as in the proof of 8.A.4, mutatis mutandis... .

(iv) Of course, the three basis theorems can be relativized: if A depends on an extra function parameter g, then the basis must be relativized w.r.t. g.

(v) *Concerning the formal provability of 7.A.1–3:*

- *7.A.1 is clearly provable in* $PRA^2 + \Sigma^0_1 - CA^*$.
- *7.A.2 and 7.A.3 are provable in* $PRA^2 + \Sigma^0_1 - CA^* + \Pi^1_1 - CA$.

Annex 8.G. Exercises

8.G.1. Preservation of pull-backs.

Let F be a functor from **ON** to **ON** preserving direct limits. We consider the functor

$$F'(x) = 2^x \cdot F(x) \qquad F'(f) = 2^f \cdot F(f) \; .$$

(i) Show that F' preserves direct limits.

(ii) Define, for all $x \in 0n$, a subset A_x of $F'(x)$ as follows: A_x consists of all $z = 2^x \cdot a + 2^{b_0} + ... + 2^{b_{n-1}}$, with: $a < F(x)$, $x > b_0 >,,, > b_{n-1}$, and $a \in rg\Big(F(f)\Big)$, where $f \in I(n,x)$ is defined by $rg(f) = \{b_0, ..., b_{n-1}\}$. Show the existence of a functor G from **ON** to **ON**, together with a natural trnasformation T from G to F', such that $rg\Big(T(x)\Big) = A_x$ for all x. Prove that G preserves direct limits and pull-backs.

(iii) Show that $F(x) \leq G(x)$ for all x.

(*Remark. The meaning of 8.G.1 is that preservation of pull-backs can be easily obtained by a slight modification of the functors... . See 8.G.15 for a better result... .*)

8.G.2. Paleodilators.

Consider functors from **OL** to **OL** preserving direct limits and pull-backs; we call then paleodilators.

(i) First we study paleodilators of dimension 1: if F is a paleodilator of dimension 1, such that $F(n) = 1$ show the existence of two disjoint subsets X^+ and X^- of $n = \{0, 1, ..., n-1\}$ such that $X^+ \cup X^- = n$, and with the property that: if $i_0, ..., i_{n-1}$, $j_0, ..., j_{n-1}$ are strictly increasing sequences of integers $< p$, and such that $\forall k \in X^+ \; i_k \leq j_k$, $\forall k \in X^- \; j_k \leq i_k$, then

$$(0\,;\, i_0, ..., i_{n-1}\,;\, p)_F \leq (0\,;\, j_0, ..., j_{n-1}\,;\, p)_F \; .$$

Show the existence of a permutation σ of the integers $0, ..., n-1$, with the following property: if $i_0 < ... < i_{n-1} < p$, $j_0 < ... < j_{n-1} < p$, $i_{\sigma(0)} = j_{\sigma(0)}, ..., i_{\sigma(k-1)} = j_{\sigma(k-1)}$ and $i_{\sigma(k)} < j_{\sigma(k)}$ then:

– if $\sigma(k) \in X^+$, then $(0\,;\, i_0, ..., i_{n-1}\,;\, p)_F < (0\,;\, j_0, ..., j_{n-1}\,;\, p)_F$.

– if $\sigma(k) \in X^-$, then $(0\,;\, i_0, ..., i_{n-1}\,;\, p)_F > (0\,;\, j_0, ..., j_{n-1}\,;\, p)_F$.

(ii) Next we study paleodilators of dimension 2: assume that $\mathsf{Tr}(F) = \{(a,n),(b,m)\}$, $\mathsf{dim}(F) = 2$; assume that X^+, X^-, σ have been associated to (a,n), and that Y^+, Y^-, τ have been associated with (b,m); show the existence of a pair (p,ε), such that: $\varepsilon = +1$ or -1, $p \leq n, m$ and for all $i < p$: $\sigma(i) \in X^+ \leftrightarrow \tau(i) \in Y^+$, for all $i, j < p$: $\sigma(i) < \sigma(j) \leftrightarrow \tau(i) < \tau(j)$ with the following property: if $i_0 < ... < i_{n-1} < q$, $j_0 < ... < j_{m-1} < q$, then:

- if $i_{\sigma(0)} = j_{\tau(0)}, ..., i_{\sigma(p-1)} = j_{\tau(p-1)}$, then
  $(a\,;\, i_0, ..., i_{n-1}\,;\, q)_F < (b\,;\, j_0, ..., j_{n-1}\,;\, q)_F \quad \mathsf{iff}\ \varepsilon = +1$.
- if $i_{\sigma(0)} = j_{\tau(0)}, ..., i_{\sigma(k-1)} = j_{\tau(k-1)}$, $i_{\sigma(k)} < j_{\tau(k)}$ for some $k < p$, then
  + if $\sigma(k) \in X^+$, then $(a\,;\, i_0, ..., i_{n-1}\,;\, q)_F < (b\,;\, j_0, ..., j_{m-1}\,;\, q)_F$.
  + if $\sigma(k) \in X^-$, then $(a\,;\, i_0, ..., i_{n-1}\,;\, q)_F > (b\,;\, j_0, ..., j_{m-1}\,;\, q)_F$.

(iii) Finally, we study paleodilators of dimension 3: assume that $\mathsf{dim}(F) = 3$ and $\mathsf{Tr}(F) = \{(a,n),(b,m),(c,p)\}$; assume that $\S(a,n\,;\,b,m) = (r,+1)$, $\S(b,m\,;\,c,p) = (s,+1)$; show that $\S(a,n\,;\,c,p) = (\mathsf{inf}\,(r,s), +1)$.

(iv) Consider *paleodendroids*, i.e. the same thing as quasi-dendroids, except that two colours (+,−) are possible when we underline elements; we assume of course that all branchings are homogeneous in colour, i.e. that $s \in D^* \wedge s * (\underset{+}{x}) \in D^* \rightarrow s * (\underset{-}{y}) \notin D^*$, etc.... The theory of paleodendroids is developed in a way akin to quasi-dendroids, except that:

– we do not bother about any kind of well-foundedness assumptions.

– when we define the order $<^+$, we say that

$s * (x) * s' <^+ s * (y) * s'' \quad$ if $x < y$

$s * (\underset{+}{x}) * s' <^+ s * (\underset{+}{y}) * s'' \quad$ if $x < y$ $s * (\underset{-}{x}) * s' <^+ s * (\underset{-}{y}) * s'' \quad$ if $x > y$.

Show that if D is a homogeneous paleodendroid of type ω, then one can construct a paleodilator $F = \mathbf{LIN}_p(D)$, by $F(x) =$ ordertype of $D^\circ(x)$ w.r.t. $<^+$.

(v) Using (iv) shows that:

- given $n = X^+ \cup X^-$, $X^+ \cap X^- \neq \emptyset$, given σ permutation of n, there is a paleodilator F of dimension 1 corresponding to the data X^+, X^-, σ.
- given $n = X^+ \cup X^-$, $X^+ \cap X^- = \emptyset$, given σ permutation of n, given $m = Y^+ \cup Y^-$, $Y^+ \cap Y^- = \emptyset$, given τ permutation of m, given p such that: $\forall i < p\ \sigma(i) \in X^+ \leftrightarrow \tau(i) \in X^+$, $\forall i \forall j < p\ \sigma(i) < \sigma(j) \leftrightarrow \tau(i) < \tau(j)$ show the existence of a paleodilator F of dimension 2 corresponding to the data X^+, X^-, σ, Y^+, Y^-, τ, p, $+$.

(*One would show that paleodilators and homogeneous paleodendroids of type ω form equivalent categories.*)

8.G.3. Study of some dilators.
We define a category $\mathcal{C}$ (very close to $\mathbf{ON}^{\mathbf{ON}}$):

objects: families $(x_i)_{i<y}$ where $y \in 0n$ and (x_i) is a family of *nonzero* ordinals.
morphisms: families $(f_i)_{i \in g}$ ($g \in I(g, y')$, $f_i \in I(x_i, x'_{g(i)})$).

Consider the full subcategory $\mathcal{C}'$ of **DIL** consisting of *linear* dilators:

$$D \in |\mathcal{C}'| \leftrightarrow \mathsf{Tr}(D) = D(1) \times \{1\}\ .$$

Prove that the categories $\mathcal{C}$ and $\mathcal{C}'$ are isomorphic.
(*Remark. This shows that* **DIL** *is a very big category!* $\mathbf{ON}^{\mathbf{ON}}$ *is practically a subcategory of* **DIL**!)

8.G.4. Normalization of quasi-dendroids.
If $s = (a_0, ..., a_n)$ is a sequence consisting of ordinals and of underlined ordinals, one defines $Occ(s)$ by:

$$Occ(s) = (x_0, ..., x_{p-1})\ ,$$

where

- x_i are pairwise distinct ordinals.
- there is a function $f \in I(p, n+1)$ such that, for all $j < n+1$, if a_j is underlined, there exists $i < p$ such that $a_j = \underline{x}_i = a_{f(i)}$, and $f(i) \leq j$.

If D is a quasi-dendroid, then we define a function Occ^D on the ordinal $h(D)$, as follows:

$$Occ^D(z) = Occ\Big(\varphi_D^{-1}(z)\Big) \ .$$

If D is a quasi-dendroid, and p is an integer, define equivalence relations $\sim_p$ on D by:

- if $s \in D$, then $s \ \sim_p \ s$.
- if $s, s' \in D$ and $s \neq s'$, then: $s \ \sim_{2p+1} \ s'$ if one can write $s = t * u$, $s' = t * u'$, and $Occ(t)$ has $p+1$ elements (i.e. $Occ(t) = (x_0, ..., x_p)$) $s \ \sim_{2p} \ s'$ if one can write $s = t * (\underline{x}) * u$, $s' = t * (\underline{x}') * u'$, and $Occ(t)$ has p elements, and for all $z \in Occ(t)$, $z \notin [x, x']$.

These equivalences are transferred to $h(D)$ by

$$z \ \sim_p^D \ z' \leftrightarrow \varphi_D^{-1}(z) \ \sim_p \ \varphi_D^{-1}(z') \ .$$

We define an equivalence relation $\asymp$ between quasi-dendroids of the same type by: $D \asymp D'$ iff

1. $h(D) = h(D')$.
2. the functions Occ^D and $Occ^{D'}$ coincide.
3. the equivalence relations $\sim_p^D$ and $\sim_p^{D'}$ are equal.

(i) Show that the equivalence classes modulo $\sim_p^D$ are intervals.

(ii) Prove that in any equivalence class modulo $\asymp$ there is exactly one dendroid $N(D)$. Hence $D \asymp D'$ iff $N(D) = N(D')$.
(*Hint. Construct the dendroid $N(D)$ by adapting the construction of the functor* **BCH** *to this new context.... For the unicity of the dendroid D' such that $D \asymp D'$, observe that: $N(D') = D'$, and $N(D)$ is completely determined by the data $h(D)$, Occ^D, $\sim_p^D$.*)

(iii) If D is a dendroid, show that:

$$f^{-1}(D) = N({}^f D) \ .$$

(iv) Assume that $(D_i)_{i<a}$ is a family of quasi-dendroids of the same type; then define a new quasi-dendroid $D = \sum\limits_{i<a} D_i$ of the same type, by:

$$D = \{(i) * s \, ; \, s \in D_i\} \ .$$

Prove that $\sum$ is compatible with $\asymp$.

(v) Assume that $(D_i)_{i<a}$ is a family of quasi-dendroids of type a; then define a new quasi-dendroid $D = \sum\limits_{i<a} D_i$ of type a by:

$$D = \{(\underline{i}) * s \, ; \, s \in D_i\} \ .$$

Prove that $\sum^*$ is compatible with $\asymp$.

(vi) Prove the following principle of induction; this principle has nothing to do with induction on dilators! Assume that $P(\cdot)$ is a property of quasi-dendroids of type x, which is compatible with $\asymp$ (i.e. $D \asymp D' \to \big(P(D) \to P(D')\big)$). Suppose that the following hold:

- $P(0)$ (0 is the void dendroid).
- $P(1)$ (1 is a quasi-dendroid $\{(a)\}$ for some $a \in 0n$).
- if $P(D_i)$ for all $i < z$, then $P\Big(\sum\limits_{i<z} D_i\Big)$.
- if $P(D_i)$ for all $i < x$, then $P\Big(\sum\limits_{i<x}{}^* D_i\Big)$.

Conclude that $P(\cdot)$ holds for all D of type x.
(*Further information can be found in [5], 7.2.*)

8.G.5. <u>Dendra</u>.
Let K be a finite or denumerable set (the set of **colours**), let $(x_k)_{k \in K}$ be a family of ordinals. A **dendron** (plural: dendra) of type (x_k) is a pair $\big((x_k, D\big)$ such that D is a set of finite sequences $(a_0, \ldots, a_n)$ and:

1. $(a_0, ..., a_n) \in D \wedge m < n \to (a_0, ..., a_m) \notin D$.

2. If $a_0, ..., a_n) \in D$, then for all $i \leq n$, a_i is either an ordinal or a pair (x, k), $x \in 0n$, $k \in K$; such a pair will be dnoted by $\dfrac{x}{k}$. One says that $\dfrac{x}{k}$ is of colour k.

3. If $s * (t) \in D^*$ and $s * (u) \in D^*$ and t is of colour k, then u is of colour k.

4. There is no sequence (a_n) such that, for all n, $(1_0, ..., a_n) \in D^*$.

A dendron $\big((x_k), D\big)$ is a **multi-dendroid** iff the following extra conditions hold:

5. $(a_0, ..., a_n) \in D \to n$ is even.

6. $(a_0, ..., a_n) \in D \wedge 2i \leq n \to a_{2i}$ is not coloured.
   $(a_0, ..., a_n) \in D \wedge 2i + 1 \leq n \to a_{2i+1}$ is coloured.

7. If $s * (x) \in D^*$, $x \in 0n$ and $x' < x$, then $s * (x') \in D^*$.

8. If $\Big(a_0, ..., a_{2i}, \dfrac{x}{k}\Big) \in D^*$, $\Big(a_0, ..., a_{2i}, \dfrac{x'}{k}\Big) \in D^*$, and if for some $j < 2i$, $a_j = \dfrac{x''}{k}$, then $x'' \notin [x, x']$.

(i) Define an equivalence relation $\asymp$ between dendra of the same type, in such a wau that each equivalence class contains one and only one multi-dendroid.

(ii) Assume that $(f_k)_{k \in K}$ is such that: $f_k \in I(x_k, y_k)$; if D is a dendron of type (g_k), define a dendron ${}^{(f_k)}D$ of type (x_k) by:

$$\Big(..., (\frac{z}{k}), ..., z', ...\Big) \in {}^{(f_k)}D \leftrightarrow \Big(..., \Big(\frac{f_k(z)}{k}\Big), ..., z', ...\Big) \in D\ .$$

Show that ${}^{(f_k)}\cdot$ is compatible with the equivalence $\asymp$; from this deduce an operation $(f_k)^{-1}\cdot$ defined between multi-dendroids.

(iii) Let D be a dendron of type $(x_k)_{k \in K}$, and let $K' \subset K$; D is said to be K'-**homogeneous** iff given any families $(y_k)_{k \in K}$, $(f_k)_{k \in K}$, $(g_k)_{k \in K}$ such that $f_k, g_k \in I(y_k, x_k)$ for all $k \in K$, and $x_k = y_k$, $f_k = g_k = \mathbf{E}_k$ for all $k \in K'$, then

$$^{(f_k)}D = {}^{(g_k)}D \ .$$

Assume that D is K'-homogeneous and that $h_k \in I(z_k, x_k)$ for all $k \in K$; show that D is K'-homogeneous.

(iv) Let D be a dendron of type $(x_k)_{k\in K}$, and let $K' \subset K$; D is **strongly** K'-**homogeneous** iff for all $k \in K'$, $x_k = \omega$, and if for all $(y_k)_{k\in K}$ with $y_k \geq \omega$ for all $k \in K'$ and $y_k = x_k$ for all $k \in K - K'$, one can find a K'-homogeneous dendron D' of type (y_k) such that

$$D = (\mathbf{E}_{x_k y_k})^{-1} D' \ .$$

When considered as a strongly K'-homogeneous dendron (K'-sh. dendron), the type of D is the family $(x_k)_{k\in K-K'}$. If $(z_k)_{k\in K'}$ is a family of ordinals, give the definition of a dendron denoted by $F^\circ\big((z_k)\big)$, when D is an arbitrary K'-sh. dendron. From this show that, if D is a K'-sh. dendron, one can associate to D a functor $\mathbf{LIN}^{K'}(D)$ from $\mathbf{ON}^{K'}$ to $\mathbf{ON}$, preserving direct limits and pull-backs. Show that if $(f_k)_{k\in K-K'}$ is such that $f_k \in I(x_k, y_k)$, and D is a K'-sh. dendron of type (y_k), show the existence of a K'-sh. dendron $^{(f_k)}D$ of type (x_k), together with a natural transformation $\mathbf{LIN}^{K'}_D\big((f_k)\big) \in I^1\Big(\mathbf{LIN}^{K'}(^{(f_k)}D), \mathbf{LIN}^{K'}(D)\Big)$.
(*Remark. Here lies the interest of dendra; they enable us to get rid of the category-theoretic framework. Precisely if D is strongly homogeneous w.r.t. $K' \subset K$, then the "homogeneous" colours $\in K'$ represent a certain "K'-ary dilator", whereas the "non homogeneous" ones $\in K - K'$ represent certain natural transformations.... It is possible to give a non category-theoretic description of Λ and related operations; see [5].*)

8.G.6. Homogeneous trees (Jervell, [78], 1979).
Let α be an ordinal; a **Jervell tree** is a set S of finite sequences $\langle \xi_0, ..., \xi_{n-1}\rangle$, enjoying the following properties:

1. for $i < n$, $\xi_i \in \{-1\} \cup \alpha \cup \{+\infty\}$.

2. S is nonempty.

3. $s \in S \wedge t^* > s \to t \notin S$; let $S^* = \{t\,;\, \exists s \in S\ s\ \leq^*\ t\}$.

4. $s * \langle u \rangle \in S^* \to \{x\,;\, s * \langle x \rangle \in {}^*\} = \{-1\} \cup \beta \cup \{+\infty\}$ for some $\beta \leq \alpha$.

5. $s * \langle -1 \rangle \in S^* \to s * \langle -1 \rangle \in S$.

6. there is no sequence $(\xi_i)_{i \in I\!N}$ s.t., for all n, $\langle \xi_0, ..., \xi_n \rangle \in S^*$.

(i) Define a linear order $\leq^+$ on S, and prove that S is a well-order, with a topmost element $\hat{S}$. Let $\|S\|$ = order type of $S - \{\hat{S}\}$. When $f \in I(\alpha, \beta)$ and S is a jervell tree of type β, define a Jervell tree $f^{-1}(S)$... . Define the concepts of homogeneous, strongly homogeneous Jervell trees ... in a way similar to dendroids; of course the definition of dendroids is a reformulation of Jervell trees... .

(ii) If x is a denumerable ordinal $\geq \omega$, build a family of Jervell trees $(S_{x'})_{x' < x}$ with the properties that:

- $f \in I(x', x'')$, $x' \leq x'' < x \to f^{-1}(S_{X''}) = S_{x'}$.
- there is no way of defining S_x s.t. $f^{-1}(S_x) = S_{x'}$ for all $f \in I(x', x)$ and $x' < x$, in such a way that x is a Jervell treel more precisely Condition 6 fails. (*Hint. Let $f\ :\ I\!N \to x$ be an enumeration of x; if y is any ordinal, define S'_y as follows: $(a_0, ..., a_{n-1}) \in S'_y \leftrightarrow n \neq 0 \wedge (a_{n-1} = -1 \vee a_{n-1} = +\infty \wedge \forall i, j < n-1 (f(i) < f(j) \leftrightarrow a_i < a_j)$. Then modify S'_y to ensure Condition 4; observe that the resulting S_y enjoys 1–5, and enjoys 6 iff $y < x$... .)*

(Many operations, including a variant of Λ, and a proof of the hierarchy theorem, can be performed by means of strongly homogeneous Jervell trees... .)

8.G.7. <u>Homogeneous trees and ladders</u> (Masseron, [81], 1980).

(i) Assume that S is a Jervell tree of type α; we define a structure of rung R of type α and height $\|S\|$: define, when $s \in S^*$, $Z_S(s) = s^*\langle -1 \rangle$ if $s \in S$, $Z(s) = s, * \langle -1 \rangle$ otherwise; if $s \in S$, write $s = Z_S(s_0)$ with s_0

not ending by $\{-1\}$; if $s_0 = \langle\,\rangle$, then $T(s) = 0$; if $s_0 \neq \langle\,\rangle$, let $s_0 = s_1 * \langle a \rangle$ and let β be such that $\{x\,;\, s_1 * \langle x \rangle \in S^*\} = \{-1\} \cup \beta \cup \{+\infty\}$; we define $T(s) = \mathsf{inf}\,(\beta, a)$, and for $\xi < T(s)$: $[s]\,\xi = Z_S(s * \langle \xi \rangle)$. Show that this defines a structure of rung of the form $1 + R'$, or of the form $\underline{0}_\alpha$ (see 9.A for all these notions...).

(ii) Assume conversely that R is a rung of type α, and of the form $1+R'$; if $x \leq \|R\|$, we define $[x]\,(+\infty = 1)$ by: $[x]\,(+\infty + 1) = \mathsf{inf}\,\{[y]\,(\eta + 1)\,;\, [y]\,\eta < x < [y]\,(\eta + 1)$ for some y, $\eta\}$ (if this set is void, let $[x]\,(+\infty) = x+1$). We define $[x]\,(+\infty) =$ the greatest y s.t. $[y]\,T(x) = x$. Show that any two intervals $\Big[[x_i]\,\xi_i, [x_i]\,\xi_i + 1\Big[$, with $i = 1, 2$, and $\xi_i \in \alpha \cup \{+\infty\}$ are either disjoint or comparable for inclusion. We define $[x]\,(-1) = \mathsf{inf}\Big\{[y]\,(\eta)\,;\, [y]\,\eta < x < [y]\,(\eta + 1)$ for some y and $\eta \in \alpha \cup \{+\infty\}\Big\}$ (if this set is void, let $[x]\,(-1) = 0$). Prove that $[x]\,(-1) = ([x]\,0) - 1$ when $x \neq 0$. Given $x \leq \|R\|$ consider all intervals of the form $I = \Big[[y]\,\eta, [y]\,(\eta+1)\Big[$ s.t. $x \in I$; show that these intervals form a finite linearly ordered set. Write

$$x \in I_n \neq \ldots \neq I_0 \ , \qquad \mathsf{with}\ I_i = \Big[[y_i]\,\eta_i, [y_i]\,(\eta_i + 1)\Big[\ .$$

To x we associate $x^* = \langle \eta_0, \ldots, \eta_n \rangle$. Show that $\{x^*\,;\, x \in \|R\| + 1\}$ is a Jervell tree... .

(iii) Show that the operations defined in (i) and (ii) are inverse isomorphisms identifying Jervell trees $\neq \{\langle\,\rangle\}$ with rungs of the form $1+R'$.

(iv) Extend (iii) into an isomorphisms between homogeneous Jervell trees and ladders; from this we can for instance transfer Λ on homogeneous Jervell trees... .

8.G.8. The category $\mathbf{SET}_{tr}$

We define a category $\mathbf{SET}_{tr}$ as follows:

– objects: sets x which are transitive (i.e. $t \in t' \wedge t' \in x \rightarrow t \in x$).

– morphisms from x to y: functions from x to y s.t.:

$$\forall t, t' \Big(t \in t' \leftrightarrow f(t) \in f(t') \Big) \ .$$

(i) Prove that, in $\mathbf{SET}_{tr}$, every set is a direct limit of finite sets.
(*Hint. The result is non trivial, because of the extensionality axiom that must be fulfilled by the finite substructures of x that we construct ... if $X \subset x$ we say that X is extensional if $\forall a \forall b \in X \Big(a \neq b \to \exists c \in X \ \neg(c \in a \leftrightarrow c \in b) \Big)$. Then show that, if $X \subset x$ and x is finite, there is an extensional Y s.t. $X \subset Y \subset x$, and which is finite. We work by induction on $\mathsf{max}\, rk(X) = \mathsf{sup}\, \{rk(t)\, ;\, t \in X\}$, and we consider the closure X' of X under boolean operations ($x \cup y$, $x \cap y$, $x - y$); them $\mathsf{max}\, rk(X') = \mathsf{max}\, rk(X)$. Write $X' = X_1 \cup Y$, where Y consists of all points of X' of maximal rank. Let $a_1, ..., a_p$ be elements of Y minimal for inclusion, and choose $c_1, ..., c_p$ in $a_1, ..., a_p \ldots$. Apply the induction hypothesis to $X' \cup \{c_1, ..., c_p\} \ldots$.*)

(ii) Assume that (x_i, f_{ij}) is a direct system in $\mathbf{SET}_{tr}$, and that its direct limit, as a system of orders, is well-founded; conclude that the system has a direct limit in $\mathbf{SET}_{tr}$.

8.G.9. Exercise.
Find two finite dimensional dilators D and D' such that $D(x) = D'(x)$ for all $x \in 0n$, but $D \neq D'$.

8.G.10. Abstract description of predilators (see 8.4.25).

(i) Show that the following concept is equivalent to the concept of predilator:

1. A linear oder X.
2. For any two $x, y \in X$ s.t. $x \leq y$, an integer n_{xy}, together with a linear order on σ_{xy} on $n_{xy} = |\sigma_{xy}|$.
3. If $x \leq y \leq z$, then $n_{xz} = \mathsf{inf}\,(n_{xy}, n_{yz})$ and $\sigma_{xz} = \sigma_{xy}\lceil n_{xz} = \sigma_{yz}\lceil n_{xz}$ (denoted by $\sigma_{xz} = \sigma_{xy} \wedge \sigma_{yz}$).

(ii) Show how finite dimensional predilators can be generated by induction on their dimension, as in (i).

(iii) Find the analogue of the concepts of two variable predilators, prebilators.
(*Hint.* n_{xy} *is equipped with two linear orders,* σ^1_{xy}, σ^2_{xy}, *with:* $|\sigma^1_{xy}| \cap |\sigma^2_{xy}| = \emptyset$, $|\sigma^1_{xy}| \cup |\sigma^2_{xy}| = n_{xy}$; *in the case of prebilators, we require that* 0 *is the topmost element in* $|\sigma^2_{xy}|$.)

(iv) Express separation and unification of variables (Chapter 9), by means of operations on the representations (i) and (iii).... This is the most elegant description of **SEP** and **ON**.

8.G.11. Decreasing permutations.
Let F be a predilator such that:

1. all permutations relative to F are of the form

$$\sigma(0) = n-1, ..., \sigma(n-1) = 0\ .$$

2. $F(\omega^2)$ is well-founded.

Show that F is a dilator.
(*Hint. Use* $\mathbf{BCH}_q(F)\ldots$.)

8.G.12. Differential equations for dilators (Boquin).
Let H be a dilator; show that the "differential equation"

$$\begin{cases} F(x+1) = F(x) + H(F(x), x) \\ F(f + \mathbf{E}_1) = F(f) + H(F(f), f) \\ F \text{ flower} \\ F(0) = a \end{cases}$$

admits a unique solution.

8.G.13. Amalgamation in **DIL**
Assume that $T_1 \in I^1(A, B_1,\ T_2 \in I^1(A, B_2)$; show the existence of B, $U_1 \in I^1(B_1, B)$, $U_2 \in I^2(B_2, B)$ such that the diagram

$$\begin{array}{ccccc} & & B_1 & & \\ & T_1 & & U_1 & \\ A & & & & B \\ & T_2 & & U_2 & \\ & & B_2 & & \end{array}$$

is commutative and cartesian (i.e. $U_1 \wedge U_2 = U_1 T_1 = U_2 T_2$).
(*Hint. One can directly work on an abstract description of dilators (8.G.10), and obtain a solution B which is a predilator ... then show that the B constructed must be a dilator.... Another possibility is to imitate what is done in Chapter 12.*)

8.G.14. About traces.
If Y is a finite set, what can be ascertained as to the cardinality of the set:

$$\{D\,;\ D \in |\mathbf{DIL}| \wedge \mathsf{Tr}(D) = Y\}\ .$$

In particular, if X is a finite set of finite dimensional dilators, define X' by:

$$D \in X' \leftrightarrow \exists D_1, D_2, D_3\ \exists T_1, T_2, T_3 \text{ s.t. } T_i \in I^1(D_i, D)$$
$$(i = 1, 2, 3) \wedge \mathsf{Tr}(D) = rg\Big(\mathsf{Tr}(T_1)\Big) \cup rg\Big(\mathsf{Tr}(T_2)\Big) \cup rg\Big(\mathsf{Tr}(T_3)\Big)\ .$$

Prove that X' is a finite set.

8.G.15. Preservations of pull-backs.
Let F be a functor from **ON** to **ON** preserving direct limits.

(i) Prove a "normal form theorem": $z = F(f)(z_0)$ with $rg(f)$ minimal; notation: $z = (z_0\,;\, x_0, ..., x_{n-1}\,;\, x)_F$, with $rg(f) = \{x_0, ..., x_{n-1}\}$. Prove that 8.2.20 still holds for F.

(ii) Define $\mathsf{Tr}(F) = \{(z_0, n)\,;\, z_0 = (z_0\,;\, 0, ..., n-1\,;\, n_F\}$; define a_i $(i = 0, ..., n-1)$ when $(z_0, n) \in \mathsf{Tr}(F)$ as in 8.4.10; consider $I_{z_0,n} = \{i < n\,;\, a_i \neq (z_0\,;\, 0, 2, ..., 2n-2\,;\, 2n)_F\}$; prove that

$$(z_0\,;\, x_0, ..., x_{n-1}\,;\, x)_F = (z_0\,;\, x'_0, ..., x'_{n-1}\,;\, x)_F$$
$$\text{iff } x_i = x'_i\ \forall i \in I_{z_0,n}\ .$$

Show that, if $(z_0\,;\, x_0, ..., x_{n-1}\,;\, x)_F = (z_1\,;\, y_0, ..., y_{m-1}\,;\, x)_F$, then

$$\{x_i\,;\, i \in I_{z_0,n}\} = \{y_j\,;\, j \in I_{z_1,m}\}\ .$$

(iii) Prove that $F_0\Big(\underline{\omega}(\underline{1} + \mathsf{Id})\Big)$ is a dilator.

Conclude that there is a natural transformation from F into a dilator.

(*This result expresses, in a more satisfactory way than 8.G.1, the fact that "preservation of pull-backs costs nothing"; this is precisely a reason to restrict to functors with this preservation property: we are essentially at the same level of generality, but the theory is really simpler….*)

CHAPTER 9
DILATORS AS WELL-ORDERED CLASSES

Here again the basic reference is [5], especially Chapters 3 and 5.

9.1. Classification of dilators

The purpose of this section is to extend the familiar classification of ordinals in zero, successors, limits to dilators:

9.1.1. Proposition.

(i) Let x be an ordinal, and let $(F_z)_{z<x}$ be a family of dilators; then one can define a new dilator $G = \sum\limits_{z<x} F_z$ by

$$G(a) = \sum_{z<x} F_z(a) \ ,$$

and when $f \in I(a,b)$:

$$G(f)\Big(\sum_{z<z'} F_z(a) + u\Big) = \sum_{z<z'} F_z(b) + F_{z'}(f)(u) \ .$$

(ii) Let y be another ordinal, let $(H_z)_{z<y}$ be another family of dilators, let $f \in I(x,y)$ and (T_z) be a family of natural transformation from F_z to $H_{f(z)}$, $(z < x)$; then one can define a natural transformation $T = \sum\limits_{z<f} T_z$ from $\sum\limits_{z<x} F_z$ to $\sum\limits_{z<y} H_z$ by:

$$T(a)\Big(\sum_{z<z'} F_z(a) + u\Big) = \sum_{z<f(z')} G_z(a) + T_{z'}(a)(u) \ .$$

Proof. (i) It is immediate that G is a functor from **ON** to **ON**. The fact that G preserves direct limits and pull-backs is easy to establish; let us see how the denotations w.r.t. G look like: if $t = \sum\limits_{z<z'} F_z(x) + u$ with $u <$

$F_{z'}(x)$ write $u = (u_0\,;\, x_0, ..., x_{n-1}\,;\, x)_{F_{z'}}$; then one has: $t = \Big(\sum_{z<z'} F_z(n) + u_0\,;\, x_0, ..., x_{n-1}\,;\, x\Big)_G$.

(ii) is immediate as well; observe that $\mathsf{Tr}(T)\Big(\Big(\sum_{z<z'} F_z(n) + u_0, n\Big)\Big) = \Big(\sum_{z<z'} G_z(n) + \mathsf{Tr}(T_{z'})(u_0), n\Big)$. □

9.1.2. <u>Remark</u>.
An important case is $x = 2$ (or $f = \mathbf{E}_2$), which enables us to define the *sum* of two dilators, or of two natural transformations:

$$(F + F')(a) = F(a) + F'(a)$$
$$(F + F')(f) = F(f) + F'(f)$$
$$(T + T')(a) = T(f) + T'(f)\ .$$

9.1.3. <u>Proposition</u>.
The following conditions are equivalent:

(i) $\exists F'\ G = F + F'$.

(ii) For all $a \in 0n$ $F(a) \leq G(a)$ and the family $(\mathbf{E}_{F(a)G(a)})_{a \in 0n}$ defines a natural transformation from F to G.

<u>Proof</u>. Immediate, left to the reader. □

9.1.4. <u>Definition</u>.
If the equivalent properties of 9.1.3 hold, one will denote this situation by $F \sqsubseteq G$; and the natural transformation $T(a)$: $\mathbf{E}_{F(a)G(a)}$ will be denoted by $\mathbf{E}^1_{FG}$. $\mathbf{E}^1_{FF}$ is abbreviated into $\mathbf{E}^1_F$: $\mathbf{E}^1_F$ is the identity of F.

9.1.5. <u>Definition</u>.

(i) If F is a nonzero dilator with the property that $F = F' + F'' \to F' = \underline{0} \vee F'' = \underline{0}$, then F is said to be **connected**.

(ii) If $T \in I^1(F, G)$ where F and G are connected dilators, then T is said to be **connected**.

9.1.6. Theorem.

(i) If F is a dilator, then F can be written as a sum: $F = \sum_{z<x} F_z$, with all dilators F_z connected. (The F_z's are the connected **components** of F). The ordinal x and the dilators F_z are unique.

(ii) If F, G are dilators, and $T \in I^1(F, G)$, then T can be written as a sum: $T = \sum_{z<f} T_z$, with all natural transformations T_z (the connected **components** of T) connected. The function f, and the T_z's are unique.

Proof. We first establish a lemma.

9.1.7. Lemma.
A non zero dilator F is connected if $\forall(a, n) \in \mathsf{Tr}(F)\ \forall(b, m) \in \mathsf{Tr}(F)$ $|\S^F(a, n\,;\, b, m)| > 0$.

Proof. If $F = F' + F''$, with $F', F'' \neq \underline{0}$, let $T = \mathbf{E}^1_{F'F}$, $T' = \mathbf{E}^1_{\underline{0}F'} + \mathbf{E}^1_{F''}$, and let $(a, n) \in rg\Big(\mathsf{Tr}(T)\Big)$, $(b, m) \in rg\Big(\mathsf{Tr}(T')\Big)$; it is immediate that $(a\,;\, x_0, ..., x_{n-1}\,;\, x)_F < (b\,;\, y_0, ..., y_{m-1}\,;\, x)_F$ for all strictly increasing sequences $x_0 < ... < x_{n-1} < x$, $y_0 < ... < y_{m-1} < x$, and this implies that $\S^F(a, n\,;\, b, m) = 0$. (The converse of the lemma holds, but is of no need here.) $\square$

We define an equivalence relation $\sim^F_0$ on the set $\mathsf{Tr}(F)$ by:

$$(a, n) \ \sim^F_0\ (b, m) \leftrightarrow |\S^F(a, n\,;\, b, m)| > 0 \vee (a, n) = (b, m)\ .$$

(This is an equivalence because $|\S^F(a, n\,;\, c, l)| \geq \mathsf{inf}\,(|\S^F(a, n\,;\, b, m), \S^F(b, m\,;\, c, l)|)$; this is proved by 8.4.31 (i).) This equivalence is compatible with the order $\leq^F$, in other terms the equivalence classes C modulo $\leq^F$ are such that $x, y \in C$ and $x \ \leq^F\ z\ \leq^F\ y \rightarrow z \in C$ (if $x \ \sim^F_0\ y$ and $x\ \leq^F\ z\ \leq^F\ y$, we know that $\mathsf{inf}\,(|\S^F(x, z)|, |\S^F(z, y)|) = |\S^F(x, y)| > 0...$). Hence it is possible to define a linear order $\leq^F / \sim^F_0$. We claim that this is a well-order: if (a_i, n_i) is a strictly decreasing sequence in $\mathsf{Tr}(F)$ for $\leq^F$, such that $|\S^F(a_i, n_i\,;\, a_j, n_j)| = 0$ for $i \neq j$, it is immediate that $z_i = (a_i\,;0,...,n_i-1\,;\omega)_F$ is a s.d.s. in $F(\omega)$. Let $x = \|\leq^F / \sim_0^F\|$, and let $z \leadsto C_z$ be the order-preserving isomorphism from x to $\leq^F / \sim_0^F$. Define a dilator F_z, together with a natural transformation $U_z \in I^1(F_z, F)$ by the condition $rg\Big(\mathsf{Tr}(U_z)\Big) = C_z$. We claim that F_z is connected: this is immediate from 9.1.7, since all elements of its trace are equivalent modulo $\sim_0^F$. Given $u \in F_z(z)$, we prove, by induction on z, that $U_z(x)(u) = \sum\limits_{z'<z} F_{z'}(x) + u$; this is immediate from the following remark: $v < U_z(x)(u) \leftrightarrow (v = U_z(x)(u')$ for some $u' < u$ or $v = U_{z'}(x)(u')$ for some u' and some $z' < z)$. From that it easily follows that $F = \sum\limits_{z<x} F_z$. Now assume that $F = \sum\limits_{z<x'} G_z$, with the G_z's connected. Then the sets $D_z = \Big\{\Big(\sum\limits_{z'<z} G_{z'}(n) + u\,; n\Big)\,; (u,n) \in \mathsf{Tr}(G_z)\Big\}$ must be equivalence classes modulo $\sim_0^F$, hence $C_z = D_z$, $x = x'$, $G_z = F_z$. In order to prove (ii), observe that $\mathsf{Tr}(T)$ is a strictly increasing function from $(\mathsf{Tr}(F), \leq^F)$ to $(\mathsf{Tr}(G), \leq^G)$, compatible with the equivalences $\sim_0^F$ and $\sim_0^G$; hence $\mathsf{Tr}(T)$ induces a strictly increasing function f from $\|\leq^F / \sim_0^F\|$ to $\|\leq^F / \sim_0^G\|$. If $(D_z)_{z<y}$ are the equivalence classes modulo $\sim_0^G$, it is clear that $\mathsf{Tr}(T)$ maps C_z into $D_{f(z)}$, for all $z < x$. Define G_z and $V_z \in I^1(G_z, G)$ by $rg\Big(\mathsf{Tr}(V_z)\Big) = D_z$; then one can define $T_z \in I^1(F_z, G_{f(z)})$ by: $V_{f(z)}T_z = TU_z$ (this definition is possible because of Remark 8.3.11 (ii), since $rg\Big(\mathsf{Tr}(TU_z)\Big) \subset rg\Big(\mathsf{Tr}(V_{f(z)})\Big)$). Then T_z is obviously connected, and clearly $T = \sum\limits_{z<f} T_z$. The unicity of the decomposition is immediate. □

9.1.8. Corollary ("splitting lemma").
Assume that $T \in I^1(F, G' + G'')$; then one can define F' and F'', $T' \in I^1(F', G')$, $T'' \in I^1(F'', G'')$, such that $F = F' + F''$, $T = T' + T''$. This decomposition is unique.

Proof. Write $F = \sum\limits_{z<x} F_z$, $G' = \sum\limits_{z<y'} G_z$; $G'' = \sum\limits_{z<y''} G_{y'+z}$; then there exists $f \in I(x, y' + y'')$ and a family (T_z), $T_z \in I^1(F_z, G_{f(z)})$ such that $T = \sum\limits_{z<f} T_z$. Define x', x'', $f' \in I(x', y')$, $f'' \in I(x'', y'')$, by the condition $f = f' + f''$ (for instance x' is the smallest ordinal z such that $z = x$ or $(z < x$ and $f(z) \geq y')$...). Define $F' = \sum\limits_{z<x'} F_z$, $F'' = \sum\limits_{z<x''} F_{x'+z}$,

$T' = \sum_{z<f'} T_z$, $T'' = \sum_{z<f''} T_{x'+z}$: then $F = F' + F''$, $T = T' + T''$. The unicity of the solution is immediate. □

9.1.9. Remarks.

(i) One would easily verify that $F' \sqsubseteq F$, where $F = \sum_{z<x} F_z$ iff $F' = \sum_{z<x'} F_z$ for some $x' \leq x$.

(ii) One can define a functor **LH** (length) from **DIL** to **ON** by:

$$\mathbf{LH}\Big(\sum_{z<x} F_z\Big) = x$$
$$\mathbf{LH}\Big(\sum_{z<f} T_z\Big) = f \ .$$

It is easy to verify that **LH** preserves direct limits. Bug **LH** does not preserve pull-backs (a remark by Daniel Boquin): if G is a two-dimensional connected dilator, with $rg(G) = \{a, b\}$, define T_a and T_b by $rg\Big(\mathsf{Tr}(T_a)\Big) = \{a\}$, $rg\Big(\mathsf{Tr}(T_b)\Big) = \{b\}$; $\mathbf{LH}(T_a) = \mathbf{LH}(T_b) = \mathbf{E}_1$; but $T_a \wedge T_b = \mathbf{E}^1_{\underline{0}G}$; $\mathbf{LH}(T_a \wedge T_b) = \mathbf{E}_{01} \neq \mathbf{E}_1 \wedge \mathbf{E}_1$.

9.1.10. Definition.

(i) The dilator F is of **kind**

- **0** iff $F = \underline{0}$.
- **1** iff $F = F' + \underline{1}$ for some F'.
- $\boldsymbol{\omega}$ iff $F = \sum_{z<x} F_z$, with x limit and $F_z \neq \underline{0}$ for all $z < x$.
- $\boldsymbol{\Omega}$ iff $F = F' + F''$ for some F', F'', with F'' connected and $\neq \underline{1}$.

(ii) The natural transformation $T \in I^1(F, G)$ is **deficient** if $T = T' + \mathbf{E}^1_{\underline{0}G'}$ for some $G' \neq \underline{0}$; otherwise F and G are of the same kind and T is of kind **0** (resp. **1**, $\boldsymbol{\omega}$, $\boldsymbol{\Omega}$) iff F and G are of kind **0** (resp. **1**, $\boldsymbol{\omega}$, $\boldsymbol{\Omega}$).

9.1.11. Theorem.

(i) If F is a dilator, then F is of one (and only one) of the kinds $\mathbf{0}$, $\mathbf{1}$, $\boldsymbol{\omega}$, $\boldsymbol{\Omega}$.

(ii) If T is a non deficient natural transformation, then T is of one of the kinds $\mathbf{0}$, $\mathbf{1}$, $\boldsymbol{\omega}$, $\boldsymbol{\Omega}$).

Proof. (i) Write $F = \sum_{z<x} F_z$; then

– if $x = 0$, $F = \underline{0}$ hence F is of kind $\mathbf{0}$.

– if x is limit, then F is of kind $\boldsymbol{\omega}$.

– if $x = x' + 1$, and $F_{x'} = \underline{1}$, F is of kind $\mathbf{1}$.

– if $x = x' + 1$, and $F_{x'} \neq \underline{1}$, F is of kind $\boldsymbol{\Omega}$.

(ii) Write $G = \sum_{z<y} G_z$, $T = \sum_{z<f} T_z$; if $\hat{f}(x) < y$, then $f = f' + E_{0y'}$ for some $y' \neq 0$, and $T = \sum_{z<f'} T_z + \mathbf{E}^1_{\underline{0}G'}$ with $G' \neq \underline{0}$, hence T is deficient.
If $\hat{f}(x) = y$, then T is not deficient; if $x = 0$, then $y = \hat{f}(x) = 0$, hence F and G are both of kind $\mathbf{0}$. If $x = x' + 1$, then $y = f(x') + 1$: if $T_{x'} \in I^1(F_{x'}, G_{f(x')})$, then:

– if $F_{x'} = \underline{1}$, then $G_{x'} = \underline{1}$: F and G are both of kind $\mathbf{1}$.

– if $F_{x'} \neq \underline{1}$, then $G_{x'} \neq \underline{1}$: F and G are both of kind $\boldsymbol{\Omega}$.

(If $(z, 0) \sim^H_0 (z', n)$, then $(z, 0) = (z', n)$; hence a connected dilator H such that $(z, 0) \in \mathsf{Tr}(H)$ is necessarily $\underline{1}$; so if $I^1(F, G) \neq \emptyset$ and $F = \underline{1}$ or $G = \underline{1}$, then $F = G$.) If x is limit, then y is limit as well: F and G are both of kind $\boldsymbol{\omega}$.

9.1.12. Remark.
If x is an ordinal, then $\underline{x}$ is of kind

(i) $\mathbf{0}$ if $x = 0$.

(ii) $\mathbf{1}$ if x is a successor.

(iii) $\boldsymbol{\omega}$ if x is limit.

The kind $\mathbf{\Omega}$ is something new, with no analogue in the case of ordinals.

9.2. Flowers

9.2.1. Definition. A **flower** is a dilator which enjoys the property (FL) for all x, y in $0n$ with $x \leq y$, $F(\mathbf{E}_{xy}) = \mathbf{E}_{F(x)F(y)}$.

9.2.2. Proposition.
A dilator F is a flower iff for all denotation $(z_0\,;\,x_0,...,x_{n-1}\,;\,x)_F$ and all $y \geq x$:

$$(z_0\,;\,x_0,...,x_{n-1}\,;\,x)_F = (z_0\,;\,x_0,...,x_{n-1}\,;\,y)_F\ .$$

Proof. This equality exactly expresses that

$$F(\mathbf{E}_{xy})\Big((z_0\,;\,x_0,...,x_{n-1}\,;\,x)_F\Big) = (z_0\,;\,x_0,...,x_{n-1}\,;\,x)_F$$

i.e. $F(\mathbf{E}_{xy})(z) = z$ for all $z < F(x)$. □

9.2.3. Theorem.
If F is a flower, then the function $x \rightsquigarrow F(x)$ is topologically continuous.

Proof. If x is a limit ordinal, then $x = \underset{\substack{\longrightarrow \\ x'<x}}{\lim}{}^{*}\,(x', \mathbf{E}_{x'x''})$, hence $F(x) = \underset{\substack{\longrightarrow \\ x'<x}}{\lim}{}^{*}\,(F(x'), \mathbf{E}_{F(x')F(x'')}) = \sup_{x'<x} F(x')$. □

9.2.4. Remarks.
Assume that F is a flower; we shall use the notation $(z_0\,;\,x_0,...,x_{n-1})_F$ to mean $(z_0\,;\,x_0,...,x_{n-1}\,;\,y)_F$ for any $y > x_{n-1}$. We assume that F is non constant; then

(i) $z = (z_0\,;\,x_0,...,x_{n-1})_F < F(x) \leftrightarrow x_0,...,x_{n-1} < x$ (because $z < F(x) \leftrightarrow z \in rg\big(F(\mathbf{E}_x)\big)$).

(ii) Since F is non constant, there is a point $(z,m) \in \mathsf{Tr}(F)$, with $m \neq 0$; let z be the smallest point of the form $(z_0\,;\,x_0,...,x_{n-1})_F$ for some pair (z_0,n), with $n \neq 0$; then $x_0 = 0,...,x_{n-1} = n-1$, and by (i) above $z = F(0) = ... = F(n-1) \neq F(n)$. If $x \geq n-1$, then $F(x) < F(x+1)$, because $(z_0\,;\,0,...,n-2,x) \in F(x+1)$, but does not belong to $F(x)$.

(iii) Another consequence of (i) is that if $f \in I(x, y)$, $z \leq x$, and $f(z') = z'$ for all $z' < z$, then $F(f)(t) = t$ for all $t < F(z)$.

(iv) Yet another consequence of (i) is that $F(x)$ is the smallest z of the form $(z_1\,;\, x_0, ..., x_{p-1})_F$ such that $p \neq 0$ and $x_{p-1} \geq x$; if $x \geq n-1$ (n is the integer defined in (ii)), then it suffices to look for $z = (z_1\,;\, x_0, ..., x_{p-1})_F$ with $x_{p-1} = x$. It is immediate that $F(x)$ is of the form: $F(x) = \Big(z_1\,;\, 0, ..., p-2, \mathsf{sup}\,(x, p-1)\Big)_F$. (But z_1 and p may depend on x.)

9.2.5. <u>Theorem</u>.
Let F be a connected dilator $\neq \underline{1}$; then F is a flower iff for all $(z_0, n) \in \mathsf{Tr}(F)$, $\sigma^F_{z_0,n}(0) = n-1$.

<u>Proof</u>. If F is a connected dilator $\neq \underline{1}$ and $(z_0, n) \in \mathsf{Tr}(F)$, then $n \neq 0$. Assume that F is a flower; then consider $a_i = (z_0\,;\, 0, ..., 2i-2, 2i+1, 2i+2, ..., 2n-2)_F$, by 9.2.4 (i), $a_0, ..., a_{n-2} < F(2n-1)$, whereas $a_{n-1} \geq F(2n-1)$, hence $\sigma^F_{z_0,n}(0) = n-1$. Conversely assume that $\sigma^F_{z_0}(0) = n-1$ for all $(z_0, n) \in \mathsf{Tr}(F)$; since F is connected, $|\S(z_0, n\,;\, z_1, m)| > 0$ for all $(z_0, n), (z_1, m) \in \mathsf{Tr}(F)$. Let $t = (z_0\,;\, x_0, ..., x_{n-1}\,;\, x)_F$, and let $t' = (z_0\,;\, x_0, ..., x_{n-1}\,;\, y)_F$ with $y \geq x$, i.e. $t' = F(\mathbf{E}_{xy})(t)$; assume that $u' = (z_1\,;\, y_0, ..., y_{m-1}\,;\, y)_F$ is such that $u' \leq t'$; then $y_{m-1} > x_{n-1}$ (together with $|\S(z_0, n\,;\, z_1, m)| > 0$, $\sigma^F_{z_0}(0) = n-1$, $\sigma^F_{z_1}(0) = m-1$) would imply $t' < u'$, hence $y_{m-1} \leq x_{n-1}$. This proves that $y_0, ..., y_{m-1} < x$, hence $u' = F(\mathbf{E}_{xy})(u)$, with $u = (z_1\,;\, y_0, ..., y_{m-1}\,;\, x)_F$. We have proved that $t' \in rg\Big(F(\mathbf{E}_{xy})\Big) \wedge u' \leq t' \to u' \in rg\Big(F(\mathbf{E}_{xy})\Big)$: hence $rg\Big(F(\mathbf{E}_{xy})\Big)$ is an ordinal (necessarily $F(x)$), i.e. $F(\mathbf{E}_{xy}) = \mathbf{E}_{F(x)F(y)}$. $\square$

9.2.6. <u>Remarks</u>.

(i) One would easily check that, if F is a flower, F can be uniquely written as $F = \underline{x} + F'$, with $x = F(0)$, and F' is either $\underline{0}$ or a connected flower $\neq \underline{1}$.

(ii) One would easily prove that F is a flower iff $F(\mathbf{E}_{nm}) = \mathbf{E}_{F(n)F(m)}$ for all integers $n \leq m$. (This is a simple direct limit argument: see [5], 2.4.2.)

9.2.7. Examples.

(i) The functors Id, $\underline{x}$, are flowers.

(ii) If F is one of the binary functors sum, product, exponential, then the functors F_x defined by:

– $F_x(y) = F(x, y)$

– $F_x(g) = F(\mathbf{E}_x, g)$

are flowers; on the other hand the functors F^y defined by:

– $F^y(x) = F(x, y)$

– $F^y(f) = F(f, \mathbf{E}_y)$

are not (in general) flowers.

In other terms these functors are flowers "in y", but not "in x"; this can be directly seen from the definition: for instance the Cantor Normal Form of an element $z < (1 + x)^y$ mentions x explicitly, but not y: this means that the Cantor Normal Form is independent of y. This surely indicates (see 9.2.2) that the functor exponential is a flower "in y".

9.2.8. Definition.

(i) Assume that F is a dilator, and define $G = \displaystyle\int F(y)dy$ by:

$$G(x) = \sum_{y<x} F(y) \ ,$$

and when $f \in I(x, x')$, $y < x$ and $z < F(y)$:

$$G(f)\Big(\sum_{y'<f(y)} F(y') + z\Big) = \sum_{y'<f(y)} F(y') + F(g)(z) \ ,$$

where $g \in I\Big(y, f(y)\Big)$ is defined by $g(t) = f(t)$ for all $t < y$.

(ii) Assume that F' is another dilator, with $G' = \int F'(y)dy$, and that $T \in I^1(F, F')$; then one defines a natural transformation $U = \int T(y)dy$ by:

$$U(x)\Big(\sum_{y'<y} F(y') + z\Big) = \sum_{y'<y} F'(y') + T(y)(z) \ .$$

9.2.9. Proposition.
$\int F(y)dy$ is a flower.

Proof. We leave the precise proof to the reader; we only indicate how denotations w.r.t. $G = \int F(y)dy$ are computed: if $z = \sum_{y'<y} F(y') + z'$, with $z' = (z_0 \,;\, y_0, ..., y_{n-1} \,;\, y)_F$ then $z = \Big(sum_{n'<n} F(n') + z_0 \,;\, y_0, ..., y_{n-1}, y\Big)_G$; observe theat the sequence $y_0, ..., y_{n-1} \,;\, y$ has been replaced by $y_0, ..., y_{n-1}, y ...$.
□

9.2.10. Proposition.

(i) Assume that G is a flower; then one can define a dilator $F = \dfrac{dG(y)}{dy}$ by:

$$F(x) = G(x+1) - G(x) \quad , \quad \text{i.e. } G(x+1) = G(x) + F(x)$$
$$F(f) = G(f + \mathbf{E}_1) - G(f) \quad , \quad \text{i.e. } G(f + \mathbf{E}_1) = G(f) + F(f) \ .$$

(ii) Assume that G' is another flower with $F' = \dfrac{dG'(y)}{dy}$ and that $U \in I^1(G, G')$; then one can define a natural transformation $T = \dfrac{dU(y)}{dy} \in I^1(F, F')$ by:

$$T(x) = U(x+1) - U(x) \quad , \quad \text{i.e. } U(x+1) = U(x) + T(x) \ .$$

Proof. (i) $G(x) \leq G(x+1)$, hence $F(x)$ can be defined. If $f \in I(x, y)$, then $G(f + \mathbf{E}_1)(z) = G(f + \mathbf{E}_1)G(\mathbf{E}_{xx+1})(z) = G(f)(z)$ hence $G(f + \mathbf{E}_1)$ can be put in the form $G(f) + F(f)$. F is easily shown to be a dilator: if $z \in F(x)$, write $G(x) + z = (z_0\,;\, y_0, ..., y_n)_G$; then $z = (z_1\,;\, y_0, ..., y_{n-1}\,;\, y_n)_F$, with $z_1 = z_0 - G(n)$.

(ii) is left to the reader. □

9.2.11. Definition.
FL is the category of flowers, i.e. the full subcategory of **DIL** whose objects are flowers.

9.2.12. Theorem.
The following functors establish an isomorphism between the categories **FL** and **ON** × **DIL**:

(i) $$\mathbf{\Phi}(F) = F(0) \otimes \frac{dF(y)}{dy}$$

$$\mathbf{\Phi}(T) = T(0) \otimes \frac{dT(y)}{dy}\ .$$

(ii) $$\mathbf{\Psi}(x \otimes F) = \underline{x} + \int F(y)dy$$

$$\mathbf{\Psi}(f \otimes T) = \underline{f} + \int T(y)dy\ .$$

Proof. Immediate. □

9.2.13. Definition.
A **bilator** is a functor from **ON** × **ON** to **ON**, such that:

(i) F preserves direct limits and pull-backs.

(ii) For all $x \in 0n$, the partial functor F_x $(= F(x, \cdot))$ is a flower.

(iii) F actually depends on the second variable (i.e. there is no dilator G such that $F(x, y) = G(x)$, $F(f, g) = G(f)$ for all x, y, f, g).

9.2.13. Remark.
With bilators, we are considering two-variable analogues of dilators. Most of the results concerning dilators can be adapted, mutatis mutandis, to their two-variable analogues; we shall adapt results without further justification, when needed.

9.2.14. Definition.
The following data define a category **BIL**:

objects: bilators.
morphisms from F to G: the set $I^b(F,b)$ of all natural transformations from F to G.

9.2.15. Remark.
In **BIL**, pull-backs do not necessarily exist; the reader will easily find an example. This is due to the fact that functors which do not depend on y are not considered as elements of **BIL**. The same thing will be true for the isomorphic category $\mathbf{\Omega}\,$**DIL**.

9.2.16. Examples.
The functors sum, product, and exponential are typical examples of bilators. In general any non-constant flower can be considered as a bilator (if F is a flower, associate to F the functor $G(x,y) = F(y)$, $G(f,g) = F(g)$...).

9.2.17. Notation.
A Normal Form theorem holds for bilators, and it is therefore possible to use denotations w.r.t. bilators; $z = (z_0\,;\, x_0, ..., x_{n-1}\,;\, x\,;\, y_0, ..., y_{m-1}\,;\, y)_F$ will mean that $z = F(f,g)(z_0)$ where $z_0 < F(n,m)$ and $f \in I(n,x)$, $g \in I(m,y)$ are such that $f(0) = x_0, ..., f(n-1) = x_{n-1}$, $g(0) = y_0, ..., g(m-1) = y_{m-1}$ and z_0 is uniquely defined by the condition: if $f' \in I(n',n)$, $g' \in I(m',m)$ are such that $z_0 \in rg\Big(F(f',g')\Big)$, then $n' = n$ and $m' = m$. In fact, since F is a flower in y, the datum y is redundant, and we use the notation $z = (z_0\,;\, x_0, ..., x_{n-1}\,;\, x\,;\, y_0, ..., y_{m-1})_F$.

9.2.18. Definition.
The following data define a category $\mathbf{\Omega}\,$**DIL**:

objects: dilators of kind $\mathbf{\Omega}$.
morphisms from F to G: the set $\mathbf{\Omega}\, I^1(F,G)$ of all natural transformations from F to G of kind $\mathbf{\Omega}$.

9.2.19. Remark.
The essential achievement of the next section (and of this chapter) is to establish an isomorphism between **BIL** and $\mathbf{\Omega\,DIL}$. Hence in $\mathbf{\Omega\,DIL}$ pull-backs do not necessarily exist: if one takes the Example 9.1.9 (ii): T_a and T_b are morphisms in $\mathbf{\Omega\,DIL}$, but they have no pull-back in this category, since $\underline{0}$ is not an object of $\mathbf{\Omega\,DIL}$.

9.3. The functor **SEP** and **UN**

The results of this section are crucial: they establish the isomorphism between the categories **BIL** and **Ω DIL**, and this will be used in the next section to define a *predecessor* relation between dilators. Many equivalent definitions of the isomorphisms are possible; I have chosen the one which is the closest to practice, i.e. the one using denotations.

9.3.1. Definition.

Assume that F is a connected dilator $\neq \underline{1}$; if x and y are ordinals, we define a subset $\overline{F(x,y)}$ of $F(y+x)$ as follows: $z = (z_0\,;\, x_0, ..., x_{n-1}\,;\, y+x)_F \in \overline{F(x,y)}$ iff (with $q = \sigma^F_{z_0,n}(0)$, remark that $n \neq 0\,!$).

(i) $x_q < y$.

(ii) if $q < n-1$, then $y \leq x_{q+1}$.

9.3.2. Theorem.

(i) If F is a connected dilator $\neq \underline{1}$, then there exists a binary functor $\mathbf{SEP}(F)$ from $\mathbf{ON}^2$ to $\mathbf{ON}$, together with a natural transformation $\mathbf{\Theta}_F$ from $\mathbf{SEP}(F)$ to the binary functor F_+: $F_+(x,y) = F(y+z)$, $F_+(f,g) = F(g+f)$ such that $rg\Big(\mathbf{\Theta}_F(x,y)\Big) = \overline{F(x,y)}$.

(ii) If G is another connected dilator $\neq \underline{1}$, if $T \in I^1(F,G)$, then there is a unique natural transformation $\mathbf{SEP}(T)$ from $\mathbf{SEP}(F)$ to $\mathbf{SEP}(G)$ making the following diagram commutative:

$$\begin{tikzcd} \mathbf{SEP}(F) \arrow[r, "\mathbf{\Theta}_F"] \arrow[d, "\mathbf{SEP}(T)"'] & F_+ \arrow[d, "T_+"] \\ \mathbf{SEP}(G) \arrow[r, "\mathbf{\Theta}_G"'] & G_+ \end{tikzcd}$$

(with $T_+(x,y) = T(y+x)...$).

Proof. It suffices to prove that:

(i) $F(g+f)$ maps $\overline{F(x,y)}$ into $\overline{F(x',y')}$, when $f \in I(x,x')$, $g \in I(y,y')$.

(ii) $T(y+x)$ maps $\overline{F(x,y)}$ into $\overline{G(x,y)}$

because it will be possible to define:

$$\mathbf{SEP}(F)(x,y) = \|\overline{F(x,y)}\| \ , \qquad \mathbf{SEP}(F)(f,g) = \|\overline{F(f,g)}\|$$

where $\overline{F(f,g)}$ is the function from $\overline{F(x,y)}$ to $\overline{F(x',y')}$ obtained by restriction of $F(g+f)$, $\mathbf{SEP}(T)(x,y) = \|\overline{T(x,y)}\|$ where $\overline{T(x,y)}$ is the function from $\overline{F(x,y)}$ to $\overline{G(x,y)}$ obtained by restriction of $T(x,y)$, and $\mathbf{\Theta}_F(x,y)$ is the order-preserving isomorphism from $\mathbf{SEP}(F)(x,y) = \|\overline{F(x,y)}\|$ to $\overline{F(x,y)}$.

Now, Property (i) is immediate: if $x_q < y \leq x_{q+1}$, then $g(x_q) < y' \leq y' + f(x_{q+1} - y)$. For (ii), if $x_q < y \leq x_{q+1}$, then observe that $\sigma^F_{z_0,n}(0) = q = \sigma^G_{\mathsf{Tr}(T)(z_0,n)}(0)$. □

9.3.3. Theorem.
If F is a connected dilator $\neq \underline{1}$, then $\mathbf{SEP}(F)$ is a bilator; if $x \in 0n$, then $\mathbf{SEP}(F)(x,0) = 0$.

Proof. (i) $\mathbf{SEP}(F)$ preserves direct limits and pull-backs: if $z < \mathbf{SEP}(F)(x, y)$, we show the existence of a unique normal form for z; consider the denotation of $\mathbf{\Theta}_F(x,y)(z) = (z_0\,;\, x_0, ..., x_{n-1}\,;\, y+x)_F$, and let $q = \sigma^F_{z_0,n}(0)$; then $z_0 \in \overline{F(n-q-2, q+2)}$, and if $z_0 = \mathbf{\Theta}_F(n-q-2, q+2)(z_1)$, then one can write

$$z = (z_1\,;\, x_{q+1} - y, ..., x_{n-1} - y\,;\, x\,;\, x_0, ..., x_q\,;\, y)_{\mathbf{SEP}(F)} \ .$$

From that preservation of $\varinjlim$ and $\wedge$ easily follows.

(ii) We prove that $\mathbf{SEP}(F)(x,\cdot)$ is a flower on the model of 9.2.5: assume that $z, z' \in \overline{F(x,y')}$, $z \leq z'$ and $z' \in rg\Big(F(\mathbf{E}_{yy'} + \mathbf{E}_x)\Big)$; we show that $z \in rg\Big(F(\mathbf{E}_{yy'} + \mathbf{E}_x)\Big)$, and this will establish that $\mathbf{SEP}(F)(\mathbf{E}_x, \mathbf{E}_{yy'}) = \mathbf{E}_{\mathbf{SEP}(F)(x,y)\mathbf{SEP}(F)(x,y')}$: write $z = (z_0\,;\, x_0, ..., x_{m-1}\,;\, y'+x)_F$, $z' = (z_1\,;\, x'_0, ..., x'_{n-1}\,;\, y'+x)_F$, let $q = \sigma^F_{z_0,m}(0)$, $r = \sigma^F_{z_1,n}(0)$; we have$x_q < y'$, and since $z \in rg\Big(F(\mathbf{E}_{yy'} + \mathbf{E}_x)\Big)$, one gets $x_q < y$. Now, since F is connected $|\S^F(z_0, m\,;\, z_1, n)| > 0$, and the hypothesis $z' \leq z$ entails $x'_r \leq x_q$; hence $x'_r < y$; if $r < n-1$, then $x'_{r+1} \geq y]$ hence $z' \in rg\Big(F(\mathbf{E}_{yy'} + \mathbf{E}_x)\Big)$.

(iii) $\overline{F(x,y)}$ is void when $y = 0$ (because $x_q \not< 0$),hence $\mathbf{SEP}(F)(x,0) = 0$; but if $F(n) \neq 0$, one clearly gets $\overline{F(n,n)} \neq \emptyset$, hence $\mathbf{SEP}(F)(n,n) \neq 0$, i.e. $\mathbf{SEP}(F)$ actually depends on y.

(i), (ii) and (iii)prove that F is a bilator. □

9.3.4. Remark.
In order to state the next theorem, it is necessary to extend the basic concepts of Chapter 8 to a binary functor F; we do it below, without proof:

(i) If F is a binary functor from $\mathbf{ON}^2$ to $\mathbf{ON}$ preserving direct limits and pull-backs, then $\mathsf{Tr}(F)$ is the set of all 3-uples (z_0, n, m) such that $z_0 < F(n,m)$ and for all $n' \leq n$, $f \in I(n', n)$, all $m' \leq m$, $g \in I(m', m)$, $z_0 \notin rg\Big(F(f,g)\Big)$. In the obvious normal form theorem w.r.t. F, one will number the coefficients as follows: $z = (z_0\,;\, x_m, ..., x_{m+n-1}\,;\, x\,;\, x_0, ..., x_{m-1}\,;\, y)_F$ (with $x_m < ... < x_{m+n-1} < x$, $x_0 < ... < x_{m-1} < y$).

(ii) To $(z_0, n, m) \in \mathsf{Tr}(F)$, it is possible to associate a permutation $\sigma^F_{z_0,n,m}$ of $n+m$ as follows: consider $a_i = (z_0\,;\, x^i_m, ..., x^i_{m+n-1}\,;\, 2(m+m)\,;\, x^i_0, ..., x^i_{m-1}\,;\, 2m)_F$, with $x^i_j = 2j$ when $j \neq i$, $x^i_i = 2i+1$. Then

$$a_i < a_j \leftrightarrow \sigma^F_{z_0,n,m}(i) > \sigma^F_{z_0,n,m}(j)\ .$$

One can prove in this context the exact analogue of 8.4.11.

(iii) To (z_0, n, m) and (z'_0, n', m'), distinct points in $\mathsf{Tr}(F)$, it is possible to associate $\S^F(z_0, n, m\,;\, z'_0, n', m') = (p, \varepsilon)$, with $p \leq n+m$ and $\varepsilon = \pm 1$, in such a way that the exact analogue of 8.4.20 holds.

9.3.5. Theorem.
Assume that F is a connected dilator $\neq \underline{1}$; then there exists a function sep_F from $\mathsf{Tr}(F)$ to $\mathsf{Tr}\Big(\mathbf{SEP}(F)\Big)$, with the following properties:

(i) sep_F is a bijection.

(ii) $\mathrm{sep}_F\Big((z_0, n)\Big) = (z'_0, n-q-1, q+1)$ with $q = \sigma^F_{z_0,n}(0)$.

(iii) $\sigma^{\mathbf{SEP}(F)}_{\mathsf{sep}(z_0,n)} = \sigma^F_{z_0,n}$.

(iv) $\S^{\mathbf{SEP}(F)}\Big(\mathsf{sep}_F(z_0,n), \mathsf{sep}_F(z_1,m)\Big) = \S^F(z_0,n\,;\,z_1,m)$.

(v) If $T \in I^1(F,G)$, then the following diagram is commutative:

$$\begin{tikzcd} \mathsf{Tr}(F) \arrow[r, "\mathsf{sep}_F"] \arrow[d, "\mathsf{Tr}(T)"'] & \mathsf{Tr}\Big(\mathbf{SEP}(F)\Big) \arrow[d, "\mathsf{Tr}\Big(\mathbf{SEP}(T)\Big)"] \\ \mathsf{Tr}(G) \arrow[r, "\mathsf{sep}_G"'] & \mathsf{Tr}\Big(\mathbf{SEP}(G)\Big) \end{tikzcd}$$

Proof. sep_F is defined as follows: given $(z_0,n) \in \mathsf{Tr}(F)$, let $q = \sigma^F_{z_0,n}(0)$; then let z'_0 be such that $z_0 = \mathbf{\Theta}_F(n-q-1,q+1)(z'_0)$, then $\mathrm{sep}_F(z_0,n) = (z'_0, n-q-1, q+1)$.

Properties (ii) and (v) are immediate. If a_i is defined by: $a_i = (z'_0\,;\,x^i_m, ..., x^i_{m+p-1}\,;\,2(m+p)\,;\,x^i_0, ..., x^i_{m-1}\,;\,2m)_{\mathbf{SEP}(F)}$ with $x^i_j = 2j$ for $j \neq i$, $x^i_i = 2i+1$, and $(z'_0,p,m) = \mathsf{sep}_F(z_0,n)$. Consider the points $b_i = \Big(z_0\,;\,x^i_0, ..., x^i_{m+p-1}\,;\,2(m+p)\Big)_F$; then $a_i < a_j \leftrightarrow b_i < b_j$: from that (iii) easily follows.

(iv) is proved in the same way. Finally sep_F is a bijection; this is clear from the way of passing from a F-denotation to a $\mathbf{SEP}(F)$-denotation. □

9.3.6. Definition.
One defines the functor $+$ from $\mathbf{DIL} \times \mathbf{BIL}$ to $\mathbf{BIL}$ by:

$$(F+G)(x,y) = F(x) + G(x,y)$$
$$(F+G)(f,g) = F(f) + G(f,g)$$
$$(T+U)(x,y) = T(x) + U(x,y)\ .$$

9.3.7. Definition.
One defines the functor $\mathbf{SEP}$ (**separation**) from $\mathbf{\Omega\,DIL}$ to $\mathbf{BIL}$ as follows:

(i) If $F = F' + F''$, with F'' connected, then $\mathbf{SEP}(F) = F' + \mathbf{SEP}(F'')$.

(ii) If $T = T' + T''$ with T'' connected, then $\mathbf{SEP}(T) = T' + \mathbf{SEP}(T'')$.

9.3.8. Remark.

The results and constructions which were made for connected dilators $\neq \underline{1}$ easily extend to the case of dilators of kind $\mathbf{\Omega}$; let us mention:

(i) The subset $\overline{F(x,y)}$ of $F(y+x)$ is defined to be $rg\Big(F'(\mathbf{E}_{0y}+\mathbf{E}_x)\Big) \cup \{F'(x+y)+z\,;\, z \in \overline{F''(x,y)}\}$. One defines a natural transformation $\mathbf{\Theta}_F$ from $\mathbf{SEP}(F)$ to F_+ by $rg\Big(\mathbf{\Theta}_F(x,y)\Big) = \overline{F(x,y)}$.

(ii) The bijection sep_F is defined as follows: write $\mathsf{Tr}(F) = S_1 \cup S_2$, where S_2 is the "topmost" equivalence class modulo $\sim_0^F$ (w.r.t. $\leq^F / \sim_0^F$), (and if $F = F' + F''$, F'' connected, $S_1 = \mathsf{Tr}(F')$). Then $\mathrm{sep}_F(z_0,n)$ is defined exactly as in 9.3.5 when $(z_0,n) \in S_2$, whereas $\mathrm{sep}_F(z_0,n) = (z_0,n,0)$ when $(z_0,n) \in S_1$. The exact analogue of 9.3.5 holds.

9.3.9. Examples.

(i) Assume that F is a non constant flower; then F is of kind $\mathbf{\Omega}$; let us look at the case $F(0) = 0$, i.e. F connected; now recall that $\sigma^F_{z_0,n}(0) = n-1$ by 9.2.5, hence $\overline{F(x,y)} = rg\Big(F(\mathbf{E}_{yy+x})\Big)$, hence $\mathbf{\Theta}_F(x,y) = F(\mathbf{E}_{yy+x})$; the commutativity of the diagram

$$\begin{tikzcd} F(y) \arrow[r, "F(\mathbf{E}_{yy+x})"] \arrow[d, "F(g)"'] & F(y+x) \arrow[d, "F(g+f)"] \\ F(y') \arrow[r, "F(\mathbf{E}_{y'y'+x})"'] & F(y'+x) \end{tikzcd}$$

implies that $\mathbf{SEP}(F)(x,y) = F(y)$, $\mathbf{SEP}(F)(f,g) = F(g)$. The case $F(0) \neq 0$ is immediate: if $F = \underline{x} + F''$, with $a = F(0)$ then $\mathbf{SEP}(F)(x,y) = a + F''(y) = F(y)$, $\mathbf{SEP}(F)(f,g) = \mathbf{E}_a + F''(g)$. Hence separation of variables on non-constant flowers is just a renaming of variables.

(ii) Suppose that $F = \mathsf{Id} \cdot \mathsf{Id} = \mathsf{Id}^2$, then $G = \mathbf{SEP}(F)$ satisfies the following equations: (if $f \in I(x,x')$, $g \in I(y,y')$)

(1) $G(x,0) = 0$ $\qquad G(f,\mathbf{E}_0) = \mathbf{E}_0$

(2) $G(x,y+1) = G(x,y)+y+1+x$ $\qquad G(f,g+\mathbf{E}_1) =$

$$G(f,g)+g+\mathbf{E}_1+f$$

(3) $\qquad G(f,g+\mathbf{E}_{01}) \ = $

$$G(f,g)+\mathbf{E}_{0y'+1+x'}$$

(4) $G\Big(x,\mathsf{sup}(y_i)\Big)=\mathsf{sup}\Big(G(x,y_i)\Big) \qquad G(f,\mathbf{U}g_i)=\mathbf{U}G(f,g_i)$.

We only prove (1) and (2) since (3) and (4) are general properties of bilators: (3) follows from (FL) and (4) from preservation of direct limits and (FL).

It is immediate that $\overline{F(x,0)}=\emptyset$: this proves (1). Let us now compute the permutations associated with the elements of $\mathsf{Tr}(F)=\{(1,2),(0,1),(2,2)\}$: $\sigma^F_{0,1}(0)=0$, and

$$(1\,;\,0,3\,;\,4)_F=4\,\cdot\,0+3=3<(1\,;\,1,2\,;\,4)_F=4\,\cdot\,1+2=6:$$

$$\sigma^F_{1,2}(0)=0,\ \sigma^F_{1,2}(0)=1$$

$$(2\,;\,1,2\,;\,4)_F=4\,\cdot\,2+1=9<(2\,;\,0,3\,;\,4)_F=4\,\cdot\,3+0=12:$$

$$\sigma^F_{2,2}(0)=1,\ \sigma^F_{2,2}(1)=0$$

$$\begin{aligned}\overline{F(x,y)} \ &= \ \{(0\,;\,u\,;\,y+x)_F\,;\,u<y\}\,\cup\\ &\quad\ \{(1\,;\,u,v\,;\,y+x)_F\,;\,u<y\le v<y+x\}\,\cup\\ &\quad\ \{(2\,;\,u,v\,;\,y+x)\,;\,v<u<y\}\ .\end{aligned}$$

If $z\in\overline{F(x,y)}$, then $F(\mathbf{E}_{yy+1}+\mathbf{E}_x)(z)\in\overline{F(x,y+1)}$; if $z\in\overline{F(x,y+1)}$, but $z\notin rg\Big(F(\mathbf{E}_{yy+1}+\mathbf{E}_x)\Big)$, one easily sees that $z=(y+1+x)y+v$ for some $v<y+1+x$); conversely all points $z=(y+1+x)+v$ with $v<y+1+x$ are in $\overline{F(x,y+1)}$, but not in $rg\Big(F(\mathbf{E}_{yy+1}+\mathbf{E}_x)\Big)$: this is immediate from $(1\,;\,u,v\,;\,y+1+x)_F=(y+1+x)-u+v$ and $(2\,;\,u,v\,;\,y+1+x)_F=(y+1+x)v+u$, and $(0\,;\,u\,;\,y+1+x)_F=(y+1+x)u+u$. Hence $G(x,y+1)=G(x,y)+y+1+x$.

Let $f\in I(x,x')$, $g\in I(y,y')$; if $z\in\overline{F(x,y+1)}$ and $z\in rg\Big(F(\mathbf{E}_{yy+1}+\mathbf{E}_x)\Big)$, then

$$\begin{aligned}F(g+\mathbf{E}_1+f)(z) \ &= \ F(g+\mathbf{E}_1+f)F(\mathbf{E}_{yy+1}+\mathbf{E}_x)(t) \ =\\ &\quad\ F(\mathbf{E}_{y'y'+1}+\mathbf{E}_{x'})F(g+f)(t)\ .\end{aligned}$$

If $z = (y+1+x)+v$, then $F(g+\mathbf{E}_1+f)(z) = (y'+1+x')y' + (g+\mathbf{E}_1+f)(v)$: hence $G(f, g+\mathbf{E}_1) = G(f,g) + g + \mathbf{E}_1 + f$.

(iii) There exists a dilator F such that $G = \mathbf{SEP}(F)$ is the bilator product: F is the prime dilator corresponding to the permutation $\sigma(0) = 0$, $\sigma(1) = 1$ of 2: the bilator $\cdot$ has a trace consisting of the only point (0,1,1) ($(0\,;\,a\,;\,x\,;\,b\,;\,y). = x \cdot b + a$); clearly $a_1 = (0\,;\,3\,;\,4\,;\,0\,;\,2). < (0\,;\,2\,;\,4\,;\,1\,;\,2). = g$, hence the associated permutation is the identity permutation, and the result follows from 9.3.5.

9.3.10. Definition.
If F is a bilator, define $\overline{F(x)} \subset F(x,x)$ by:

$$(z_0\,;\,x_0,\dots,x_{m-1}\,;\,x\,;\,y_0,\dots,y_{n-1})_F \in \overline{F(x)} \;\leftrightarrow$$
$$\Big(nm = 0 \lor (nm \neq 0 \land y_{n-1} < x_0)\Big)\;.$$

9.3.11. Theorem.

(i) If F is a bilator, then there is a functor $\mathbf{UN}(F)$ from **ON** to **ON** (**unification**), together with a natural transformation Ξ_F from $\mathbf{UN}(F)$ to the dilator F_δ: $F_\delta(x) = F(x,x)$, $F_\delta(f) = F(f,f)$ such that $rg\Big(\Xi_F(x)\Big) = \overline{F(x)}$.

(ii) If G is another bilator, if $T \in I^b(F,G)$, then there is a unique natural transformation $\mathbf{UN}(T)$ from $\mathbf{UN}(F)$ to $\mathbf{UN}(G)$ which makes the following diagram commutative:

$$\begin{tikzcd}
\mathbf{UN}(F) \arrow[r, "\Xi_F"] \arrow[d, "\mathbf{UN}(T)"'] & F_\delta \arrow[d, "T_\delta"] \\
\mathbf{UN}(G) \arrow[r, "\Xi_G"'] & F_\delta
\end{tikzcd}$$

(with $T_\delta(x) = T(x,x)$).

Proof. It suffices to prove that

(i) $F(f,f)$ maps $\overline{F(x)}$ into $\overline{F(y)}$ when $f \in I(x,y)$.

(ii) $T(x,x)$ maps $\overline{F(x)}$ into $\overline{G(x)}$.

Because if $\overline{F(f)}$ denotes the function from $\overline{F(x)}$ to $\overline{F(y)}$ obtained by restriction of $F(f,f)$, and if $\overline{T(x)}$ denotes the function from $\overline{F(x)}$ to $\overline{G(x)}$ obtained by restriction of $T(x,x)$, then $\mathbf{UN}(F)(x) = \|\overline{F(x)}\|$, $\mathbf{UN}(F)(f) = \|\overline{F(f)}\|$, $\mathbf{UN}(T)(x) = \|\overline{T(x)}\|$ are the solutions of the theorem, provided $\Xi_F(x)$ is chosen to be the order-preserving isomorphism from $\mathbf{UN}(F)(x)$ to $\overline{F(x)}$.

Property (i) is immediate: if $nm = 0$ or $y_{n-1} < x_0$, then clearly $nm = 0$ or $f(y_{n-1}) < f(x_0)$. For (ii) if $nm = 0$ or $y_{n-1} < x_0$, observe that $T(x,x)(z_0\,;\,x_0,...,x_{m-1}\,;\,x\,;\,y_0,...,y_{n-1})_F = (T(m,n)(z_0)\,;\,x_0,...,x_{m-1}\,;\,x\,;\,y_0,...,y_{n-1})_G$. □

9.3.12. <u>Theorem</u>.
UN is a functor from **BIL** to **Ω DIL**.

<u>Proof</u>. We show that $\mathbf{UN}(F)$ is a dilator, and as usual we need to express denotations w.r.t. $\mathbf{UN}(F)$; if $z < \mathbf{UN}(F)(x)$, let $z' = \Xi_F(z)$, and write $z' = (z_0\,;\,x_0,...,x_{n-1}\,;\,x\,;\,y_0,...,y_{m-1})_F$; then the point $z'_0 = (z_0\,;\,m,...,m+n+1\,;\,m+n\,;\,0,...,m-1)_F$ belongs to $\overline{F(m+n)}$, hence $z'_0 = \Xi_F(m+n)(z_1)$ for some $z_1 \in \mathbf{UN}(F)(m+n)$, and clearly one can write $z = (z_1\,;\,y_0,...,y_{m-1},x_0,...,x_{n-1}\,;\,x)_{\mathbf{UN}(F)}$, and unicity of such a notation is easily established. From that it follows that **UN** is a functor from **BIL** to **DIL**. It is necessary to show that $\mathbf{UN}(F)$ and $\mathbf{UN}(T)$ are of kind $\mathbf{\Omega}$. This is a consequence of the following theorem:

9.3.13. <u>Theorem</u>.
Assume that F is a bilator; then there exists a function un_F from $\mathsf{Tr}(F)$ to $\mathsf{Tr}\Big(\mathbf{UN}(F)\Big)$, such that:

(i) un_F is a bijection.

(ii) $\mathrm{un}_F\Big((z_0,n,m)\Big) = (z'_0, n+m)$.

(iii) $\sigma^{\mathbf{UN}(F)}_{\mathsf{un}_F(z_0,n,m)} = \sigma^F_{z_0,n,m}$.

(iv) $\S^{\mathbf{UN}(F)}\Big(\mathsf{un}_F(z_0,n,m), \mathsf{un}_F(z'_0,n',m')\Big) = \S^F(z_0,n,m\,;\,z'_0,n,m)$.

(v) If $T \in I^b(F,G)$ the diagram

$$\begin{tikzcd} \mathsf{Tr}(F) \arrow[r, "\mathsf{un}_F"] \arrow[d, "\mathsf{Tr}(T)"'] & \mathsf{Tr}\big(\mathbf{UN}(F)\big) \arrow[d, "\mathsf{Tr}(\mathbf{UN}(T))"] \\ \mathsf{Tr}(G) \arrow[r, "\mathsf{un}_G"'] & \mathsf{Tr}\big(\mathbf{UN}(G)\big) \end{tikzcd}$$

is commutative.

Proof. If $(z_0,n,m) \in \mathsf{Tr}(F)$, then $(z_0\,;\, m,...,m+n-1\,;\, m+n\,;\, 0,...,m-1)_F \in \overline{F(m+n)}$, and define z'_0 by $z_0 = \Xi_F(m{+}n)(z'_0)$: then $\mathsf{un}_F(z_0,n,m) = (z'_0, m+n)$. Properties (ii) and (v) are immediate; (i) is immediate if one looks at the way $\mathbf{UN}(F)$-denotations are obtained. (iii) and (iv) are left as exercise for the reader... . □

End of the proof of 9.3.12. – If $F(x,0)=0$ for all x, then $|\S^F(z_0,n,m\,;\,z'_0,n',m')| \neq 0$ for all (z_0,n,m), (z'_0,n,m) in $\mathsf{Tr}(F)$ (since the comparison between $(z_0\,;\,x_0,...,x_{n-1}\,;\,x\,;\,y_0,...,y_{m-1})_F$ and $(z'_0\,;\,x'_0,...,x'_{n-1}\,;\,x\,;\,y'_0,...,y'_{m'-1})_F$ depends on the relative orders of the points y_{m-1} and $y'_{m'-1}$). By 9.3.13 (iv) $|\S^F(z_1,p\,;\,z'_1,p')| \neq 0$ for all (z_1,p) and (z'_1,p') in $\mathsf{Tr}\big(\mathbf{UN}(F)\big)$.

– In the general case one can observe that there is a maximum equivalence class for $\sim_0^F$ in $\mathsf{Tr}(F)$, and that this class is transferred into a maximum equivalence class for $\sim_0^{(F)}$ in $\mathsf{Tr}\big(\mathbf{UN}(F)\big)$. □

But one can also use the following proposition:

9.3.14. Proposition.

(i) If F' is a dilator, if F'' is a bilator, then $\mathbf{UN}(F'{+}F'') = F'{+}\mathbf{UN}(F'')$.

(ii) If $T' \in I^1(F',G')$, if $T'' \in I^b(F'',G'')$, then $\mathbf{UN}(T'+T'') = T' + \mathbf{UN}(T'')$.

Proof. Clearly $\overline{(F'+F'')(x)}$ consists of all points $z < F'(x)$ (because $z = (z_0\,;\,x_0,...,x_{n-1}\,;\,x)_{F'+F''}$, i.e. $m=0$) and all points $F'(x)+z$, with $z \in \overline{F''(x)}$... . □

9.3.15. Theorem.
SEP and **UN** are inverse functor.

Proof. (i) We show that $\mathbf{SEP}\,\mathbf{UN}(F) = F$, and $\mathbf{UN}\,\mathbf{SEP}(F) = F$, when F is an object. The function $\mathrm{sep}_{\mathbf{UN}(F)}\mathsf{un}_F = g$ is a bijection from $\mathsf{Tr}(F)$ to $\mathsf{Tr}\Big(\mathbf{SEP}\,\mathbf{UN}(F)\Big)$ which has the following properties: $g\Big((z_0, n, m)\Big) = (z_0'', n, m)$, $\sigma^{\mathbf{SEP}\,\mathbf{UN}(F)}_{g(z_0,n,m)} = \sigma^F_{z_0,n,m}$, $\S^{\mathbf{SEP}\,\mathbf{UN}(F)}(z_0, n, m\,;\, z_0', n, m) = \S^F(z_0, n, m\,;\, z_0', n, m)$. This means that F and $\mathbf{SEP}\, bfUN(F)$ have the same invariants. By 8.4.23 (or rather its analogue for two-variable functors) F and $\mathbf{SEP}\,\mathbf{UN}(F)$ are isomorphic functors, hence equal. Exactly the same argument yields $\mathbf{UN}\,\mathbf{SEP}(F) = F$.

(ii) In order to show that $\mathbf{SEP}\,\mathbf{UN}(T) = T$, observe that the diagram

$$\begin{array}{ccc} \mathsf{Tr}(F) & \xrightarrow{g_F\ (=\text{identity})} & \mathsf{Tr}(F) \\ \mathsf{Tr}(T) & & \mathsf{Tr}\Big(\mathbf{SEP}\,\mathbf{UN}(T)\Big) \\ \mathsf{Tr}(G) & \xrightarrow[g_G\ (=\text{identity})]{} & \mathsf{Tr}(G) \end{array}$$

is commutative by 9.3.5 (v) and 9.3.12 (v), hence T and $\mathbf{SEP}\,\mathbf{UN}(T)$ have the same trace, and this implies $T = \mathbf{SEP}\,\mathbf{UN}(T)$. One gets $\mathbf{UN}\,\mathbf{SEP}(T) = T$ by a similar argument. □

9.3.16. Corollary.
UN and **SEP** preserve direct limits and pull-backs.

Proof. Because they are isomorphisms. □

The exceptional importance of **SEP** and **UN** in the theory of dilators makes it necessary to give an alternative approach; most of the proofs will be omitted; one can find some of these in [5], 3.6.

9.3.17. Definition.

(i) Assume that F is a bilator; then ∂F is the following functor from $\mathbf{ON}^2$ to $\mathbf{ON}$:

$$\partial F(x, y) = F(x, y+1) - F(x, y)$$

$$\partial F(f,g) = F(f, g + \mathbf{E}_1) - F(f,g)$$

(i.e. $\partial F = \dfrac{dF(\cdot, z)}{dz}$; one easily checks that ∂F preserves direct limits and pull-backs).

(ii) If G is another bilator and $T \in I^b(F,G)$, then one defines a natural transformation γT from ∂F to ∂G by:

$$\partial T(x,y) = T(x, y+1) - T(x,y) \ .$$

9.3.18. Theorem.

(i) If F is a bilator then $G = \mathbf{UN}(F)$ can be defined by:

$$G(x) = F(x,0) + \sum_{y<x} \partial F(x - (y+1), y) \ ,$$

and if $f \in I(x,x')$

$$G(f) = F(f, \mathbf{E}_0) + \sum_{g<f} \partial F(f - (g + \mathbf{E}_1), g)$$

(this means that

$$G(f)\Big(F(x,0) + \textstyle\sum_{y'<y} \partial F(x - (y'+1), y') + z\Big) \ = \ F(x',0) + \displaystyle\sum_{y'<f(y)} \partial F(x' - (y'+1), y') + \partial F(f^y, f_y)(z)$$

where $f = f_y + \mathbf{E}_1 + f^y$ and $f_y \in I\big(y, f(y)\big)$).

(ii) If $T \in I^b(F,F')$, then $\mathbf{UN}(T)$ is the natural transformation U defined by:

$$U(x) = T(x,0) + \sum_{y<x} \partial T(x - (y+1), y) \ .$$

Proof. See [5], 3.6.2. □

9.3.19. Proposition.

(i) If F is a dilator of kind $\mathbf{\Omega}$ and y is an ordinal, then $F \circ (\underline{y} + \mathsf{Id})$ is of kind $\mathbf{\Omega}$.

(ii) If F is a dilator of kind $\mathbf{\Omega}$, if $g \in I(y, y')$, then the natural transformation $\mathbf{E}^1_F \circ (\underline{g} + \mathbf{E}^1_{\mathsf{Id}})$ from $F \circ (\underline{y} + \mathsf{Id})$ to $F \circ (\underline{y}' + \mathsf{Id})$ is of kind $\mathbf{\Omega}$.

(iii) If $U \in \mathbf{\Omega}\, I^1(F, G)$, then the natural transformation $U \circ (\mathbf{E}^1_{\underline{y}} + \mathbf{E}^1_{\mathsf{Id}})$ from $F \circ (\underline{y} + \mathsf{Id})$ to $G \circ (\underline{y} + \mathsf{Id})$ is of kind $\mathbf{\Omega}$.

($\circ$ denotes **composition**; composition is defined by $(F \circ F')(x) = F\big(F'(x)\big)$, $(F \circ F')(f) = F\big(F'(f)\big)$, $(T \circ T')(x) = T\big(T'(x)\big)$).

Proof. This result can be established by computing the equivalence relation $\sim_0^{F \circ (\underline{y}+\mathsf{Id})}$. But a more direct proof can be obtained by means of the characterization: F is of kind $\mathbf{\Omega}$ iff $F(0n)$ is of cardinality $0n$ (9.4.6), because $y + 0n = 0n$, hence $F(0n) = \big(F \circ (\underline{y} + \mathsf{Id})\big)(0n)\ldots$. $\square$

9.3.20. Definition.

(i) If F is a dilator of kind $\mathbf{\Omega}$, if y is an ordinal, define a dilator ${}^*_yF$ by:

$$F \circ (\underline{y} + \mathsf{Id}) = {}^*_yF + F' \quad \text{for some } F' \text{ connected .}$$

(ii) If F is a dilator of kind $\mathbf{\Omega}$, if $g \in I(y, y')$, define ${}^*_gF \in I^1({}^*_yF, {}^*_{y'}F)$ by:

$$\mathbf{E}^1_F \circ (\underline{g} + \mathbf{E}^1_{\mathsf{Id}}) = {}^*_gF + T' \quad \text{for some } T' \text{ connected .}$$

(iii) If $T \in \mathbf{\Omega}\, I^1(F, G)$, define ${}^*_yT \in I^1({}^*_yF, {}^*_yG)$ by:

$$T \circ (\mathbf{E}^1_{\underline{y}} + \mathbf{E}^1_{\mathsf{Id}}) = {}^*_yT + T' \quad \text{for some } T' \text{ connected .}$$

9.3.21. <u>Definition</u>.

(i) If F is a dilator of kind $\mathbf{\Omega}$, define a two- variable functor from $\mathbf{ON}^2$ to $\mathbf{ON}$, $\partial\,\mathbf{SEP}(F)$, by

$$\begin{aligned} {}^{*}_{y+1}F(x) &= {}^{*}_{y}F(1+x) + \partial\,\mathbf{SEP}(F)(x,y) \\ {}^{*}_{g+\mathbf{E}_1^F}(f) &= {}^{*}_{g}F(\mathbf{E}_1+f) + \partial\,\mathbf{SEP}(F)(f,g)\ . \end{aligned}$$

(ii) If $T \in \mathbf{\Omega}\,I^1(F,G)$, define a natural transformation $\partial\,\mathbf{SEP}(T)$ from $\partial\,\mathbf{SEP}(F)$ to $\partial\,\mathbf{SEP}(G)$ by:

$${}^{*}_{y+1}T(x) = {}^{*}_{y}T(1+x) + \partial\,\mathbf{SEP}(T)(x,y)\ .$$

(Such a definition is made possible because, for instance, if $F \circ (y+1+\mathsf{Id}) = {}^{*}_{y+1}F + F'$ and $F \circ (y+\mathsf{Id}) = {}^{*}_{y}F + F''$ for some F' and F'' connected, then $F \circ (\underline{y+1}+\mathsf{Id}) = {}^{*}_{y}F \circ (\underline{1}+\mathsf{Id}+F'' \circ (\underline{1}+\mathsf{Id})$, hence ${}^{*}_{y+1}F = {}^{*}_{y}F \circ (\underline{1}+\mathsf{Id}) + F''$ for some F''' $(= \partial\mathbf{SEP}(F)(\cdot,y))$.)

9.3.22. <u>Theorem</u>.

(i) Assume that F is a dilator of kind $\mathbf{\Omega}$; then

$$\begin{aligned} \mathbf{SEP}(F)(x,y) &= {}^{*}_{0}F(x) + \sum_{y'<y} \partial\,\mathbf{SEP}(F)(x,y') \\ \mathbf{SEP}(F)(f,g) &= {}^{*}_{0}F(f) + \sum_{y'<y} \partial\,\mathbf{SEP}(F)(f,g_{y'})\ , \end{aligned}$$

where $g_{y'} \in I\big(y', g(y')\big)$ is defined by $g_{y'}(z) = g(z)$ for all $z < y'$.

(ii) Assume that $T \in \mathbf{\Omega}\,I^1(F,G)$; then

$$\mathbf{SEP}(T)(x,y) = {}^{*}_{0}T(x) + \sum_{y'<y} \partial\,\mathbf{SEP}(T)(x,y')\ .$$

<u>Proof</u>. See [5], 3.6.6. □

9.3.23. Remarks.

(i) ${}^*_yF(x)$, ${}^*_gF(f)$ defines a binary functor from $\mathbf{ON}^2$ to $\mathbf{ON}$; in fact one can write: $F_+ = {}^*_\cdot F(\cdot) + F'$ for some "connected" F'. Similar properties hold for ${}^*_\cdot T(\cdot)$.

(ii) It is possible to write:

$$\mathbf{SEP}(F) = {}^*_0F + \int \frac{\partial\,\mathbf{SEP}(F)(\cdot, z)}{\partial z}$$
$$\mathbf{SEP}(T) = {}^*_0T + \int \frac{\partial\,\mathbf{SEP}(T)(\cdot, z)}{\partial z} \ .$$

(iii) The idea of the proof of 9.3.22 is as follows: one restricts to the case when F is connected; then ${}^*_yF(x)$ is exactly the set of all elements $z = (z_0\,;\, x_0, ..., x_{n-1}\,;\, y + x)_F$ such that $x_q < y$, with $q = \sigma^F_{z_0,n}(0)$. Then the set of points which are in ${}^*_{y+1}F(x)$, but not in ${}^*_yF(1+x)$, is exactly the set of all elements$z = (z_0\,;\, x_0, ..., x_{n-1}\,;\, y+1+x)_F$, with $x_q = y ....$

9.4. Induction on dilators

9.4.1. Ordinal classes.

The class $0n$ of ordinals is well-ordered by the membership relation; the Burali-Forti paradox states that $0n$ is not an ordinal, i.e. not a set (if $0n$ were a set, then $0n \in 0n$, and $x_n = 0n$ would be a s.d.s. for $\in$ in $0n$). However, there is no reason against a reasonable use of $0n$. In practice $0n$ can be considered as an ordinal. A typical example is when F is a dilator: then $F(0n)$ can be defined by the usual direct limit process: $0n$ is the direct limit of $(x, \mathbf{E}_{xx'})$, when x varies through $0n$: the index "set" here is a proper class.... Concretely $F(0n)$ consists of all formal denotations $(z_0\,;\, x_0, ..., x_{n-1}\,;\, 0n)_F$, with $(z_0, n) \in \mathsf{Tr}(F)$ and $x_0 < ... < x_{n-1}$, the ordering between $t = (z_0\,;\, x_0, ..., x_{n-1}\,;\, 0n)_F$ and $t = (z_1\,;\, x'_0, ..., x'_{m-1}\,;\, 0n)_F$ is determined as usual from the relative orders of the x_i's and x_j's, see ?? p. 37. $F(0n)$ is a class (in general proper) and is a well-order: this is the reason why we shall speak of an **ordinal class**. In practice $0n$ can often be "relativized", i.e. replaced by some ordinal:

- very often $0n$ can be replaced by $\aleph_1$: this measn that the only objects that we acknowledge as ordinals are denumerable.
- In some cases $0n$ can even be replaced by admissible ordinals (ω_1^{ck}, or the first stable σ_0).
- In some situations $0n$ can even be replaced by any limit ordinal of the form ω^x.

Dilators yield a new approach to the question of proper classes (in the context of ordinals): F enables us to compute the ordinal class $F(0n)$ "in function of $0n$": this is a *dynamic* theory of ordinal classes.

9.4.2. Definition.

The **predecessor** relation $\ll$ between dilators is defined as follows:

(i) $F \ll F + G$ when $G \neq \underline{0}$.

(ii) If F is of kind $\mathbf{\Omega}$, let $G = \mathbf{SEP}(F)$; then $G^y \ll F$ for all $y \in 0n$ (recall that $G^y = G(\cdot, y)$).

(iii) $F \ll G$ and $G \ll H$ imply $F \ll H$.

9.4.3. Theorem.
Let $\mathbf{\Delta}$ be the following functor (**diagonal** functor) from **BIL** to **DIL**: $(\mathbf{\Delta}F)(x,x) = F(x,x)$, $(\mathbf{\Delta}F)(f,f) = F(f,f)$, $(\mathbf{\Delta}T)(x,x) = T(x,x)$; then

(i) $\Big(\mathbf{\Delta}\,\mathbf{SEP}(F)\Big) \circ (\underline{\omega}^{1+\mathsf{Id}}) = F \circ (\underline{\omega}^{1+\mathsf{Id}})$

(ii) $\Big(\mathbf{\Delta}\,\mathbf{SEP}(T)\Big) \circ (\mathbf{E}^1_{\underline{\omega}^{1+\mathsf{Id}}}) = F \circ (\mathbf{E}^1_{\underline{\omega}^{1+\mathsf{Id}}})$

when F and T are of kind $\mathbf{\Omega}$.

Proof. (i) means that, when F is a bilator, then $F(x,x) = \mathbf{UN}(F)(x)$ for all $x = \omega^{1+x'}$, $F(f,f) = \mathbf{UN}(F)(f,f)$ for all $f = \underline{\omega}^{\mathbf{E}_1+f'}$. Define a function φ_x from $F(x,x)$ to itself by: $\varphi_x\Big((z_0\,;\,x_0,...,x_{n-1}\,;\,x\,;\,y_0,...,y_{m-1})\Big)_F = (z_0\,;\,y+x_0,...,y+x_{n-1}\,;\,x\,;\,y_0,...,y_{m-1})_F$ with $y=0$ if $m=0$, $y = y_{m-1}+1$ otherwise. This definition is made possible because $y<z$ and $y+x=x$. Now observe that φ_x is a strictly increasing function: if $t = (z_0\,;\,x_0,...,x_{n-1}\,;\,x\,;\,y_0,...,y_{m-1})_F < (z_1\,;\,x'_0,...,x'_{p-1}\,;\,x\,;\,y'_0,...,y'_{q-1})_F = u$, then consider $y=0$ if $m=0$, $y=y_{m-1}+1$ otherwise, $y'=0$ if $q=0$, $y'=y'_{q-1}+1$ otherwise. If $y<y'$, then $(z_0\,;\,y+x_0,...,y+x_{n-1}\,;\,x\,;\,y_0,...,y_{m-1})_F < (z_1\,;\,y'+x'_0,...,y'+x'_{p-1}\,;\,x\,;\,y'_0,...,y'_{q-1})_F$ by general properties of flowers; if $y=y'$, then

$$\varphi_x(t) = F(\mathbf{E}_{0y}+\mathbf{E}_x,\mathbf{E}_x)(t) < \varphi_x(u) = F(\mathbf{E}_{0y}+\mathbf{E}_x,\mathbf{E}_x)(u)\ .$$

Obviously $rg(\varphi_x) = \overline{F(x)}$, hence $F(x,x) = \mathbf{UN}(F)(x)$. In order to prove that $F(f,f) = \mathbf{UN}(F)(f)$, it will be sufficient to prove the commutativity of the diagram:

$$\begin{tikzcd}
F(x,x) \arrow[r, "\varphi_x"] \arrow[d, "F(f,f)"'] & F(x,x) \arrow[d, "F(f,f)"] \\
F(x',x') \arrow[r, "\varphi_{x'}"'] & F(x',x')
\end{tikzcd}$$

$$\varphi_{x'}(f,f)\Big((z_0\,;\,x_0,...,x_{n-1}\,;\,x\,;\,y_0,...,y_{m-1})_F\Big) =$$
$$\Big(z_0\,;\,y'+f(x_0),...,y'+f(x_{n-1})\,;\,x'\,;\,f(y_0),...,f(y_{m-1})\Big)_F\ ,$$

with $y' = 0$ if $n = 0$, $y' = f(y_{m-1}) + 1$ otherwise. On the other hand we have:

$$F(f,f)\varphi_x\Big((z_0\,;\,x_0,...,x_{n-1}\,;\,x\,;\,y_0,...,y_{m-1})_F\Big) \;=\; \Big(z_0\,;\,f(y+x_0),...,f(y+x_{n-1})\,;\,x'\,;\,f(y_0),...,f(y_{m-1})\Big)_F\,,$$

with $y = 0$ if $m = 0$, $y = y_{m-1} + 1$ otherwise. If $n = 0$, then the two expressions coincide. If $m \neq 0$, then we must show that $f(y_{m-1}+1+x_i) = f(y_{n-1})+1+f(x_i)$; but $f(1) = 1$ (because $f = \underline{\omega}^{\mathbf{E}_1+f'}$), hence it suffices to prove that f is "linear", i.e. that $f(a+b) = f(a)+f(b)$: this is immediate if one looks at the Cantor Normal Forms of a and b.

(ii) means that, if F and G are bilators and $T \in I^1(F,G)$, then $T(x,x) = \mathbf{UN}(T)(x)$. In order to prove this property, it will suffice to show that $\varphi^G_x T(x,x) = T(x,x)\varphi^F_x$: this is left to the reader. □

9.4.4. Remark.
9.4.3 expresses that $\mathbf{UN}$ and $\boldsymbol{\Delta}$ are very close to one another: they coincide on ordinals of the form $\omega^{1+x'}$ and function of the form $\underline{\omega}^{\mathbf{E}_1+f'}$.

9.4.5. Theorem.
Let F be a dilator; then the predecessors of F for $\ll$ form a well-ordered class of order type $F(0n)$.

Proof. First observe that $F(0n)$ is a well-order: if $u_n = (z_n\,;\,y^n_0,...,y^n_{p_n-1}\,;\,0n)_F$ is a s.d.s. in $F(0n)$, let y be an ordinal $>$ all ordinals $y^n_{p_n-1}$; then $u'_n = (z_n\,;\,y^n_0,...,y^n_{p_n-1}\,;\,y)_F$ is a s.d.s. in $F(y)$.... We prove the theorem by induction on $F(0n)$:

- If F is of kind $\mathbf{0}$ (i.e. $F(0n) = 0$), then F has no predecessor.
- If F is of kind $\mathbf{1}$, write $F = F' + \underline{1}$, and $F(0n) = F'(0n) + 1$; the predecessors of F for $\ll$ are F' and its predecessors; the induction hypothesis yields that the class of all predecessors of F' is a well-order of order type $F'(0n)$, hence the class of all predecessors of F is a well-order of order type $F(0n)$.
- If F is of kind $\boldsymbol{\omega}$, write $F = \sum\limits_{i<x} F_i$ with x limit; let $G_i = \sum\limits_{i'<i} F_i$; then

$F' \ll F$ iff $F' \ll G_i$ for some i (observe that $G_i \ll G_j$ when $i < j$); the induction hypothesis yields that the class of predecessors of G_i is a well-order of order type $G_i(0n)$ for all i, hence the class of predecessors of F is a well-order of order type $\sup_i G_i(0n) = F(0n)$.

– If F is of kind $\mathbf{\Omega}$, then $F' \ll F$ iff $F' \ll \mathbf{SEP}(F)^y$ for some $y \in 0n$; observe that $\mathbf{SEP}(F)^y \underset{=}{\ll} \mathbf{SEP}(F)^{y'}$ when $y \leq y'$. The function $y \rightsquigarrow \mathbf{SEP}(F)(0n, y)$ is strictly increasing for $y \geq n$, hence $\mathbf{SEP}(F)(0n, y) < \mathbf{SEP}(F)(0n, 0n)$; but $\mathbf{SEP}(F)(0n, 0n) = F(0n)$ by 9.4.3 (because $0n = \omega^{1+0n}$; the theorem still holds when x is not a set). Hence we can apply the induction hypothesis which yields that the class of predecessors of $\mathbf{SEP}(F)^y$ is a well-order of order type $\mathbf{SEP}(F)(0n, y)$. Hence the union of all these mutually compatible well-orders is a well-order of order type $F(0n) = \mathbf{SEP}(F)(0n, 0n) = \sup_{y \in 0n} \mathbf{SEP}(F)(0n, y)$. □

9.4.6. Remark.
We have obtained the following characterization of the kind of a dilator:

(i) F is of kind $\mathbf{0}$ iff $F(0n) = 0$.

(ii) F is of kind $\mathbf{1}$ iff $F(0n)$ is a successor.

(iii) F is of kind $\boldsymbol{\omega}$ iff $F(0n)$ is limit and of cofinality $< 0n$, i.e. the supremum of a sequence of length $< 0n$ (i.e. indexed by a set).

(iv) F is of kind $\mathbf{\Omega}$ iff $F(0n)$ is limit and of cofinality $0n$, i.e. the supremum of a sequence of length $0n$.

In fact, in this characterization, $0n$ can be replaced by any regular cardinal $> F(\omega)$. In practice, $F(\omega)$ is often denumerable, hence one can replace $0n$ by $\aleph_1$: F will be of kind $\mathbf{0}$, $\mathbf{1}$, $\boldsymbol{\omega}$ or $\mathbf{\Omega}$ when $F(\aleph_1)$ is 0, successor, of cofinality ω, of cofinality $\aleph_1$. This is the origin of our terminology, since $\mathbf{\Omega}$ is the obsolete way of denoting $\aleph_1$.

9.4.7. Theorem Induction on dilators (Girard, [5]).
Let P be a property defined on dilators, and assume that:

(i) $P(\underline{0})$.

(ii) $P(F) \to P(F + \underline{1})$.

(iii) If x is limit, and if $F_i \neq \underline{0}$ for all $i < x$, and if for all $y < x$, $P\Bigl(\sum_{i<y} F_i\Bigr)$, then $P\Bigl(\sum_{i<x} F_i\Bigr)$.

(iv) If F is of kind $\mathbf{\Omega}$ and if for all $y \in 0n$, $P(\mathbf{SEP}(F)^y)$, then $P(F)$.

Then $P(F)$ holds for all F.

Proof. If $\neg P(F)$, then one constructs, using (i)–(iv), a sequence F_n such that $\neg P(F_n)$, and the values $F_n(0n)$ are strictly decreasing.... $\square$

9.4.8. Remark.
The well-foundedness of the predecessor relation can be obtained from a more general result: when F, G are functions from $0n$ to $0n$, let $F <_\infty G$ mean that for some $a \in 0n$, $F(x) < G(a + x)$. Then

9.4.9. Theorem.
$<_\infty$ is a well-founded order relation.

Proof. First observe that $<_\infty$ is an order relation: if $F(x) < G(a + x)$ for all $x \in 0n$ and $G(x) < H(b+x)$ for all $x \in 0n$, then $F(x) < H(b+a+x)$ for all $x \in 0n$. Assume that F_n is a s.d.s. for $<_\infty$; then $F_{n+1}(x) < F_n(a_n + x)$. Let b be a limit ordinal of the form ω^x, and strictly greater than all a_n's; then $F_{n+a}(b) < F_n(a_n + b) = F_n(b)$, hence $F_n(b)$ is a s.d.s. in $0n$.... $\square$

End of the remark.

Observe that, when $F \ll G$, we have $F <_\infty G$ (here F and G are considered as functions from $0n$ to $0n$): if $G = F + F'$, with $F' \neq \underline{0}$, then $F'(x) > 0$ for all $x > n$ for a certain n. Then $F(x) \leq F(n+x) < G(n+x)$ for all $x \in 0n$. If G is of kind $\mathbf{\Omega}$ and $F = \mathbf{SEP}(G)^y$, then $F(x) = \|\overline{G(x,y)}\| \leq G(y+x)$; in fact, if $(z_0, n) \in \mathsf{Tr}(G)$ is an element of the topmost equivalence class modulo $\sim_0^G$ (for the order $\leq^G$ / $\sim_0^G$), then the image of $\overline{G(x,y)}$ under $G(\mathbf{E}_{yy+\omega+1} + \mathbf{E}_1)$ is bounded above by $(z_0\,;\, 0, ..., q-1, \omega, \omega+1, ..., \omega+n-q-1\,;\, y+\omega+1+x)_G$, hence $F(x) < G(y+\omega+1+x)$. The well-foundedness of $<_\infty$ implies therefore the well-foundedness of $\ll$.

9.4.10. Remark.

The task achieved in these sections is the following:

(i) The idea is to express an induction principle on the well-ordered class $F(0n)$. Of course such a principle could be directly formulated by considering denotations $(z_0\,;\, x_0, ..., x_{n-1}\,;\, 0n)_F$. The disadvantage of this formulation is that $(z_0\,;\, x_0, ..., x_{n-1}\,;\, 0n)_F$ denotes a point in $F(0n)$, whereas in practice, we would rather need "a functor". The problem is therefore to "fill" the space $F(0n)$ with dilators, that we shall style as the predecessors of F.

(ii) Some of the predecessors of F are already known, namely the dilators F' such that $F' \sqsubset F$ (9.1.4). Essentially, the relation $\sqsubset$ is sufficient to determine the immediate predecessors of dilators of kinds $\mathbf{0}$, $\mathbf{1}$ or $\boldsymbol{\omega}$.

(iii) The essential difficulty is to find the predecessors of a connected dilator of kind $\mathbf{\Omega}$: the idea is to proceed as follows: if $z < F(0n)$, i.e. $z = (z_0\,;\, x_0, ..., x_{n-1}\,;\, 0n)_F$, let $a = x_{n-1} + 1$; then $F_a = F \circ (\underline{a} + \mathsf{Id})$ has the following property: if $f \in I(0n, 0n)$, then $F_a(f)(z) = z$, hence F_a maps z into z. Then the idea is to take as predecessor of F, corresponding to z, the dilator G defined by: $G(x) = (z_0\,;\, x_0, ..., x_{n-1}\,;\, a+x)_F$, and, when $z < G(x)$, $G(f)(z) = F_a(f)(z) = F(\mathbf{E}_a + f)(z)$. Unfortunately, this definition does not give a linear order, and we are led to (slightly) modigy this picture, and this yields

SEP: when $F(0n)$ is of cofinality $0n$, one constructs a "fundamental sequence" $\check{F}(x)$, as follows: $\check{F}(x)$ is the smallest $z < F(0n)$ such that the values $F(\mathbf{E}_x + f)(z)$ are cofinal in $F(0n)$, when $f \in I(0n, 0n)$. The sequence $\check{F}(x)$ is strictly increasing for $x \geq n$ (for some integer n), and continuous at limit points; if one associates to each $\check{F}(x)$ a dilator G_x by the process explained above (use the fact that $z' < z \rightarrow F(\mathbf{E}_x + f)(z') < z$); then G_x is essentially a restriction (i.e. a predecessor for $\sqsubset$) of $F \circ (\underline{x} + \mathsf{Id})$. The final solution is the dilator $\mathbf{SEP}(F)(\cdot, y)$ which coincides with G_1 between $\check{F}(0)$ and $\check{F}(1), \ldots$, with G_{x+1} between $\check{F}(x)$ and $\check{F}(x+1)$.

$\check{F}(x)$ $\qquad$ $\check{F}(x+1)$ $\qquad$ $\check{F}(y)$

if $z \in [\check{F}(x), \check{F}(x+1)[$, then $\mathbf{SEP}(F)(\cdot, y)$ coincides on z with $F \circ (\underline{y+1} + \mathsf{Id}) \ldots$. (Of course $\check{F}(x) = \mathbf{SEP}(F)(0n, x) \ldots$.)

This construction was expressed in more abstract terms in Theorem 9.3.22; $\check{F}(y) = \overset{*}{y}\ F(0n)$, etc... .

9.4.11. Examples.

(i) If F is a flower then the predecessors of F are constants; since $\mathbf{SEP}(F)(x, y) = F(y)$, $\mathbf{SEP}(F)(f, g) = g$ it is clear that $\mathbf{SEP}(F)(\cdot, y) = \underline{F(y)}$. Hence, if F and G are two distinct non constant flowers, then F and G are not comparable for the relation $\ll$.

(ii) For instance the predecessors of Id are the constants $\bar{x}$, the predecessors of $\mathsf{Id} + \mathsf{Id}$ are the $\underline{x}$'s and the $\mathsf{Id} + \underline{x}$'s, etc... . In some sense Id is what is "after" having exhausted all ordinals (but Id is not the only such point: any non constant flower would do as well). Once again, the superiority of our approach to this question w.r.t. the traditional conception of a proper class mainly rests upon the following facts:

– the “actual” $0n$ is not needed.

– The “finitary” control.

But of course it cannot be expected that all proper classes are of the form $F(0n)$... .

9.5. Generalized products

9.5.1. Definition.
A flower F is **nice** iff for all $x, y, z \in 0n$, for all $f \in I(x, y)$:

$$z < x \to F(z) < F(x) \wedge F(f)\Big(F(z)\Big) = F\Big(f(z)\Big) \ .$$

9.5.2. Remark.
If F is nice, then $T(x)(z) = F(z)$ defines a natural transformation from Id to F, denoted by ξ_F.

9.5.3. Proposition.
F is nice iff F enjoys 9.5.1 for all $x, y, z < \omega$.

Proof. Left to the reader. □

9.5.4. Remark.
One would easily show the equivalence:

$$F \text{ nice} \leftrightarrow \frac{dF(y)}{dy} = \underline{1} + G \quad \text{for some } G \ .$$

(The 1 in $\underline{1} + G$ corresponds to the points $F(x)$.... .)

9.5.5. Definition.
The following data define a category $\mathbf{FL}_n$:

objects: nice flowers.
morhpisms from F to G: the set $I_n^1(F, G)$ of all **nice** morphisms from F to G, i.e. all $T \in I^1(F, G)$ such that

$$T(x)\Big(F(z)\Big) = G(z) \quad \text{for all } z < x \in 0n \ .$$

9.5.6. Theorem.
Assume that (F_i, T_{ij}) is a direct system in $\mathbf{FL}_n$, with the following property: if $i \preceq j$, then $F_j = F_i \circ F_{ij}$ for some F_{ij} in $\mathbf{FL}_n$ and $T_{ij} = \mathbf{E}^1_{F_i} \circ \xi_{F_{ij}}$; $\circ$ is the functor **composition**; $(F \circ G)(x) = F\Big(G(x)\Big)$, $(F \circ G)(f) = F\Big(G(f)\Big)$, $(T \circ U)(x) = T\Big(U(x)\Big)$.

Then:

(i) (F_i, T_{ij}) has a direct limit in $\mathbf{FL}_n$, (F, T_i).

(ii) if $C_i = \{F_i(x)\,;\, x \in 0n\}$, if $C = \{F(x)\,;\, x \in 0n\}$ then $C = \bigcap\limits_{i \in I} C_i$.

<u>Proof</u>. (i) can be replaced by (i)$'$ and (i)$''$:

(i)$'$ $\quad \big(F_i(x), T_{ij}(x)\big)$ has a direct limit in $\mathbf{ON}$ for all $x \in 0n$.

(i)$''$ $\quad$ if $z < x \in 0n$, then $T_i(x)\big(F_i(z)\big) = F(z)$.

We define by induction strictly increasing functions f_i from $0n$ to $0n$, as follows:

$$f_i(0) = \sup_{j \succeq i} \ F_{ij}(0)$$

$$f_i(x+1) = \sup_{j \succeq i} \ F_{ij}(f_j(x)+1)$$

$$f_i(x) = \sup_{y < x} \ f_i(y) \quad \text{ if } y \text{ is limit }.$$

We prove by induction on x that $i \preceq j \to F_{ij}\big(f_j(x)\big) = f_i(x)$:

+ $f_i(0) = \sup\limits_{k \succeq j} \Big(F_{ij}\big(F_{jk}(0)\big)\Big)$; but F_{ij} is *topologically* continuous, (9.2.3), hence $f_i(0) = F_{ij}\Big(\sup\limits_{k \succeq j} \big(F_{jk}(0)\big)\Big) = F_{ij}\big(f_j(0)\big)$.

+ The cases x successor and x limit are similar... .

We now define functions g_{ix} from $F_i(x)$ to $0n$ by: $g_{ix}\Big((a\,;\, x_0, ..., x_{n-1})_{F_i}\Big) = \Big(a\,;\, f_i(x_0), ..., f_i(x_{n-1})\Big)_{F_i}$. g_{ix} is strictly increasing; moreover assume that $z_i = (a\,;\, x_0, ..., x_{n-1})_{F_i}$ and let $z_j = T_{ij}(x)(z_i)$; hence $z_j = (b\,;\, x_0, ..., x_{n-1})_{F_j}$ for some b and also $z_j = \Big(a\,;\, F_{ij}(x_0), ..., F_{ij}(x_{n-1})\Big)_{F_i}$ (since by hypothesis $T_{ij} = \mathbf{E}^1_{F_i} \circ \xi_{F_{ij}}$); then we get:

$$\begin{aligned} g_{jx}(z_j) &= \Big(b\,;\, f_j(x_0), ..., f_j(x_{n-1})\Big)_{F_j} = \\ &= \Big(a\,;\, F_{ij}\big(f_j(x_0)\big), ..., F_{ij}\big(f_j(x_{n-1})\big)\Big)_{F_i} = \\ &= \Big(a\,;\, f_i(x_0), ..., f_i(x_{n-1})\Big)_{F} = g_{ix}(z_i)\ . \end{aligned}$$

We have shown that $g_{ix} = g_{jx}T_{ij}(x)$, and from that it follows that $\big(F_i(x), T_{ij}(x)\big)$ has a direct limit in **ON** (8.1.21). This establishes (i)$'$.

(i)$''$ is an immediate consequence of the following lemma:

9.5.7. <u>Lemma</u>.
If (F_i, T_{ij}) is a direct system in $\mathbf{FL}_n$ such that, considered as a system in **DIL**, we have: $(F, T_i) = \varinjlim (F_i, T_{ij})$, then (F, T_i) is the direct limit of (F_i, T_{ij}) in $\mathbf{FL}_n$ (i.e. $\mathbf{FL}_n$ is "closed" in **DIL**).

<u>Proof</u>. The first thing (left to the reader) is to verify that F is a flower and that $x < y \to F(x) < F(y)$! Then, given $z < x \in 0n$, choose i such that $F(z) \in rg\big(T_i(x)\big)$ hence $F(z) = T_i(x)\big(a_i(z)\big)$; $F(z)$ is the smallest object of the form $(a\,; x_0, ..., x_{m-1})_F$ with $m \neq 0$ and $x_{m-1} \geq z$, and necessarily $a_i(z)$ is the smallest object $(b\,; x_0, ..., x_{m-1})_{F_i}$, with $m \neq 0$ and $x_{m-1} \geq z$, i.e. $a_i(x, z) = F_i(x, z)$ (see 9.2.4 (iv)). This shows that $T_i(x)\big(F_i(z)\big) = F(z)$. Hence $F(f)\big(F(z)\big) = F(f)T_i(x)\big(F_i(z)\big) = T_i(y)F_i(f)\big(F_i(z)\big) = T_i(y)\Big(F_i\big(f(z)\big)\Big) = F\big(f(z)\big)$: F and the T_i's are *nice*. $\square$

Let $C(z)$ be the z^{th} element of $\bigcap_i C_i$; we establish (ii) by providing that $C(z) \leq F(z)$ and $F(z) \leq C(z)$:

+ $F(z) = \varinjlim{}^{*} \big(F_i(z), T_{ij}(x)\big) = \varinjlim_{K}{}^{*} \bigg(F_k\Big(F_{ki}(z)\Big), F_k\Big(F_{ki}\big(\xi_{F_{ij}}(z)\big)\Big)\bigg) =$ $F_k\bigg(\varinjlim_{K}{}^{*} \Big(F_{ki}(z), F_{ki}\big(\xi_{F_{ij}}(z)\big)\Big)\bigg)$ with $K = \{i\,; i \in I \wedge i \succeq k\}$. We have established that $F(z) \in rg(F_k)$; hence $F(z) \in \bigcap_i C_i$, and so: $F(z) \geq C(z)$.

+ Conversely, we show that $C(z) = F_i\big(f_i(z)\big)$ for all $i \in I$, by induction on z: first remark that:

$$i \preceq j \to F_i\big(f_i(z)\big) = F_i\Big(F_{ij}\big(f_j(z)\big)\Big) = F_j\big(f_j(z)\big)\ ;$$

this establishes that the value $F_i\big(f_i(z)\big)$ is independent of i; hence it will be sufficient to show that $C(z) = F_i\Big(f_i(z)\Big)$ for some $i \in I$! The cases $z = 0$ and z limit are left to the reader, and we only prove the property for $z+1$: if for all $i \in I$, $C(x,z) = F_i\Big(f_i(z)\Big)$, observe that $i \preceq j \to f_i(z) \geq f_j(z)$ (because $f_i(z) = F_{ij}\Big(f_j(z)\Big)$ and $z \rightsquigarrow F_{ij}(z)$ is strictly increasing) hence, for some $k \in I$, $f_i(z)$ takes a constant value t for all $i \succeq k$ and we have, if $k \preceq i \preceq j$:

$$t = f_i(z) = F_{ij}\Big(f_j(z)\Big) = F_{ij}(t)\ .$$

Let $C_k^* = \bigcap_{j \succeq k} rg\Big(F_{kj}(\cdot)\Big)$; obviously

$$a \in C \leftrightarrow \exists u\ \Big(u \in C_k^* \wedge a = F_k(u)\Big)\ .$$

From this it follows that $C(z+1) = F_k(u)$, where u is the smallest element of C_k^* strictly greater than t; the expression of u is easy to obtain:

$$u = \sup_{j \succeq k} F_{kj}(t+1) = \sup_{j \succeq k} F_{kj}(f_j(z)+1) = f_k(z+1)\ .$$

Hence $C(z+1) = F_k\Big(f_k(z+1)\Big)$.

Now observe that $T_i(x)\Big(F_i(z)\Big) \leq g_{ix}\Big(f_i(z)\Big)$; but the definition of g_{ix} yields $g_{ix}\Big(F_i(z)\Big) = F_i\Big(f_i(z)\Big)$, and this gives us: $F(z) = T_i(x)\Big(F_i(z)\Big) \leq g_{ix}\Big(F_i(z)\Big) = F_i\Big(f_i(z)\Big) = C(z)$. $\square$

9.5.8. Remark.
A traditional ordinal technique is the use of so-called **normal functions**: a normal function is a function from $0n$ to $0n$ which is strictly increasing and topologically continuous. If F is a nice flower, then the function $x \rightsquigarrow F(x)$ is normal, but the converse is false, i.e. a normal function is not necessarily induced by a nice flower (for instance $x \rightsquigarrow \aleph_x$). One traditionally considers the following operations on *normal* functions:

(i) composition: if F and G are normal, so is $F \circ G$.

(ii) intersection: if $(F_i)$is a family of normal functions such that: $i \leq j \rightarrow rg(F_i) \subset rg(F_i)$, then the set $\bigcap\limits_i rg(F_i)$ is the range of a normal function.ss

(In the literature (i) is often replaced by (i)$'$: (i)$'$ fixed point: if F is normal, so is the function F' whose range is the class $\{z\,;\, F(z) = z\}$.

In fact (i)$'$ can be obtained from (i) and (ii): let $F^n = F \circ ... \circ F$ (n times); then $rg(F^m) \subset rg(F^n)$ when $n \leq m$, and if $rg(F') = \bigcap\limits_n rg(F^n)$, it is plain that $rg(F') = \{z\,;\, z = F(z)\}$; so the composition is more general than the fixed point... .)

Constructions (i) and (ii) can be carried out in the more delicate context of nice flowers: (i) is just composition of functors, whereas (ii) requires, in order that one can apply 9.5.6, that $F_j = F_i \circ F_{ij}$ for some F_{ij}, when $i \preceq j$. A practical consequence is that all constructions involving normal functions can be adapted, mutatis mutandis, to nice flowers; of course such an adaptation has the advantage that we have a "finitary" control on the construction, whereas this is not the case for normal function... .

9.5.9. Definition.

Assume that $(F_t)_{t<x}$ is a family of nice flowers; then one defines a new nice flower $\prod\limits_{t<x} F_t$, the generalized **product** of the family (F_t); if $(G_u)_{u<y}$ is another such family, if $f \in I(x, y)$ and $T_t \in I^1_n(F_t, G_{f(t)})$ for all $t < x$, then one defines $\prod\limits_{t<f} T_t \in I^1_n\Big(\prod\limits_{t<x} F_t, \prod\limits_{u<y} G_u\Big)$ as follows: (the definition is by induction on y)

(i) If $x = y = 0$, then
$$\prod_{t<x} F_t = \prod_{u<y} G_u = \mathsf{Id},\ \prod_{t<f} T_t = \mathbf{E}^1_{\mathsf{Id}}$$

(ii)
$$\prod_{u<y+1} G_u = \Big(\prod_{u<y} G_u\Big) \circ G_y$$
$$\prod_{t<f+\mathbf{E}_1} T_t = \Big(\prod_{t<f} T_t\Big) \circ T_x$$
$$\prod_{t<f+\mathbf{E}_{01}} T_t = \Big(\prod_{t<f} T_t\Big) \circ \xi_{G_y}$$

(iii) $$\prod_{u<y} G_u = \varinjlim_{y'<y}{}^{*} \Big(\prod_{u<y'} G_u, \prod_{u<\mathbf{E}_{y'y''}} \mathbf{E}^1_{G_u}\Big)$$

$$\prod_{t<\bigcup_i f_i} T_t = \varinjlim \Big(\prod_{t<f_i} T_t\Big)$$

when y is a limit ordinal.

9.5.10. Example.
Assume that $x = 3$, $y = 7$, and that $f(0) = 2$, $f(1) = 3$, $f(2) = 5$; then

$$\begin{array}{lcccccccccccccc}
\prod_{t<3} F_t & = & \mathrm{Id} & \circ & \mathrm{Id} & \circ & F_0 & \circ & F_1 & \circ & \mathrm{Id} & \circ & F_2 & \circ & \mathrm{Id} \\
 & & | & & | & & | & & | & & | & & | & & | \\
\prod_{t<f} T_t & = & \xi_{G_0} & \circ & \xi_{G_1} & \circ & T_0 & \circ & T_1 & \circ & \xi_{G_4} & \circ & T_2 & \circ & \xi_{G_6} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\prod_{u<7} G_u & = & G_0 & \circ & G_1 & \circ & G_2 & \circ & G_3 & \circ & G_4 & \circ & G_5 & \circ & G_6
\end{array}$$

9.5.11. Theorem.
Definition 9.5.9 is sound.

Proof. We show by induction on y that $\prod_{u<y} G_u$ and $\prod_{t<f} T_t$ exist, and that the following *associativity* property holds: if $y = y' + y''$, $f = f' + f''$, then (if $f' \in I(x', y')$)

$$\prod_{u<y} G_u = \prod_{u<y'} G_u \circ \prod_{u<y''} G_{y'+u}$$

and

$$\prod_{t<f} T_t = \prod_{t<f'} T_t \circ \prod_{u<f''} T_{x'+t}$$

(i) If $y = 0$, everything is trivial.

(ii) If the properties hold for y, they hold for $y + 1$.

(iii) If y is limit, then the system $\Big(\prod_{u<y'} G_u, \prod_{u<\mathbf{E}_{y'y''}} \mathbf{E}_{G_u}\Big)_{y'<y''<y}$ enjoys the hypotheses of 9.5.6: this is an immediate consequence of the

associativity property applied to $y'' < y$. From this, the existence of the $\prod\limits_{u<y} G_u$ and $\prod\limits_{t<f} T_t$ is ensured. It remains to prove associativity; assume that $y = y' + y''$ and one can suppose that $y'' \neq 0$; one can write

$$\prod_{u<y} G_u = \underset{\substack{\longrightarrow \\ z<y''}}{\lim}{}^{*}\Big(\prod_{u<y'+z} G_u, \prod_{u<\mathbf{E}_{y'}+\mathbf{E}_{zz'}} \mathbf{E}^1_{G_u}\Big) =$$

$$\lim_{z<y}{}^{*}\Big(\prod_{u<y'} G_u \circ \prod_{u<z} G_{y'+u}, \prod_{u<\mathbf{E}_{y'}} \mathbf{E}^1_{G_u} \circ \prod_{u<\mathbf{E}_{zz'}} \mathbf{E}^1_{G_{y'+u}}\Big) =$$

(this equality uses the fact that $\circ$ preserves direct limits, see 9.5.12)

$$\prod_{u<y'} G_u \circ \underset{\substack{\longrightarrow \\ z<y''}}{\lim}{}^{*}\Big(\prod_{u<z} G_{y'+u}, \prod_{u<\mathbf{E}_{zz'}} \mathbf{E}^1_{G_{y'+u}}\Big) =$$

$$= \prod_{u<y'} G_u \circ \prod_{u<y''} G_{y'+u} \ .$$

Associativity for natural transformations is proved in the same way. □

9.5.12. <u>Theorem</u>.,
$\prod$ is a functor from the category of sequences $\mathbf{FL}_n^{\mathbf{ON}}$ to $\mathbf{FL}_n$; $\prod$ preserves direct limits and pull-backs.

<u>Proof</u>. The verification is tedious and straightforward. In particular, this means that the functor $\circ$ from $\mathbf{FL}_n^2$ to $\mathbf{FL}_n$ preserves direct limits and pull-backs. This is a particular case of

9.5.13. <u>Proposition</u>.
The functor $\circ$ from $\mathbf{DIL}^2$ to $\mathbf{DIL}$ preserves direct limits and pull-backs.

<u>Proof</u>. Preservation of direct limits is proved as follows: given $(a, n) \in \mathsf{Tr}(F \circ G)$, we show the existence of T, F', U, G', such that $T \in \mathbf{I}^1(F', F)$, $U \in \mathbf{I}^1(G', G)$, with F' and G' finite dimensional, and $(a, n) \in rg\Big(\mathsf{Tr}(T \circ U)\Big)$; $a = (a\,;\, 0, ..., n-1\,;\, n)_{F\circ G} = \Big(b\,;\, x_0, ..., x_{m-1}\,;\, G(n)\Big)_F$; assume that

$x_i = (c_i\,;\, p_i^0, ..., p_i^{k_i-1}\,;\, n)_G$; if one defines T by $rg\Big(\mathsf{Tr}(T)\Big) = \{(b\,;\, m)\}$ and U by $rg\Big(\mathsf{Tr}(U)\Big) = \{(c_i\,;\, k_i)\,;\, i < m\}$, then $(a,n) \in rg\Big(\mathsf{Tr}(T \circ U)\Big)...$. We establish preservation of pull-backs: with the notations just introduced, one sees that $(a,n) \in rg\Big(\mathsf{Tr}(T_i \circ U_i)\Big)$ iff $(b,m) \in rg\Big(\mathsf{Tr}(T_i)\Big)$ and $(c_j,k_j) \in rg\Big(\mathsf{Tr}(U_i)\Big)$ for all $j < m$; from that $rg\Big(\mathsf{Tr}(T_3 \circ U_3)\Big) = rg\Big(\mathsf{Tr}(T_1 \circ U_1)\Big) \cap rg\Big(\mathsf{Tr}(T_2 \circ U_2)\Big)$ when $T_3 = T_1 \wedge T_2$ and $U_3 = U_1 \wedge U_2$.□

We sketch a proof of 9.5.12: preservation of direct limits essentially means that, given $(a,n) \in \mathsf{Tr}\Big(\prod_{t<x} F_t\Big)$, it is possible to find a family $(G_i)_{i<n}$ of finite dimensional nice flowers, a function $f \in I(n,x)$, and $T_i \in I_n^1(G_i, F_{f(i)})$ such that $(a,n) \in rg\Big(\mathsf{Tr}\Big(\prod_{i<f} T_i\Big)\Big)$; we argue by induction on x:

+ The case $x = 0$ is perfectly trivial.

+ Assume that the property holds for x, and let (a,n) be a point in $\mathsf{Tr}\Big(\prod_{t<x+1} F_t\Big)$; then one can associate to (a,n) a point (b,m) in $\mathsf{Tr}\Big(\prod_{t<x} F_t\Big)$ and apsoints (c_i,k_i) in $\mathsf{Tr}(F_x)$ $(i = 0, ..., m-1)$, by the construction of 9.5.13. We apply the induction hypothesis: $(b,m) \in rg\Big(\mathsf{Tr}\Big(\prod_{i<f} T_i\Big)\Big)$ for some $f \in I(p,x)$, with $rg\Big(\mathsf{Tr}(T_i)\Big)$ finite for all $i < p$; define T_p and G_p by $T_p \in I^1(G_p, F_x)$ and $rg\Big(\mathsf{Tr}(T_p)\Big) = \{(c_0,k_0), ..., (c_{m-1},k_{m-1})\}$. Then $(a,n) \in rg\Big(\mathsf{Tr}\Big(\prod_{i<f+\mathbf{E}_1} T_i\Big)\Big)$.

+ If x is limit, $\Big(\prod_{t<x} F_t, \prod_{t<E_{x'x}} \mathbf{E}^1_{F_t}\Big) = \varinjlim_{x'<x}{}^{*} \Big(\prod_{t<x'} F_t, \prod_{t<\mathbf{E}_{x'x''}} \mathbf{E}^1_{F_t}\Big)$ and this implies that $(a,n) = \mathsf{Tr}\Big(\prod_{t<\mathbf{E}_{x'x}} \mathbf{E}^1_{F_t}\Big)\Big((b\,;\, m)\Big)$ for some $x' < x$ and $(b\,;\, m) \in \mathsf{Tr}\Big(\prod_{t<x'} F_t\Big)$. Assume now that (induction hypothesis) the function $f \in I(n,x')$, the finite dimensional G_i's and $T_i \in I^1(G_i, F_{f(i)})$ are such that $(b,m) \in rg\Big(\mathsf{Tr}\Big(\prod_{i<f} T_i\Big)\Big)$; if $g = \mathbf{E}_{x'x} f$, then clearly $(a,n) \in rg\Big(\mathsf{Tr}\Big(\prod_{i<g} T_i\Big)\Big)$.

In fact the function f, the family T_i which have been constructed above have the following property (analogue of the normal form theorem): if $(a,n) \in rg\Big(\prod_{t<g} U_t\Big)$, then:

+ $rg(f) \subset rg(g)$.
+ If $f(t) = g(t')$, then $rg\Big(\mathsf{Tr}(T_t)\Big) \subset rg\Big(\mathsf{Tr}(U_{t'})\Big)$ and from this, preservation of pull-backs easily follows. □

9.5.14. Example.
The traditional example from the theory of normal functions is the **Veblen hierarchy**:

$$V_0 = \mathsf{Id} \qquad \text{(the identity function from } 0n \text{ to } 0n\text{)}$$
$$V_{\alpha+1} = V_\alpha \circ \vartheta \qquad \text{(where } \vartheta \text{ is a fixed normal function)}$$

V_λ, for λ limit, enumerates the intersection of the ranges of the function $V_{\lambda'}$, for $\lambda' < \lambda$.

This hierarchy was introduced in [84]; the traditional presentation is $V'_0 = \vartheta$, $V'_{\alpha+1}$ enumerates the fixed points of V'_α, and for λ limit V'_λ enumerates the intersection of the range of the $V'_{\lambda'}$, $\lambda' < \lambda$; V and V' are connected as follows: $V'_\alpha = V_{\omega^\alpha}$; if $\alpha = \omega^{\beta_1} + ... + \omega^{\beta_n}$ with $\beta_1 \geq ... \geq \beta_n$, then $V_\alpha = V'_{\beta_1} \circ ... \circ V'_{\beta_n}$; hence the two definitions are trivial variants one of another. However, the formal properties of V_α are more satisfactory, and it is the reason why we modify the traditional definition on a minor point.)

In practice, ϑ will be (defined by) a nice flower; hence it makes sense to write

$$V_\alpha = \prod_{t<a} \vartheta \qquad (\text{hence } V'_\alpha = \prod_{t<\omega^\alpha} \vartheta)\ .$$

This means that V_α is the product of "α copies" of ϑ. But also, by 9.5.12. $\alpha \rightsquigarrow V_\alpha$, $f \rightsquigarrow \prod_{t<f} \mathbf{E}^1_\vartheta$, defines a functor from **ON** to $\mathbf{FL}_n$, preserving direct limits and pull-backs, and of course such a functor can be identified with a functor V from $\mathbf{ON}^2$ to **ON**:

$$V(\alpha, \beta) = V_\alpha(\beta)\ ; \qquad V(f, g) = V_f(g) = \prod_{t<f} \mathbf{E}^1_\vartheta\ .$$

The functor V is a bilator, and is such that $V(f,g)\Big(V(x,z)\Big) = V\Big(x', g(z)\Big)$ when $f \in I(x,x')$, $g \in I(y,y')$ and $z < y$.

More details on this construction can be found in [85]. Observe that the Veblen hierarchy appears, in this framework, as a finitistic construction. In fact, we shall extend this construction, and the result will be the functor $\mathbf{\Lambda}$.

9.5.15. Remark.
Our task is now to transfer all our concepts to bilators, using the fact that a bilator can be viewed as a functor from **ON** to **FL**:

(i) The category $\mathbf{BIL}_n$ is defined by:
objects: **nice** bilators, i.e. bilators such that $y < y' \to F(x,y) < F(x,y')$
and $F(f,g)\Big(F(x,z)\Big) = F\Big(x', g(z)\Big)$, when $f \in I(x,x')$, $g \in I(y,y')$, $z < y$;
morphisms from F to G: the set $I^b_n(F,G)$ of all $T \in I^b(F,G)$ such that $T(x,y)\Big(F(x,z)\Big) = G(x,z)$ for all x, y, z with $z < y$.

(ii) The **semi-product** is a functor from $\mathbf{BIL}^2_n$ to $\mathbf{BIL}_n$

$$(F \circ_s G)(x,y) = F\Big(x, G(x,y)\Big)$$
$$(F \circ_s G)(f,g) = F\Big(f, G(f,g)\Big)$$
$$(T \circ_s U)(x,y) = T\Big(x, U(x,y)\Big) .$$

One easily checks that $\circ_s$ preserves direct limits and pull-backs (remark that, in $\mathbf{BIL}_n$, pull-backs always exist).

(iii) Generalized semi-products of nice bilators are defined in a way akin to 9.5.9; we use $\mathbf{\Pi}$ instead of $\prod$ to indicate that we are working with semi-products.

All results concerning nice flowers, products and generalized products, can be adapted, mutatis mutandis, to nice bilators, semi-products, and generalized semi-products; the idea is to use the identification between $\mathbf{BIL}_n$ and the category of functors from **ON** to $\mathbf{BIL}_n$ preserving direct limits and pull-backs. The reader will find in [5], Section 5.3, a detailed definition of these concepts for bilators.

9.6. <u>The functor $\mathbf{\Lambda}$</u>

$\mathbf{\Lambda}$ is a kind of exponential, which transforms sums into (semi-) products; $\mathbf{\Lambda}$ is a functor from **DIL** to $\mathbf{BIL}_n$, preserving direct limits and pull-backs. In fact, since $\mathbf{BIL}_n$ can be identified with a subcategory of **DIL** (by means of the unifying functor **UN**), then $\mathbf{\Lambda}$ can also be viewed as a functor from **DIL** to itself; then $\mathbf{\Lambda}$ is a *ptyx* of type $(\mathbf{O} \to \mathbf{O}) \to (\mathbf{O} \to \mathbf{O})$ (see Chapter 12). When we say that $\mathbf{\Lambda}$ is a kind of exponential, this suggests that $\mathbf{\Lambda}$ is akin to the familiar ordinal exponential (e.g. the dilator 2^{Id}): in fact we shall see that the formula which expresses that $\mathbf{\Lambda}$ is a ptyx of the appropriate type, i.e. that $\mathbf{\Lambda}$ maps dilators on dilators, is formally equivalent to the Π^1_1-comprehension axion; (see Sec. 1.1.6) and recall that the fact that 2^{Id} is a dilator (i.e. that 2^X is a well-order when X is a well-order) is formally equivalent to $(\Sigma^0_1 - CA)^*$ (see Sec. 5.4). Hence, the analogy with oridinal exponentiation will be enhanced by this result.

Technically speaking, the main feature in the definition of $\mathbf{\Lambda}$ is the use of induction on dilators; to be more precise, induction on dilators is not necessary in the *definition* of $\mathbf{\Lambda}$, but only in the proof of the *soundness* of the definition, namely that $\mathbf{\Lambda}$ maps dilators on nice bilators ... this is a very familiar situation.

In the sequel, we shall encounter many $\mathbf{\Lambda}$-like objects, for instance:

(i) Variants of $\mathbf{\Lambda}$ based upon iteration (9.B.3).

(ii) Variants of $\mathbf{\Lambda}$ based upon neighbouring concepts such as rungs and ladders (annex 9.A).

(iii) The functor that performs cut-elimination in Chapter 11, which is also very close to $\mathbf{\Lambda}$.

In fact $\mathbf{\Lambda}$ is a "civilized" version of *Bar-recursion of type 2*, just as induction on dilators is a "civilized" version of *Bar-induction of type 2*.

9.6.1. <u>Definition</u>.
One defines a functor $\mathbf{\Lambda}$ from **DIL** to $\mathbf{BIL}_n$, as follows:

(i) If $G = \sum\limits_{t<y} G_t$, with G_t connected for all t, then

$$\mathbf{\Lambda} G = \prod_{t<y} \mathbf{\Lambda} G_t \ .$$

If $T \in I^1(F,G)$ and $F = \sum_{t<x} F_t$, with F_t connected for all t, if $f \in I(x,y)$ and the family $T_t \in I^1(F_t, G_{f(t)})$ is such that $T = \sum_{t<f} T_t$, then

$$\mathbf{\Lambda} T = \prod_{t<f} \mathbf{\Lambda} T_t \ .$$

(ii) $\mathbf{\Lambda}\underline{1} = \mathsf{Id} + \mathsf{Id}$ (hence $\mathbf{\Lambda}\mathbf{E}_{\underline{1}} = \mathbf{E}^b_{\mathsf{Id}+\mathsf{Id}}$); strictly speaking, $\mathsf{Id} + \mathsf{Id}$ is a dilator; what we denote by $\mathsf{Id} + \mathsf{Id}$ is the bilator $\mathbf{SEP}(\mathsf{Id} + \mathsf{Id})$, i.e. the bilator *sum*; this kind of abuse of notations will be frequent... .

(iii) If G is connected and $\neq \underline{1}$, if $f \in I(x,x')$, $g \in I(y,y')$, then write

$$\mathbf{SEP}(G)(\cdot,y) = \sum_{t<y} {}_tG \qquad \mathbf{SEP}(G)(\cdot,g) = \sum_{t<g} {}_{gt}G$$

with ${}_{gt}G = I({}_tG, {}_{g(t)}G)$: with the notations of 9.3.22, ${}_tG$ is $\partial\mathbf{SEP}(G)(\cdot,t)$ and ${}_{gt}G$ is $\partial\mathbf{SEP}(G)(\cdot,g_t)$; then

$$(\mathbf{\Lambda} G)(x,y) = \Big(\prod_{t<y} \big(\underline{1} + \mathbf{\Lambda}_t G)\Big)(x,0)$$

$$(\mathbf{\Lambda} G)(f,g) = \Big(\prod_{t<g} \Big(\mathbf{E}^b_{\underline{1}} + \mathbf{\Lambda}_{gt} G\Big)\Big)(f,\mathbf{E}_0) \ .$$

If F is connected and $T \in I^1(F,G)$, then write $\mathbf{SEP}(T)(\cdot,y) = \sum_{t<\mathbf{E}_y} {}_tT$ (hence ${}_tT = \partial\mathbf{SEP}(T)(\cdot,t)$); then

$$(\mathbf{\Lambda} T)(x,y) = \Big(\prod_{t<\mathbf{E}_y} \Big(\mathbf{E}^b_{\underline{1}} + \mathbf{\Lambda}_t T\Big)\Big)(x,0) \ .$$

9.6.2. Theorem.
Definition 9.6.1 is sound; more precisely there exists one and only one functor $\mathbf{\Lambda}$ from **DIL** to $\mathbf{BIL}_n$ which enjoys 9.6.1 (i)–(iii): furthermore, this functor has the following features (equivalent to 9.6.1 (i)–(iii)):

(i) $\mathbf{\Lambda}$ is a functor from **DIL** to **BIL**: more precisely if G is a dilator, $\mathbf{\Lambda}G$ is a bilator; if $T \in I^1(F,G)$, $\mathbf{\Lambda}T \in I^b(\mathbf{\Lambda}F, \mathbf{\Lambda}G)$; $\mathbf{\Lambda}^1_G = \mathbf{E}^b_{\mathbf{\Lambda}G}$; $\mathbf{\Lambda}(TU) = (\mathbf{\Lambda}T)(\mathbf{\Lambda}U)$.

(ii) $\mathbf{\Lambda}\underline{0} = \mathsf{Id}$ $\qquad \mathbf{\Lambda}\mathbf{E}^1_{\underline{0}} = \mathbf{E}^b_{\mathsf{Id}}$.

(iii) $\mathbf{\Lambda}\underline{1} = \mathsf{Id} + \mathsf{Id}$ $\qquad \mathbf{\Lambda}\mathbf{E}^1_{\underline{1}} = \mathbf{E}^b_{\mathsf{Id}+\mathsf{Id}}$.

(iv) If $T \in \mathbf{\Omega}I^1(F,G)$, with F and G connected, then

$$(\mathbf{\Lambda}G)(x,y) = \Big(\underset{t<y}{\mathbf{\Pi}}\ (\underline{1} + \mathbf{\Lambda}_t G)\Big)(x,0)$$

$$(\mathbf{\Lambda}G)(f,g) = \Big(\underset{t<g}{\mathbf{\Pi}}\ (\mathbf{E}^b_{\underline{1}} + \mathbf{\Lambda}_{gt} G)\Big)(f,\mathbf{E}_0)$$

$$(\mathbf{\Lambda}T)(x,y) = \Big(\underset{t<\mathbf{E}_y}{\mathbf{\Pi}}\ (\mathbf{E}^b_{\underline{1}} + \mathbf{\Lambda}_t T)\Big)(x,0)\,.$$

(v) $\mathbf{\Lambda}(F' + F'') = (\mathbf{\Lambda}F') \circ_s (\mathbf{\Lambda}F'')$
$\mathbf{\Lambda}(T' + T'') = (\mathbf{\Lambda}T') \circ_s (\mathbf{\Lambda}T'')$.

(vi) $(\mathbf{\Lambda}\mathbf{E}^1_{\underline{0}F})(x,y)(z) = (\mathbf{\Lambda}F)(x,z)$.

(vii) $\mathbf{\Lambda}$ preserves direct limits.

(viii) $\mathbf{\Lambda}$ preserves pull-backs.

Proof. Assume that we have found a solution $\mathbf{\Lambda}$ of (i)–(viii); then this $\mathbf{\Lambda}$ is a solution of 9.6.1:
9.6.1 (i): by induction on y (in fact, using the fact that Σ preserves direct limits, one can prove the result without using induction on y: the idea is to prove the result for y finite, and then to extend it by direct limits...); if $y = 0$ apply 9.6.2 (ii); if $y = y' + 1$, then $G = G' + G_{y'}$, with $G' = \sum_{t<y'} G_t$; the induction hypothesis yields $\mathbf{\Lambda}G' = \underset{t<y'}{\mathbf{\Pi}}\ \mathbf{\Lambda}G_t$, hence, by 9.6.2 (v) we obtain:

$$\mathbf{\Lambda}G = \Big(\underset{t<y'}{\mathbf{\Pi}}\ \mathbf{\Lambda}G_t\Big) \circ_s \mathbf{\Lambda}G_{y'} = \underset{t<y}{\mathbf{\Pi}}\ \mathbf{\Lambda}G_t$$

the case of morphisms is similar (one uses 9.6.2 (vi) which can be read as: $\mathbf{\Lambda E}^1_{\underline{0}F} = \xi^b_{\mathbf{\Lambda}F}$); if y is limit, then using 9.6.2 (vii), one gets

$$\mathbf{\Lambda} G = \underset{\substack{\longrightarrow \\ y'<y}}{\lim}{}^* \left(\mathbf{\Lambda} \sum_{t<y'} G_t, \mathbf{\Lambda E}^1_{\sum_{t<y'} G_t \sum_{t<y''} G_t}\right) ;$$

using the induction hypothesis, the limit is equal to

$$\underset{\substack{\longrightarrow \\ y'<y}}{\lim}{}^* \left(\underset{t<y'}{\mathbf{\Pi}} \mathbf{\Lambda} G_t, \underset{t<\mathbf{E}_{y'y''}}{\mathbf{\Pi}} \mathbf{E}_{\mathbf{\Lambda} G_t}\right) = \underset{t<y}{\mathbf{\Pi}} \mathbf{\Lambda} G_t \ .$$

The case of morphisms is similar.
9.6.1 (ii) is exactly 9.6.2 (iii).
9.6.1 (iii) is exactly 9.6.2 (iv).

Finally, observe that 9.6.2 (vi) ensures that $\mathbf{\Lambda}$ is a functor from **DIL** into $\mathbf{BIL}_n$ (and not only **BIL**): $\mathbf{\Lambda E}^1_{\underline{0}F} = \xi^b_{\mathbf{\Lambda}F}$, hence $\mathbf{\Lambda}_F$ is nice; if $T \in I^1(F,G)$, then $T\mathbf{E}^1_{\underline{0}G}$, and this means that (9.6.2 (i)) $\mathbf{\Lambda} T \xi^b_{\mathbf{\Lambda}F} = \xi^b_{\mathbf{\Lambda}G}$: so $\mathbf{\Lambda} T$ is nice.

On the other hand, observe that, if there is a functor satisfying 9.6.1 (i)–(iii), then it is clearly unique. Hence, it remains to construct a functor $\mathbf{\Lambda}$ enjoying (i)–(viii); we shall proceed as follows: given a dilator H, define a subcategory $\mathbf{DIL} \ll H$ as follows: F is an object of $\mathbf{DIL} \ll H$ iff $\exists H'(H' \ll H$ and $I^1(F,H') \neq \emptyset)$, the morphisms in $\mathbf{DIL} \ll H$ being given by $I^1(F,G)$ as in **DIL** (i.e. $\mathbf{DIL} \ll H$ is a full subcategory of **DIL**). We also introduce $\mathbf{DIL} \underline{\ll} H = \mathbf{DIL} \ll H + \underline{1}$.

We show, *by induction on* H, the existence of a unique functor $\mathbf{\Lambda}^H$, (abbreviated as $\mathbf{\Lambda}$) from $\mathbf{DIL} \underline{\ll} H$ to **BIL** enjoying the relativization of (i)–(viii) to $\mathbf{DIL} \underline{\ll} H$. When F is in $\mathbf{DIL} \underline{\ll} H$, let us denote by $h(F)$ the smallest H' (modulo $\ll$) such that $I^1(F,H') \neq \emptyset$, H' varying through the class of all predecessors of $H + \underline{1}$ (do not forget that this class is a well-roder!). Then, when $T \in I^1(F,G)$, we shall use the notation $h(T)$ for $h(G)$. The proof splits into five cases:

9.6.3. H <u>is of kind</u> $\underline{0}$.
If $H = \underline{0}$, then $\mathbf{DIL} \underline{\ll} H$ contains exactly one object: $\underline{0}$, and one morphism: $\mathbf{E}^1_{\underline{0}}$. The $\mathbf{\Lambda}\underline{0} = \mathsf{Id}$, $\mathbf{\Lambda E}^1_{\underline{0}} = \mathbf{E}^b_{\mathsf{Id}}$ defines a functor enjoying (i)–(viii) relativised to $\mathbf{DIL} \underline{\ll} \underline{0}$.

9.6.4. $H = \underline{1}$.

$\mathbf{DIL} \lll H$ consists of two objects: $\underline{0}$ and $\underline{1}$, and three morphisms, namely $\mathbf{E}^1_{\underline{0}}$, $\mathbf{E}^1_{\underline{1}}$, $\mathbf{E}^1_{\underline{01}}$. We define $\mathbf{\Lambda}\underline{1}$, $\mathbf{\Lambda E}^1_{\underline{01}}$, $\mathbf{\Lambda E}^1_{\underline{1}}$ by means of (iii) and (vi). We establish that the relativizations of (i)–(viii) hold:

(i) $(\mathbf{\Lambda}T)(\mathbf{\Lambda}U) = \mathbf{\Lambda}(TU)$: assume that $h(T) = 1$; then either $T = \mathbf{E}^1_{\underline{1}}$ and $TU = U$ (hence $(\mathbf{\Lambda}T)(\mathbf{\Lambda}U) = \mathbf{E}^1_{\mathsf{Id}+\mathsf{Id}}(\mathbf{\Lambda}U) = \mathbf{\Lambda}U = \mathbf{\Lambda}(TU)$), or $U = \mathbf{E}^1_{\underline{0}}$, and $TU = T$, hence

$$(\mathbf{\Lambda}T)(\mathbf{\Lambda}U) = (\mathbf{\Lambda}T)\mathbf{\Lambda E}^1_{\mathsf{Id}} = \mathbf{\Lambda}T = \mathbf{\Lambda}(TU)\ .$$

(ii), (iii) and (iv) are trivially fulfilled.

(v) If $F = F' + F''$ is an object of $\mathbf{DIL} \lll \underline{1}$, then one of F' and F'' is $\underline{0}$, so one of $\mathbf{\Lambda}F'$ and $\mathbf{\Lambda}F''$ is equal to Id: so $\mathbf{\Lambda}F = \mathbf{\Lambda}F' \circ_s \mathbf{\Lambda}F''$, since Id is neutral for $\circ_s$. The case of natural transformations is similar.

(vi) By definition, when $h(F) = \underline{1}$, $(\mathbf{\Lambda E}_{01})(x, y)(z) = x + z....$

(vii) is trivial (no non-trivial direct systems in a finite category!).

(viii) If $T_1 \wedge t_2 = T_3$, with $h(T_i) = \underline{1}$, then:

– If $T_1 = \mathbf{E}^1_1$, then $T_2 = T_3$ and

$$\mathbf{\Lambda}T_1 \wedge \mathbf{\Lambda}T_2 = \mathbf{E}^b_{\mathsf{Id}+\mathsf{Id}} \wedge \mathbf{\Lambda}T_2 = \mathbf{\Lambda}T_2 = \mathbf{\Lambda}T_3\ .$$

– If $T_2 = \mathbf{E}^1_1$: symmetric.

– If $T_1 = T_2 = \mathbf{E}^1_{\underline{01}}$, then $T_1 = T_2 = T_3$, so $\mathbf{\Lambda}T_1 \wedge \mathbf{\Lambda}T_2 = \mathbf{\Lambda}T_3$.

9.6.5. $H = H' + H''$.

Assume that (i)–(viii) hold for $\mathbf{DIL} \lll H'$ and $\mathbf{DIL} \lll H''$; then they hold for $\mathbf{DIL} \lll H' + H''$. 9.6.5, used together with 9.6.4, yields the case where H is of kind 1.

Write $H = H' + H''$; we first extend $\mathbf{\Lambda}$ to $\mathbf{DIL} \lll H$:

– If $H_1 \lll H$, then either $H_1 \ll H'$ or $H_1 = H' + H''_1$, where $H''_1 \lll H$; in both cases $H_1 = H'_1 + H''_1$, with $H'_1 \lll H'$, $H''_1 \lll H''$.

– If F is an object of $\mathbf{DIL} \ll H$, then $\exists H_1 \ll H \ I^1(F, H_1) \neq \emptyset$; if $T \in I^1(F, H_1)$, then the decomposition $H_1 = H_1' + h_1''$ induces a decomposition $F = F' + F''$, $T = T' + T''$ (9.1.8) "splitting lemma") with $T' \in I^1(F', H_1')$, $T'' \in I^1(F'', H_1'')$; hence $F = F' + F''$, for some $F' \in \mathbf{DIL} \ll H'$ and $F'' \in \mathbf{DIL} \ll H''$. If $U \in I^1(G, F)$ and F is in $\mathbf{DIL} \ll H$, then by 9.1.8, a decomposition $F = F' + F''$ induces a decomposition $G = G' + G''$, $U = U' + U''$, ...; hence if U is a morphism in $\mathbf{DIL} \ll H$ U can be expressed as a sum $U' + U''$, where U' and U'' are morphisms in $\mathbf{DIL} \ll H'$ and $\mathbf{DIL} \ll H''$ respectively.

If $F = F' + F''$ is an object of $\mathbf{DIL} \ll H$, we define $\mathbf{\Lambda} F = \mathbf{\Lambda} F' \circ_s \mathbf{\Lambda} F''$; this is possible, since F' and F'' are objects of $\mathbf{DIL} \ll H'$ and $\mathbf{DIL} \ll H''$ respectively. If $F = G' + G''$ is another decomposition of F as a sum of an object of $\mathbf{DIL} \ll H'$ and an object of $\mathbf{DIL} \ll H''$, then we have for instance $\mathbf{LH}(G') \leq \mathbf{LH}(F')$, hence one can find G_1' such that $F' = G' + G_1'$, $G'' = G_1' + F''$; then

$$\mathbf{\Lambda} F = \mathbf{\Lambda} F' \circ_s \mathbf{\Lambda} F'' = \mathbf{\Lambda} G' \circ_s \mathbf{\Lambda} G_1' \circ_s \mathbf{\Lambda} F'' = \mathbf{\Lambda} G' \circ_s \mathbf{\Lambda} G''$$

(we use: the associativity of $\circ_s$, property (v) for $\mathbf{DIL} \ll H'$ and $\mathbf{DIL} \ll H''$, together with the fact that $\mathbf{\Lambda} G_1'$ is the same when computed in $\mathbf{DIL} \ll H'$ and in $\mathbf{DIL} \ll H''$). We have therefore shown our definition of $\mathbf{\Lambda} F$ to be independent of the decomposition $F = F' + F''$... .

If $T = T' + T''$, define $\mathbf{\Lambda} T = \mathbf{\Lambda} T' \circ_s \mathbf{\Lambda} T''$: one shows as above that this definition is independent of the decomposition $T = T' + T''$.

(i) We show that $\mathbf{\Lambda}(TU) = (\mathbf{\Lambda} T)(\mathbf{\Lambda} U)$: assume that $U \in I^1(F, G)$, $T \in I^1(G, K)$, and $K \in \mathbf{DIL} \ll H$; then if one writes $K = K' + K''$, one can write $T = T' + T''$, $T' \in I(G', K')$, $T'' = I(G'', K'')$, and applying 9.1.8 once more $U = U' + U''$, $U' \in I^1(F', G')$, $U'' \in I^1(F'', G'')$; then $TU = T'U' + T''U''$. Hence:

$$\begin{aligned}\mathbf{\Lambda}(TU) &= \mathbf{\Lambda}(T'U') \circ_s \mathbf{\Lambda}(T''U'') = \\ &(\mathbf{\Lambda} T')(\mathbf{\Lambda} U') \circ_s (\mathbf{\Lambda} T'')(\mathbf{\Lambda} U'') = \\ &(\mathbf{\Lambda} T' \circ_s \mathbf{\Lambda} U')(\mathbf{\Lambda} T'' \circ_s \mathbf{\Lambda} U'') = (\mathbf{\Lambda} T)(\mathbf{\Lambda} U)\ .\end{aligned}$$

(ii) and (iii) are immediate.

(iv). If G is connected, $\neq \underline{1}$ belongs to $\mathbf{DIL} \lll H$, write $G = G' + G''$, with G' in $\mathbf{DIL} \lll H'$, G'' in $\mathbf{DIL} \lll H''$; then $G = G'$ or $G = G''$, hence the property follows from the hypotheses... .

(v) If G is a sum $G' + G''$, then decompose G as $F' + F''$, with F' in $\mathbf{DIL} \lll H'$, F'' in $\mathbf{DIL} \lll H''$; then two subcases

– if $\mathbf{LH}(G') \leq \mathbf{LH}(F')$, write $F' = G' + G'_1$, so $G'' = G'_1 + F''$:

$$\mathbf{\Lambda} G = \mathbf{\Lambda} F' \circ_s \mathbf{\Lambda} F'' = \mathbf{\Lambda} G' \circ_s \mathbf{\Lambda} G'_1 \circ_s \mathbf{\Lambda} F'' = \mathbf{\Lambda} G' \circ_s \mathbf{\Lambda} G'' \ .$$

– if $\mathbf{LH}(G') \geq \mathbf{LH}(F')$, write $G' = F' + G'_1$, so $F'' = G'_1 + G''$:

$$\mathbf{\Lambda} G = \mathbf{\Lambda} F' \circ_s \mathbf{\Lambda} F'' = \mathbf{\Lambda} F' \circ_s \mathbf{\Lambda} G'_1 \circ_s \mathbf{\Lambda} G'' = \mathbf{\Lambda} G' \circ_s \mathbf{\Lambda} G'' \ .$$

The property $\mathbf{\Lambda}(T' + T'') = \mathbf{\Lambda} T' \circ_s \mathbf{\Lambda} T''$ is obtained in a similar way.

(vi) If F is in $\mathbf{DIL} \lll H$, write $F = F' + F''$, with F', F'' in $\mathbf{DIL} \lll H'$ and $\mathbf{DIL} \lll H''$ respectively. Then $\mathbf{E}_{\underline{0}F} = \mathbf{E}_{\underline{0}F'} + \mathbf{E}_{\underline{0}F''}$, hence

$$\begin{aligned}
&(\mathbf{\Lambda E}_{\underline{0}F})(x,y)(z) = (\mathbf{\Lambda E}_{\underline{0}F'})\Big(\mathbf{E}_x, (\mathbf{\Lambda E}_{\underline{0}F''})(x,y)\Big)(z) \ = \\
&(\mathbf{\Lambda} F')\Big(x, (\mathbf{\Lambda E}_{\underline{0}F''})(x,y)(z)\Big) = (\mathbf{\Lambda} F')\Big(x, (\mathbf{\Lambda} F'')(x,z)\Big) \ = \\
&(\mathbf{\Lambda} F)(x,z) \ .
\end{aligned}$$

(vii) If $(F, T_i) = \varinjlim (F_i, T_{ij})$, write $F = F' + F''$, with F', F'' respectively in $\mathbf{DIL} \lll H'$ and $\mathbf{DIL} \lll H''$; by 9.1.8, we obtain decompositions $T_i = T'_i + T''_i$, $F_i = F'_i + F''_i$, $T_{ij} = T'_{ij} + T''_{ij}$, and obviously: $(F', T'_i) = \varinjlim (F'_i, T'_{ij})$, $(F'', T''_i) = \varinjlim (F''_i, T''_{ij})$. Then

$$\begin{aligned}
&(\mathbf{\Lambda} F, \mathbf{\Lambda} T_i) = (\mathbf{\Lambda} F' \circ_s \mathbf{\Lambda} F'', \mathbf{\Lambda} T'_i \circ_s \mathbf{\Lambda} T''_i) \ = \\
&\varinjlim (\mathbf{\Lambda} F'_i \circ_s \mathbf{\Lambda} F''_i, \mathbf{\Lambda} T'_{ij} \circ_s \mathbf{\Lambda} T''_{ij}) = \varinjlim (\mathbf{\Lambda} F_i, \mathbf{\Lambda} T_{ij})
\end{aligned}$$

since $\circ_s$ preserves direct limits, and by (vii) for (F'_i, T'_{ij}) and (F''_i, T''_{ij})...).

(viii) If $T_i \in I(F_i, G)$ $(i = 1, 2, 3)$ and $T_1 \wedge T_2 = T_3$, then write $G = G' + G''$, with G', G'' in $\mathbf{DIL} \lll H'$ and $\mathbf{DIL} \lll H''$ respectively; then by 9.1.8, one may write $T_i = T'_i + T''_i$, and obviously $T'_3 = T'_1 \wedge T'_2$, $T''_3 = T''_1 \wedge T''_2$. Then

$$\boldsymbol{\Lambda}T_3 = \boldsymbol{\Lambda}T_3' \circ_s \boldsymbol{\Lambda}T_3'' \; = \; (\boldsymbol{\Lambda}T_1' \wedge \boldsymbol{\Lambda}T_2') \circ_s (\boldsymbol{\Lambda}T_1'' \wedge \boldsymbol{\Lambda}T_2'') \; = \; \boldsymbol{\Lambda}T_1 \wedge \boldsymbol{\Lambda}T_2$$

(since $\circ_s$ preserves pull-backs, and (viii) holds for the T_i's and the T_i'''s...).

9.6.6. *H* <u>of kind</u> $\boldsymbol{\omega}$.

Assume that (i)–(viii) hold for all $H' \ll H$; then it is immediate that (i)–(viii) relativized to $\mathbf{DIL} \ll H$ hold! Hence it will suffice to define $\boldsymbol{\Lambda}G$ and $\boldsymbol{\Lambda}T$ when $h(G) = h(T) = H$. Assume that $G = \sum_{i<y} G_i'$, with all G_i's connected; if $U \in I(G, H)$, one can write $U = \sum_{i<f} U_i$, for some $f \in I\big(y, \mathbf{LH}(H)\big)$; we claim that $\hat{f}(y) = \mathbf{LH}(H)$.
(<u>Proof</u>. Let $H = \sum_{i<x} H_i$ be the decomposition of H as a sum of connected dilators; then if $H' = \sum_{i<\hat{f}(y)} H_i$, one easily constructs $U' \in I(G, H')$; the hypothesis that $h(G) = H$ forces $H' = H$, hence $\hat{f}(y) = x = \mathbf{LH}(H)$. $\square$) hence y is limit. So G is of kind $\boldsymbol{\omega}$. If for $i < y$ $G_i = \sum_{j<i} G_j'$, then clearly $(G, \mathbf{E}_{G_iG}) = \varinjlim (G_i, \mathbf{E}_{G_iG_j})$. We define $\boldsymbol{\Lambda}G = \varinjlim{}^{*} (\boldsymbol{\Lambda}G_i, \boldsymbol{\Lambda}\mathbf{E}_{G_iG_j})$. The existence of such a limit in $\mathbf{BIL}_n$ is a consequence of the general results of Section 9.3: in fact we have

$$\boldsymbol{\Lambda}G = \prod_{i<y} \boldsymbol{\Lambda}G_i' \; \ldots \; .$$

By the way observe that $\boldsymbol{\Lambda}G = \varinjlim{}^{*} (\boldsymbol{\Lambda}G_i, \boldsymbol{\Lambda}\mathbf{E}_{G_iG_j})$ still holds when $h(G) \ll H$ (this is (vii) restricted to $\mathbf{DIL} \ll H$). Similarly, if $T \in I(F, G)$, with $h(T) = H$ ($= h(G)$), write $T = \sum_{i<f} T_i'$, and then (with obvious notations) $\boldsymbol{\Lambda}F = \varinjlim_{x}{}^{*} (\boldsymbol{\Lambda}F_i, \boldsymbol{\Lambda}\mathbf{E}_{F_iF_j})$; if $T_i \in I(F_i, G_{f(i)})$ is defined by $T_i = \sum_{j<f_i} T_j'$ $\big(f_i \in I\big(i, f(i)\big)$, $f_i(z) = f(z)...\big)$ then simply define $\boldsymbol{\Lambda}T = \varinjlim (\boldsymbol{\Lambda}\,T_i)$. (In other terms $\boldsymbol{\Lambda}T = \prod_{i<f} \boldsymbol{\Lambda}T_i'$.) We check (i)–(viii):

(i) $\mathbf{\Lambda}(TU) = (\mathbf{\Lambda}T)(\mathbf{\Lambda}U)$; as above write $T = \varinjlim (T_i)$, $U = \varinjlim (U_i)$, $TU = \varinjlim (V_i)$; it is immediate that $V_i = T_{g(i)}U$ with $g = \mathbf{LH}(U)$. Hence we obtain

$$\mathbf{\Lambda}(TU) = \varinjlim (\mathbf{\Lambda}T_{g(i)}U_i) = \varinjlim (\mathbf{\Lambda}T_i) \varinjlim (\mathbf{\Lambda}U_i) = (\mathbf{\Lambda}T)(\mathbf{\Lambda}U) \ .$$

(ii), (iii) and (iv) are trivial.

(v) Assume that $T' \in I(F', G')$, $T'' \in I(F'', G'')$, and let $T = T' + T''$, $G = G' + G''$; then two subcases (we restrict ourselves to the non-trivial case $h(G + G') = H$)

– If $G'' = \underline{0}$, $\mathbf{\Lambda}G = \mathbf{\Lambda}G \circ_s \mathsf{Id} = \mathbf{\Lambda}G' \circ_s \mathbf{\Lambda}G''$ and $\mathbf{\Lambda}T = \mathbf{\Lambda}T \circ_s \mathbf{E}_{\mathsf{Id}} = \mathbf{\Lambda}T' \circ_s \mathbf{\Lambda}T''$.

– If $G'' \neq \underline{0}$, let x, x', x'', y, y', y'', f, f', f'', be the respective lengths of F, F', F'', G, G', G'', T, T', T'', and observe that $x = x' + x''$, $y = y' + y''$, $f = f' + f''$; define F, F', F'', G, G', G'', T, T', T'' as above, and observe that: $F_{x+i} = F' + F''_i$, $G_{y+i} = G' + G''_i$, $T_{x+i} = T' + T''_i$; then

$$\begin{aligned} G &= \varinjlim_{y}{}^{*} (\mathbf{\Lambda}G_i, \mathbf{\Lambda}\mathbf{E}_{G_iG_j}) = \\ &\varinjlim_{y''}{}^{*} \Big(\mathbf{\Lambda}'(G' + G''_i), \mathbf{\Lambda}(\mathbf{E}_{G'} + \mathbf{E}_{G''_iG''_j})\Big) = \\ &\varinjlim_{y''}{}^{*} (\mathbf{\Lambda}G' \circ_s \mathbf{\Lambda}G''_i, \mathbf{\Lambda}\mathbf{E}_{G'} \circ_s \mathbf{\Lambda}\mathbf{E}_{G''_iG''_j}) = \\ &\mathbf{\Lambda}G' \circ_s \varinjlim_{y''}{}^{*} (\mathbf{\Lambda}G''_i, \mathbf{\Lambda}\mathbf{E}_{G''_iG''_j}) = \mathbf{\Lambda}G' \circ_s \mathbf{\Lambda}G'' \ , \end{aligned}$$

and

$$\mathbf{\Lambda}T = \varinjlim_{f} (\mathbf{\Lambda}T_i) =$$

$$\lim_{\underset{f''}{\longrightarrow}} \Big(\mathbf{\Lambda}(T' + T_i'')\Big) = \lim_{\underset{f''}{\longrightarrow}}{}^{*} (\mathbf{\Lambda}T' \circ_s \mathbf{\Lambda}T_i'') =$$

$$\mathbf{\Lambda}T' \circ_s \lim_{\underset{f''}{\longrightarrow}} (\mathbf{\Lambda}T_i'') = \mathbf{\Lambda}T' \circ_s \mathbf{\Lambda}T''$$

(using the induction hypothesis (v) for G', T' ($h(G') \ll H$) and for G_i'', T_i'', together with the preservation of direct limits by $\circ_s \ldots$.)

(vi) When $h(G) = H$, then the expression of $\mathbf{\Lambda E}_{\underline{0}G}$ as a semi-product $\mathbf{\Pi\Lambda E}_{\underline{0}G_i'}$ shows that $\mathbf{\Lambda E}_{\underline{0}G}$ is of the form $\xi^b_{\mathbf{\Lambda}G} \ldots$.

(vii) By obvious considerations (see for instance Sec. 12.3) it suffices to show that each $T \in I(F, G)$, with $\mathsf{dim}(F) < +\infty$. But, if $h(G) = H$, one may write $\mathbf{\Lambda}G = \lim\limits_{\longrightarrow}{}^{*} (\mathbf{\Lambda}G_i, \mathbf{\Lambda E}_{G_iG_j})$ hence $(z_0\,;\,n) = \mathsf{Tr}\Big(\mathbf{\Lambda}(T)\Big)(z_1\,;\,n)$, for some $T \in I(G_i, G_i)$ and some $i \in I$ (take $T = \mathbf{E}_{G_iG}$); then the induction hypothesis applied to $\mathbf{\Lambda}G$ yields: $(z_1\,;\,n) = \mathsf{Tr}\Big(\mathbf{\Lambda}(U)\Big)(z_2\,;\,n)$ for some $U \in I(F, G_i)$, with $\mathsf{dim}(F) < +\infty$. Hence $TU \in I(F, G)$, and $(z_0\,;\,n) \in rg\Big(\mathsf{Tr}(TU)\Big)$, $\mathsf{dim}(F) < +\infty$.

(viii) Still using obvious considerations we must show that, if one can express $(z_0\,;\,n)$ as: $(z_0\,;\,n) = \mathsf{Tr}(\mathbf{\Lambda}T_1)(z_1\,;\,n) = \mathsf{Tr}(\mathbf{\Lambda}T_2)(z_2\,;\,n)$, with $T_1 \in I(F_1, G)$, $T_2 \in I(F_2, G)$, $\mathsf{dim}(F_1), \mathsf{dim}(F_2) < +\infty$, then there exist F_3 and $T_{31} \in I(F_3, F_1)$, $T_{32} \in I(F_3, F_2)$ such that $T_1T_{31} = T_2T_{32}$, and $(z_3\,;\,n) \in \mathsf{Tr}(\mathbf{\Lambda}F_3)$ s.t. $(z_1\,;\,n) = \mathsf{Tr}(\mathbf{\Lambda}T_{31})(z_3\,;\,n)$, $(z_2\,;\,n) = \mathsf{Tr}(\mathbf{\Lambda}T_{32})(z_3\,;\,n)$. One can restrict to the case where G is finite dimensional.

(<u>Proof</u>. Define G' and $U \in I(G', G)$ by $rg\Big(\mathsf{Tr}(U)\Big) = rg\Big(\mathsf{Tr}(T_1)\Big) \cup rg\Big(\mathsf{Tr}(T_2)\Big)$ and define $T_1' \in I(F_1, G')$, $T_2' \in I(F_2, G')$ by $UT_1' = T_1$, $UT_2' = T_2$ ... then G' is finite dimensional.... $\square$)

I.e. the question can be reduced to a problem in the category $\mathbf{DIL}_{fd} \cap \mathbf{DIL} \underline{\ll} H$, which is obviously equal to $\mathbf{DIL}_{fd} \cap \mathbf{DIL} \ll H$; but, in that last category, the property is true by the induction hypothesis....

9.6.7. H <u>is connected and</u> $\neq \underline{1}$.

(This case, together with 9.6.5, is sufficient to handle the general case "H is of kind $\mathbf{\Omega}$" of the induction step: assume that (i)–(viii) hold for

$\mathbf{DIL} \ll H' + H''$, with H'' connected and $\neq \underline{1}$; then it is immediate that (i)–(viii) hold for $\mathbf{DIL} \lll H'$ and $\mathbf{DIL} \ll H''$; by 9.6.7, (i)–(viii) will still hold for $\mathbf{DIL} \lll H''$, and by 9.6.5 they will hold for $\mathbf{DIL} \lll H' + H''$.)

We assume to have already obtained a functor $\mathbf{\Lambda}$ defined of $\mathbf{DIL} \ll H$, and enjoying (i)–(viii); then consider G s.t. $h(G) = H$; this means that $I^1(G, H) \neq \emptyset$, and since $G \neq \underline{0}$, $\mathbf{\Omega} I^1(G, H) \neq \emptyset$, hence G is of kind $\mathbf{\Omega}$ (more precisely $\mathbf{LH}(G) \leq \mathbf{LH}(H) = 1$, and $G \neq \underline{0} \to \mathbf{LH}(G) = 1$; so G is connected; $G(0) \leq H(0) = 0$, hence $G \neq \underline{1}$). Then it is possible to define $\mathbf{\Lambda} G$ by means of (iv); if $T \in I^1(F, G)$, then either $F = \underline{0}$, and $\mathbf{\Lambda} T = \xi^b_{\mathbf{\Lambda} G}$ (this is (vi)) (in fact, we need to verify (vi) before making such a verification; the verification of (vi) made below could have been written here), or F is connected and $\neq \underline{1}$, and $T \in \mathbf{\Omega} I^1(F, G)$: in that case, one can define $\mathbf{\Lambda} T$ by means of (iv).

(i) It is immediate that $\mathbf{\Lambda} G$ defined by means of (iv) is a functor from $\mathbf{ON}^2$ to $\mathbf{ON}$; furthermore, $\mathbf{\Lambda} G$ preserves $\underset{\longrightarrow}{\lim}$ and $\wedge$: $(\mathbf{\Lambda} G)(\cdot, y) = \prod\limits_{t<y} (\underline{1} + \mathbf{\Lambda}_t G)(\cdot, 0)$, it is immediate that $(\mathbf{\Lambda} G)(\cdot, y)$ preserves $\underset{\longrightarrow}{\lim}$ and $\wedge$; hence, it will suffice to show that $(\mathbf{\Lambda} G)(x, \cdot)$ preserves $\underset{\longrightarrow}{\lim}$ and $\wedge$: if $(y, g_i) = \underset{\longrightarrow}{\lim} (y_i, g_{ij})$

$$\Big(\prod_{l<y} (\underline{1} + \mathbf{\Lambda}_l G), \prod_{l<g_i} (\mathbf{E}^b_{\underline{1}} + \mathbf{\Lambda} \mathbf{E}_{lG}) \Big) = \underset{\longrightarrow}{\lim} \Big(\prod_{l<y_i} (1 + \mathbf{\Lambda}_l G), \prod_{l<g_{ij}} (\mathbf{E}^b_{\underline{1}} + \mathbf{\Lambda} \mathbf{E}_{lG}) \Big)$$

(this is a consequence of 9.5.12), and applying both sides to the pair $(x, 0)$, we obtain:

$$\Big((\mathbf{\Lambda} G)(x, y), (\mathbf{\Lambda} G)(\mathbf{E}_x, g_i) \Big) = \underset{\longrightarrow}{\lim} \Big((\mathbf{\Lambda} G)(x, y_i), (\mathbf{\Lambda} G)(\mathbf{E}_x, g_{ij}) \Big) .$$

Similarly, when $g_i \in I(y_i, y)$ $(i = 1, 2, 3)$, and $g_3 = g_1 \wedge g_2$, then by 9.5.12 we obtain

$$\prod_{t<g_3} (\mathbf{E}^b_{\underline{1}} + \mathbf{\Lambda} \mathbf{E}_{tG}) = \Big(\prod_{t<g_1} \mathbf{E}^b_{\underline{1}} + \mathbf{\Lambda} \mathbf{E}_{tG} \Big) \wedge \Big(\prod_{t<g_2} (\mathbf{E}^b_{\underline{1}} + \mathbf{\Lambda} \mathbf{E}_{tG}) \Big)$$

and, applying both sides to $(\mathbf{E}_x, \mathbf{E}_0)$ we get

$$(\mathbf{\Lambda}G)(\mathbf{E}_x, g_3) = (\mathbf{\Lambda}G)(\mathbf{E}_x, g_1) \wedge (\mathbf{\Lambda}G)(\mathbf{E}_x, g_2) \ .$$

Then, we show that the functor $(\mathbf{\Lambda}G)(x, \cdot)$ enjoys (FL): if $y \leq y'$, then

$$(\mathbf{\Lambda}G)(\mathbf{E}_x, \mathbf{E}_{yy'}) = \mathop{\mathbf{\Pi}}_{t<\mathbf{E}_{yy'}} (\mathbf{E}^b_{\underline{1}} + \mathbf{\Lambda}\mathbf{E}_{tG})(\mathbf{E}_x, \mathbf{E}_0) \ =$$

$$\Big(\mathop{\mathbf{\Pi}}_{t<\mathbf{E}_y} \mathbf{E}^b_1 + \mathbf{\Lambda}\mathbf{E}_{tG}\Big) \circ_s \mathop{\mathbf{\Pi}}_{t<\mathbf{E}_{0y'}} (\mathbf{E}^b_{\underline{1}} + \mathbf{\Lambda}\mathbf{E}_{tG})(\mathbf{E}_x, \mathbf{E}_0) \ =$$

$$\mathbf{E}^b_F \circ_s \xi^b_K(\mathbf{E}_x, \mathbf{E}_0)$$

with

$$F = \mathop{\mathbf{\Pi}}_{t<y} (\underline{1} + \mathbf{\Lambda}_t G) \ , \qquad K = \mathop{\mathbf{\Pi}}_{y\leq t<y'} (\underline{1} + \mathbf{\Lambda}_t G) \ .$$

But:

$$(\mathbf{E}_F \circ_s \xi^b_K)(x, 0) = F\Big(\mathbf{E}_x, \xi^b_K(x, 0)\Big) = F(\mathbf{E}_x, \mathbf{E}_{0K(x,0)}) \ =$$

$$\mathbf{E}_{F(x,0)F\left(x,K(x,0)\right)} = \mathbf{E}_{(\mathbf{\Lambda}G)(x,y)(\mathbf{\Lambda}G)(x,y')} \ .$$

It is immediate that $T \in I^1(F, G) \to \mathbf{\Lambda}T \in I^1(\mathbf{\Lambda}F, \mathbf{\Lambda}G)$; then we show that $\mathbf{\Lambda}(TU) = (\mathbf{\Lambda}T)(\mathbf{\Lambda}U)$, when $h(T) = h(G) = H$, $T \in I^1(F, G)$:

– If $F \neq \underline{0}$, and $U \neq \mathbf{E}_{\underline{0}F}$, then

$$(\mathbf{\Lambda}TU)(x, y) = \mathop{\mathbf{\Pi}}_{t<\mathbf{E}_y} \Big(\mathbf{E}^b_{\underline{1}} + \mathbf{\Lambda}_t(TU)\Big)(x, 0) \ =$$

$$\mathop{\mathbf{\Pi}}_{t<\mathbf{E}_y} (\mathbf{E}^b_1 + \mathbf{\Lambda}_t T_t U)(x, 0) \ =$$

$$\Big(\Big(\mathop{\mathbf{\Pi}}_{t<\mathbf{E}_y} (\mathbf{E}^b_1 + \mathbf{\Lambda}_t T)\Big)\Big(\mathop{\mathbf{\Pi}}_{t<\mathbf{E}_y} (\mathbf{E}^b_1 + \mathbf{\Lambda}_t U)\Big)\Big)(x, 0) \ =$$

$$\Big(\Big(\mathop{\mathbf{\Pi}}_{t<\mathbf{E}_y} (\mathbf{E}^b_1 + \mathbf{\Lambda}_t T)\Big)(x, 0)\Big(\mathop{\mathbf{\Pi}}_{t<\mathbf{E}_y} (\mathbf{E}^b_1 + \mathbf{\Lambda}_t U)\Big)(x, 0)\Big) \ =$$

$$(\mathbf{\Lambda}T)(x, y)(\mathbf{\Lambda}U)(x, y) \ .$$

– In general, observe that, from the obvious equalities:

$$\mathop{\mathbf{\Pi}}_{t<y}\ (\underline{1}+\mathsf{Id})(x,z)=y+z\ ,\qquad \mathop{\mathbf{\Pi}}_{t<g}\ (\underline{1}+\mathsf{Id})(f,h)=g+h$$

the definition (iv) holds too when $G=\underline{0}$ (with ${}_tG=\underline{0}$, ${}_{gt}G=\mathbf{E}_{\underline{0}}$). From this, one can show that $(\mathbf{\Lambda}TU)=(\mathbf{\Lambda}T)(\mathbf{\Lambda}U)$ holds when $F=\underline{0}$ or $U=\mathbf{E}_{\underline{0}F}$: the proof is similar to the case just treated.

(ii), (iii), (iv) are trivial.

(v) If $h(F'+F'')=h(H)$, then $F'+F''$ must be connected, hence one of F' and F'' is $\underline{0}$ ... we conclude as in 9.6.4 (v).

(vi) If $h(F)=H$, then the property is trivial by construction; however, we have not verified the nicety of $\mathbf{\Lambda}F$: first $(\mathbf{\Lambda}F)(x,y+1)=K\Big(x,1+L(x,y)\Big)$, with $K=\mathop{\mathbf{\Pi}}_{t<y}(\underline{1}+\mathbf{\Lambda}_tF)$, $L=\mathbf{\Lambda}_yF$, whereas $(\mathbf{\Lambda}F)(x,y)=K(x,y)$, hence $K(x,y)<K(x,y+1)$. If $f\in I(x,x')$, $g\in I(z,z')$, we show that $(\mathbf{\Lambda}F)(f,g+\mathbf{E}_1)\Big((\mathbf{\Lambda}F)(x,z)\Big)=(\mathbf{\Lambda}F)(x',z')$ (from this, it will be immediate that $\mathbf{\Lambda}F$ is nice). Let

$$T=\mathop{\mathbf{\Pi}}_{t<g}\ (\mathbf{E}^b_{\underline{1}}+\mathbf{\Lambda}_{gt}F)\ \ \textsf{and}\ \ U=\mathbf{\Lambda}_{(g+\mathbf{E}_1)z}F\ ;$$

obviously

$$(\mathbf{\Lambda}F)(f,g+\mathbf{E}_1)\ =\ \Big(T\circ_s(\mathbf{E}_{\underline{1}}+U)\Big)(f,0)\ =\ T\Big(f,\mathbf{E}_1+U(f,\mathbf{E}_0)\Big)\ ,$$

and if

$$G=\mathop{\mathbf{\Pi}}_{t<z}\ (\underline{1}+\mathbf{\Lambda}_tF)\ ,\qquad G'=\mathop{\mathbf{\Pi}}_{t<z'}\ (\underline{1}+\mathbf{\Lambda}_tF)$$

(so $T\in I^b_n(G,G')$), we obtain:

$$(\mathbf{\Lambda}F)(f,g+\mathbf{E}_1)\Big(G(x,0)\Big)\ =\ T\Big(f,\mathbf{E}_1+U(f,\mathbf{E}_0)\Big)\Big(G(x,0)\Big)\ =\ G(x',0)\ .$$

Now observe that $(\mathbf{\Lambda}F)(x,z)=G(x,0)$, $(\mathbf{\Lambda}F)(x,z')=G'(x,0)$.

(vii) If $(F,T_i)=\varinjlim(F_i,T_{ij})$, and $h(F)=H$, then it is possible to assume (by restricting I to a cofinal subset), that $F_i\neq\underline{0}$ for all i: hence we start with a direct system in $\mathbf{\Omega DIL}$, with its limit in $\mathbf{\Omega DIL}$. By 9.5.12 we get

$$\Big(\prod_{t<y}(\underline{1}+\mathbf{\Lambda}_t F),\ \prod_{t<\mathbf{E}_y}(\mathbf{E}^b_1+\mathbf{\Lambda}_t T_i)\Big)\ =$$

$$\varinjlim\ \Big(\prod_{t<y}(\underline{1}+\mathbf{\Lambda}_t F_i),\ \prod_{t<\mathbf{E}_y}(\mathbf{E}^b_{\underline{1}}+\mathbf{\Lambda}_t T_{ij})\Big)$$

(using: $({}_tF,{}_tT_i) = \varinjlim\ ({}_tF_i,{}_tT_{ij})$ together with (vii) for $\mathbf{DIL} \ll H$). If one applies $(x,0)$ to both sides, one gets

$$\big((\mathbf{\Lambda}F)(x,y),(\mathbf{\Lambda}T_i)(x,y)\big) = \varinjlim\ \big((\mathbf{\Lambda}F_i)(x,y),(\mathbf{\Lambda}T_{ij})(x,y)\big)$$

which implies (8.3.7) that $(\mathbf{\Lambda}F,\mathbf{\Lambda}T_i) = \varinjlim\ (\mathbf{\Lambda}F_i,\mathbf{\Lambda}T_{ij})$.

(viii) Assume that $h(G) = H$, and that $T_i \in I(F_i,G)$ $(i = 1,2,3)$ and that $T_3 = T_1 \wedge T_2$; we have already observed, inthe proof of (i), that $T_i(x,y) = \prod_{t<\mathbf{E}_y}(\mathbf{E}_{\underline{1}}+\mathbf{\Lambda}_t T_i)(x,0)$, holds when F_i is of kind $\mathbf{\Omega}$, but also when $F_i = \underline{0}$. (In that case one defines ${}_tT = \mathbf{E}_{\underline{0}_t G}$.) Clearly ${}_tT_3 = {}_tT_1 \wedge {}_tT_2$, hence, by 9.5.12, we obtain

$$\Big(\prod_{t<\mathbf{E}_y}(\mathbf{E}^b_{\underline{1}}+\mathbf{\Lambda}_t T_3)\Big) = \prod_{t<\mathbf{E}_y}(\mathbf{E}^b_{\underline{1}}+\mathbf{\Lambda}_t T_1) \wedge \prod_{t<\mathbf{E}_y}(\mathbf{E}^b_{\underline{1}}+\mathbf{\Lambda}_t T_2)$$

(using (viii) for $\mathbf{DIL} \ll H$). If one applies $(x,0)$ to both sides, one gets

$$(\mathbf{\Lambda}T_3)(x,y) = (\mathbf{\Lambda}T_1)(x,y) \wedge (\mathbf{\Lambda}T_2)(x,y)$$

and by 8.3.10

$$\mathbf{\Lambda}T_3 = \mathbf{\Lambda}T_1 \wedge \mathbf{\Lambda}T_2\ .$$

End of the proof of 9.6.2. We have shown that, for all H in **DIL**, as soon as (i)–(viii) hold for $\mathbf{DIL} \ll H$, then they also hold for $\mathbf{DIL} \underline{\ll} H$. From this we conclude that (i)–(viii) hold for all dilators. $\square$

9.6.8. Remark.
In fact, there are many variant os $\mathbf{\Lambda}$, i.e. definitions of the same kind, but with small differences. A typical example is the function Λ defined on ladders considered in Annex A. But many other possibilities can be used. The reader will find the most important ones in the exercises of Annex B. The variant chosen here is in my opinion the most elegant, but also the most complicated, because of the use of the semi- products... .

Annex 9.A. The hierarchy theorem

In 1975, I proved a theorem relating the pointwise hierarchy $\boldsymbol{\gamma}$ to the Grzegorczyk hierarchy $\boldsymbol{\lambda}$ (see 5.B). Roughly speaking the result is:

$$\boldsymbol{\gamma}_{\eta_0} = \boldsymbol{\lambda}_{\varepsilon_0}$$

where ε_0 and η_0 are elements of O of respective heights ε_0 and η_0 (the "Howard ordinal"). Hence, since one needs "ε_0 steps" to exhaust all provably total recursive functions of **PA** when measured w.r.t. $\boldsymbol{\gamma}$, one will need "η_0 steps" if one does the same thing w.r.t. $\boldsymbol{\lambda}$. This result destroyed the belief in the monist assignment of ordinals (here ε_0) to theories (here **PA**) ... but this was already discussed in 7.C.

The proof essentially involves the functor $\boldsymbol{\Lambda}$; but in fact, *there are as many formulations of the hierarchy theorems as there are technical ways (Kleene's O, trees...) of defining the indexing sets of hierarchies.* If one considers $\boldsymbol{\Lambda}$, one gets a theorem of comparison of hierarchies, when the hierarchies are indexed by *dendroids* (see [5], Ch. 7). However, such a formulation is too far from the familiar use of Kleene's O: if we want to formulate the theorem for Kleene's O, we must adapt the construction of $\boldsymbol{\Lambda}$ to a slightly different context; this is exactly what we shall do below. Before giving the proof, it could be interesting (as a kind of cultural background) to say a few generalities on the – now obsolete – question of *Bachmann collections, Bachmann hierarchies.* The precise technical details of these questions will be omitted... . In 1950, Bachmann introduced his main concepts [86]:

(i) *Bachmann collection*: a Bachmann collection of type α and height β consists in assigning, for all *limit* $\beta' \leq \beta$, a *fundamental sequence* $([\beta']\,\xi)_{\xi<T(\beta')}$ enjoying the following properties:

- The sequence $[\beta']\,\xi$ is strictly increasing and continuous at limit points.
- $\beta' = \sup\limits_{\xi<T(\beta')} ([\beta']\,\xi)$ and $T(\beta') \leq \alpha$.
- A certain numer of technical conditions relating the various fundamental sequences, and that we omit... .

(Example: if $e \in O$, then 5.A.7 (iii) defines a Bachmann collection of height $\|e\|$ and type ω.)

(ii) *Bachmann hierarchy*: assume that B is a Bachmann collection of type V, where V is a regular cardinal; then we define a function $\boldsymbol{\lambda}_B$ from V to V; the definition is by induction on the height $\|B\|$ of B:

- If $\|B\| = 0$, let $\boldsymbol{\lambda}_B(x) = (1+x)^x$ for all $x < V$.
- If $\|B\| = a+1$, let B' be the restriction of B to a; then

$$\boldsymbol{\lambda}_B(x) = \boldsymbol{\lambda}_{B'}(x) + 1 + \boldsymbol{\lambda}_{B'}(x+1) + 1 \ .$$

- If $\|B\|$ is limit, and $T(\|B\|) < V$, define, for all $\xi < T(\|B\|)$, a Bachmann collection B_ξ, by restricting B to $[\|B\|]\,\xi$; consider the ordinals

$$A^x_\xi = \boldsymbol{\lambda}_{B_\xi}(x) + 1 + \boldsymbol{\lambda}_{B_\xi}(x+1) + 1$$

then

$$\boldsymbol{\lambda}_B(x) = \sup_{\xi < T(\|B\|)} (A^x_\xi) \ .$$

- If $\|B\|$ is limit, and $T(\|B\|) = V$, then we define for $x, \xi < V$, ordinals A^x_ξ, exactly as above; then

$$\boldsymbol{\lambda}_B(x) = \sup_{\xi < x} (A^x_\xi) \ .$$

(Example: if B is the Bachmann collection associated to the point $e \in O$ by means of 5.A.7 (iii), then it is immediate that the number-theoretic functions $\boldsymbol{\lambda}_B$ and $\boldsymbol{\lambda}_e$ coincide… .)

The Bachmann hierarchy is traditionally used to construct large recursive ordinals, for instance, assume that V is $\aleph_1$ (in fact $V = \omega_1^{(K)}$ suffices!), and consider the ordinal $\varepsilon_{\Omega+1} = \sup_n \Big(\Omega(\Omega ... \Omega)\Big)$ (n times) (with $\Omega = V = \aleph_1$). Then it is immediate to define a structure of Bachmann collection of type Ω and height $\varepsilon_{\Omega+1}$, say $\varepsilon_{\Omega+1}$. Then one easily shows that the ordinal

$$\eta_0 = \boldsymbol{\lambda}_{\varepsilon_{\Omega+1}}(\omega)$$

is recursive. This ordinal is called the *Howard ordinal*. The name comes from the fact that Howard is the first to have explicitly proposed this ordinal as "the" ordinal of ID_1 [87]. In fact, a closer look at the definition of $\boldsymbol{\lambda}$ shows that, when x is $< V$, the ordinal $\boldsymbol{\lambda}_B(x)$ is naturally equipped with a structure of Bachmann collection of type x; in particular, η_0, is equipped with a structure of Bachmann collection of type ω, η_0, and which may be identified with an element of Kleene's O.

If, instead of starting with the Bachman collection $\varepsilon_{\Omega+1}$, one starts with ε_{Ω_n+1} (with $\Omega_n = \aleph_n$, $n \neq 0$) then one can construct $\eta_n^{n-1} = \boldsymbol{\lambda}_{\varepsilon_{\Omega_n+1}}(\Omega_{n-1})$, together with a Bachmann collection of type Ω_{n-1} and height η_n^{n-1}, denoted η_n^{n-1}; we construct progressively ordinals η_n^p and Bachmann collections $-\eta_n^p$ of height η_n^p and type Ω_p, such that

$$\eta_n^p = \boldsymbol{\lambda}_{\eta_n^{p+1}}(\Omega_p)$$

(the formula is valid for $p = n-1$ if $\eta_n^n = \varepsilon_{\Omega_n+1}$, and for $p = 0$, if $\Omega_0 = \omega$).

This enables us to construct recursive ordinals η_n and elements of Kleene's O, η_n, by

$$\eta_n = \eta_n^0 \ , \qquad \eta_0 = \eta_n^0 \ .$$

These ordinals η_n are traditionally associated with the theories of $n+1$-times iterated inductive definitions ID_{n+1} (see 11.5). It is consistent with this notations to use $\eta_{-1} = \varepsilon_0$, $\eta_{-1} = \varepsilon_0$.

(We have said enough on Bachmann collections to understand the proof that follows; however, the reader may be curious to know something about the further developments of these constructions:

(i) Following Bachmann, people made constructions of more and more complicated Bachmann collections, typically Pfeiffer [99] and Isles [89], [90]. Isles's constructions involved the use of "large" cardinals (in fact their recursive analogues), inaccessibles and Mahlos. The awful complexity of the construction has clearly shown that the conceptual framework of Bachmann collections,... was terribly insufficient.

(ii) A great progress has been achieved by the introduction [91] (due to Aczel, after a suggestion of Feferman) of the so-called "ϑ- functins".

> The principal advantage of this method is that the tedious construction of fundamental sequences is avoided. Bridge [92] has shown the equivalence of this approach with Isles's. Later on, Buchholz [93] has shown that (an inessential variant of) ϑ can be used to define rather large recursive ordinals: this is the so-called *Buchholz's system*, which is widely used in Π^1_1 proof-theory. (See for instance the encyclopedic text [3] for definitions and applications... .)

Now, let us give an idea of the proof of our theorem; the ingredients are

(i) A functorialization of the concept of Bachmann collection; it is of course necessary to slightly modify the original notions: this leads to *rungs*; and *ladders* are functors from ordinals to rungs preserving $\varinjlim$ and $\wedge$.

(ii) It is possible, if L is a ladder to define ΛL – Λ is related to $\mathbf{\Lambda}$, but is not defined on the same objects – which is a function from $0n$ to $0n$ (in fact a dilator) by:

$$(\Lambda L)(x) = \boldsymbol{\lambda}_{L(x^+)}(x)$$

where x^+ is the smallest cardinal $> x$; ($L(x^+)$ is a rung of type x^+, hence something close to a Bachmann collection of type x^+, hence $\boldsymbol{\lambda}_{L(x^+)}$ can be defined...). In fact the ordinals $(\Lambda L)(x)$ are naturally equipped with structures of rungs of type x, that we shall also denote by $(\Lambda L)(x)$. In particular ΛL is a ladder.

(iii) The value $(\Lambda L)(n)$ is equal to:

- $\boldsymbol{\lambda}_{L(\omega)}(n)$, because $\omega = n^+$.
- $\boldsymbol{\gamma}_{(\Lambda L)(\omega)}(n)$, hence
  $\boldsymbol{\lambda}_{L(\omega)} = \boldsymbol{\gamma}_{(\Lambda L)(\omega)}$.

This is precisely the hierarchy theorem; as a corollary

$$\boldsymbol{\lambda}_{\varepsilon_0} = \boldsymbol{\gamma}_\eta \ ,$$

and more generally

$$\boldsymbol{\lambda}_{\eta_p} = \boldsymbol{\gamma}_{\eta_{p+1}}$$

for all p.... From this it follows that the traditional assignment

PA	$\mathbf{ID}_1$	$\mathbf{ID}_2$	$\mathbf{ID}_3$	...	$\mathbf{ID}_n$	...
ε_0	η_0	η_1	η_2	...	η_{n-1}	...

can be replaced by the new one

PA	$\mathbf{ID}_1$	$\mathbf{ID}_2$	$\mathbf{ID}_3$	...	$\mathbf{ID}_n$	...
η_0	η_1	η_2	η_3	...	η_n	...

But now, I think that the result has been sufficiently well explained and introduced, and let us give the detailed proof:

9.A.1. Definition.
Let r and a be ordinals; a **rung** R of **height** r and **type** a is a 4-uple $(a, r, T, [\cdot]\,\cdot)$ such that:

(RG1) T is a function from $r+1$ to $a+1$.
(RG2) For all $y \leq r$, $[y]\,\cdot$ is a strictly increasing and continuous function from $T(y)+1$ to $y+1$:

- $\xi \leq T(y) \rightarrow [y]\,\xi \leq y$.
- $\xi' < \xi \leq T(y) \rightarrow [y]\,\xi' < [y]\,\xi$.
- ξ limit $\rightarrow [y]\,\xi = \sup_{\xi'<\xi} [y]\,\xi'$

(RG3) For all y, b such that $y \leq r$ and $b \leq T(y)$:

(i) $T([y]\,b) = b$.

(ii) $c \leq b \rightarrow \big[[y]\,b\big]\,c = [y]\,c$.

(RG4) If $y \leq r$, then

(i) $[y]\big(T(y)\big) = y$.

(ii) If y is limit, then $T(y)$ is limit.

(RG5) Assume that $y, z \leq r$, and $b < T(y)$, and that $[y]\,b < z < [y]\,(b{+}1)$; then $[y]\,b < [z]\,0$.

9.A.2. Definition.
The function $[y]\cdot$ from $T(y)+1$ to $y+1$ is the **fundamental sequence** of y (in R).

9.A.3. Notations.

(i) $\|R\|$ will denote the *height* of R, whereas $\underline{t}(R)$ will denote its *type*.

(ii) We shall try to follow the following notational pattern: we denote a rung by capital letters, and its height by the corresponding small letter: for instance R''_n and r''_n.

9.A.4. Remark.
Compared to the traditional concept of Bachmann collection, the main improvement is that we allow non-trivial fundamental sequences for non limit points... .

9.A.5. Examples.

(i) For all ordinals a and b such that $b \leq a$, one defines a rung $\underline{b}_a$, of height b and type a, as follows: if $y \leq b$, then $T(y) = y$, and if $z \leq y$, then $[y]\,z = z$; we use the abbreviation $\underline{a}$ instead of $\underline{a}_a$.

(ii) If $R = (a, r, T, [\cdot]\,\cdot)$ and $S = (a, s, U, |[\cdot]|\,\cdot)$ are rungs of the same type a, one defines a new rung $R{+}1{+}S = (a, r{+}1{+}s, V, \{\cdot\}\,\cdot)$ as follows:

(1) If $z < r+1$, then $V(z) = T(z)$; if $c < T(z)$ $\{z\}\,c = [z]\,c$.

(2) If $z \leq s$, then $V(r{+}1{+}z) = U(z)$; if $c < U(z)$, then $\{r{+}1{+}z\}\,c = r+1+|[z]|\,c$.

$R{+}1{+}s$, the **sum** of R and S, is easily shown to be a rungof type a. A particular case is when $S = \underline{0}_a$: $R{+}1{+}S$ is, in that case, denoted by $R+1$.

(iii) If $R = (a, r, T, [\cdot]\,\cdot)$ is a rung, and if $s \leq r$, one defines $R\!\restriction s = (a, s, U, |[\cdot]|\,\cdot)$, simply by restricting T and $[\cdot]\,\cdot$ to $s \ldots$.

(iv) If $R = (a, r, T, [\cdot]\,\cdot)$ and $S = (a, s, U, |[\cdot]|\,\cdot)$ are rungs of the same type a; then we define the rung $R \times S = (a, r \times s, V, \{\cdot\}\,\cdot)$, as follows:

(1) If $s = 0$, then $R \times S = \underline{0}_a$.

(2) If $s \neq 0$, then $r \times s = \sup\limits_{z<s} (r \times z + 1 + r + 1)$, $V(r \times s) = U(s)$, and if $c \leq U(s)$ $\{r \times s\}\, c = r \times |[s]|\, c$. If $z < s$, let $Z = S\!\restriction z$; then $(R \times S)\!\restriction (r \times z + 1 + r) = R \times Z + 1 + R$. ($R \times S$ is the **product** of R and S; this is easily shown to be a rung of type a; $r \times s$ is close to the product $r \cdot s$.)

(v) If $R = (a, r, T, [\cdot]\,\cdot)$, and $S = (a, s, U, |[\cdot]|\,\cdot)$ are rungs of the same type a, then we define a rung $(1 + R)^{\cdot S} = (a, r^{\cdot s}, V, \{\cdot\}\,\cdot)$, as follows:

(1) If $s = 0$, then $(1 + R)^{\cdot S} = \underline{1}_a$.

(2) If $s \neq 0$, then $r^{\cdot s} = \sup\limits_{z<s} (r^{\cdot z} + 1 + r^{\cdot z} \times r + 1)$ $V(r^{\cdot s}) = U(s)$, and for $c \leq U(s)$, $\{r^{\cdot s}\}\, c = r^{\cdot |[s]|\, c}$; if $z < s$, let $Z = S\!\restriction z$. Then $(R^{\cdot S})\!\restriction (r^{\cdot z} + 1 + r^{\cdot z} \times r) = R^{\cdot Z} + 1 + R^{\cdot Z} \times R$.

9.A.6. Proposition.

(i) If y and z are such that $[y]\, b < [z]\, c < [y]\, (b+1)$, then $z < [y]\, (b+1)$.

(ii) If $[y]\, b = [z]\, c$ and $y \leq z$, then $b = c$ and $y = [z]\, T(y)$.

Proof. (i) $z = [y]\, (b+1)$ is impossible by (RG3) (ii); suppose that $z > [y]\, (b+1)$; then $[y]\, (b+1)$ is not of the form $[z]\, d$ (again by (RG3) (ii)). Hence for some d, we have: $[z]\, d < [y]\, (b+1) < [z]\, (d+1)$ (and $d \geq c$). By (RG5), we obtain $[z]\, d < [y]\, 0$, hence $[y]\, b < [z]\, c \leq [z]\, d < [y]\, 0$, contradiction with (RG1).

(ii) (RG3) (i) yields $b = T([y]\, b) = T([z]\, c) = c$, so $b = c$; if $y = [z]\, d$ for some d, then $T(y) = d$, $y = [z]\, \big(T(y)\big)$; otherwise, for some d, $[z]\, d < y < [z]\, (d+1)$ and $c \leq d$; so $[y]\, 0 \leq [y]\, b = [z]\, c \leq [z]\, d < [y]\, 0$, contradiction. □

9.A.7. Proposition.
For all $z \le r$, there is a greatest $y \le r$ such that $z = [y]\,T(z)$.

Proof. Assume the contrary, and let $y = \mathsf{sup}\,\{t\,;\,[t]\,T(z) = z\}$. By hypothesis, $[y]\,T(z) \neq z$, or $T(y) > T(z)$. As a first consequence, y is limit (since y is of the form $\mathsf{sup}\,(A)$, and $y \notin A$), so there is a $b < T(y)$ such that $z < [y]\,b$ and $t > [y]\,b$ such that $[t]\,T(z) = z$. If $t = [y]\,c$ for some c, then $[y]\,T(z) = \Big[[y]\,c\Big]\,T(z) = z$, contradiction; if $[y]\,c < t < [y]\,(c+1)$ for some c, with $b \le c < T(y)$, then $z < [y]\,c < [t]\,0$, a contradiction. $\square$

9.A.8. Proposition.

(i) Assume that the intervals I_i $(i = 0, 1)$ defined by $I_i = \Big[[y_i]\,b_i, [y_i]\,(b_i + 1)\Big[$ are such that $I_0 \cap I_1 \neq \emptyset$; then $I_0 \subset I_1$ or $I_1 \subset I_0$.

(ii) Suppose that the interval I of r is obtained as the union as a non void family $(I_i)_{i \in A}$, with I_i of the form $\Big[[y_i]\,b_i, y_i\Big[$; then there exists $y \le 1$ and $b \le T(y)$ such that $I = \Big[[y]\,b, y\Big[$; furthermore, the interval $[b, T(y)[$ is included in the union of the intervals $[b_i, T(y_i)[$.

Proof. (i) Assume for instance that $[y_0]\,b_0 < [y_1]\,b_1 < [y_0]\,(b_0 + 1)$, then by (RG5) and 9.A.6 (ii), we get $[y_0]\,b_0 < [y_1]\,0 < y_1 < [y_0]\,(b_0 + 1)$: hence $I_0 \subset I_1$, and the extremities of I_1 are distinct from the extremities of I_0.

(ii) Nothing is changed if one assumes that for all i, $T(y_i) = b_i + 1$. We first treat a particular case: assume that the intervals I_i are pairwise comparable for inclusion. In that case observe that (as a consequence of the fact that the extremities are distinct), if $I_i \underset{\neq}{\subset} I_j$, then $[y_i]\,b_i < [y_j]\,b_j < y_j < y_i$: if the family (I_i) would contain infinitely many distinct elements, then one of the sequences $[y_i]\,b_i$ or y_i would contain a s.d.s. of r. Hence the family contains only finitely many distinct intervals, and the property is immediate in that case. It remains now to consider the case where the intervals are pairwise incomparable w.r.t. inclusion; by (i) above, they are necessarily pairwise disjoint. Choose $i_0 \in A$, and let x be maximum with the property that $[x]\,T(y_{i_0}) = y_{i_0}$. Let B be a subset of A, maximal among those enjoying:

(1) If $i \in B$, then $y_i = [x]\,T(y_i)$.

(2) $H = \bigcup_{i \in B} I_i$ is an interval.

Since J is obviously of the form $\Big[[x]\,b, [x]\,c\Big[$, it will suffice to prove that $I = J$, i.e. that $A = B$; suppose that $[x]\,c = [y_j]\,b_j$ for some j, then by considering $J \cup \{j\}$ one gets a contradiction; similarly, if $[x]\,b = y_j$ for some $j \ldots$. $\square$

9.A.9. Definition.
Let $S = (b, s, U, |[\cdot]|\,\cdot)$ be a rung; if $f \in I(a,b)$ we define:

(i) A subset S^*_f of S:

$$z \notin S^*_f \;\leftrightarrow\; \exists x \leq b \;\exists u \;< \\ U(x)\,\Big(u \notin rg(f) \wedge |[x]|\,u \leq z < |[x]|\,(u+1)\Big)\,.$$

(Hence CS^*_f appears as a union of intervals of the form $\Big[|[y_i]|\,b_i, |[y_i]|\,(b_i+1)\Big[$.)

(ii) An ordinal $r =$ order type of S^*_f, together with a function m^s_f (in short m_f), $m^s_f \in I(r,s)$, defined by $rg(b^s_f) = S^*_f$. By abuse of notations, we shall also write: $m^s_f(r) = s$ (**mutilation function**).

(iii) A 4-uple $f^{-1}(S) = (a, r, T, [\cdot]\,\cdot)$:

(1) $\hat{f}\Big(T(z)\Big) = U\Big(\hat{m}_f(z)\Big) \qquad (z \leq r)$

(2) $m_f([z]\,c) = |[m_f(z)]|\,f(x) \qquad (x < T(z),\; z \leq r)$

(3) $[z]\,T(z) = z \qquad (z \leq r)\,.$

9.A.10. Lemma.
$\hat{m}_f([z]\,c) = |[m_f(z)]|\,\hat{f}(x)\,.$

Proof. Let $Z = \hat{m}_f([z]\,c)$, $Z' = m_f([z]\,c)$; then $[Z, Z'[$ is either void (easy: left to the reader) or non void, and maximal in $C\,S_f^*$; so by 9.A.8 (ii), this interval is of the form $\Big[|[Z']|\,u, Z'\Big[$, and $[u, U(Z')[$ is included in $C\,rg(f)$; by maximality, it follows that $u \in rg(\hat{f})$, and this is only possible when $u = \hat{f}(x)$; so

$$\hat{m}_f(|[z]|\,c) = Z = |[Z']|\,\hat{f}(c) = |[m_f([z]\,c)]|\,\hat{f}(c) = |[m_f(z)]|\,\hat{f}(c)\ .$$

(The last equality is obtained by considering separately the trivial case $z = [z]\,c$, and the case $c < T(z)$: in that case one uses Definition 9.A.9 (iii) (2) above... . □

9.A.11. Theorem.

(i) The 4-uple $(a, r, T, [\cdot]\,\cdot)$ of 9.A.9 (iii) is a rung.

(ii) $f^{-1}\Big(g^{-1}(S)\Big) = (gf)^{-1}(S)$.

Proof. (RG1), (RG2), (RG3) (ii) and (RG4) (i) are immediate; we verify the remaining properties:

– (RG3) (i): By 9.A.10 $\hat{m}_f([z]\,c) = m_f(z)\,|[\hat{f}(c)]|$, hence we get: $\hat{f}\Big(T([z]\,c)\Big) = U\Big(\hat{m}_f([z]\,c)\Big) = U\Big(|[m_f(z)]|\,\hat{f}(c)\Big) = \hat{f}(c)$, hence $T([z]\,c) = c$.

– (RG4) (ii): If $z \leq r$ is limit, then $\hat{m}_f(z)$ is limit, hence we get: $\hat{m}_f(z) = \sup\limits_{c<U\left(\hat{m}_f(z)\right)} (|[\hat{m}_f(z)]|\,c)$; since $U\Big(\hat{m}_f(z)\Big) = \hat{f}\Big(T(z)\Big)$, one gets $\hat{m}_f(z) = \sup\limits_{d<T(z)} |[\hat{m}_f(z)]|\,f(d) = \sup\limits_{d<T(z)} |[m_f(z)]|\,f(d) = \sup\limits_{d<T(z)} \Big(m_f([z]\,d)\Big)$. Hence $z = \sup\limits_{d<T(z)} [z]\,d$.

– (RG5): If $[z]\,c < t < [z]\,(c+1)$, then $m_f([z]\,c) < m_f(t) < \hat{m}_f\Big([z]\,(c+1)\Big)$; let $v = m_f(z)$; then $|[v]|\,f(c) < m_f(t) < |[v]|\,(f(c)+1)$; hence $m_f([z]\,c) = |[v]|\,f(c) < |[m_f(t)]|\,f(0) = m_f([t]\,0)$: hence $[z]\,c < [t]\,0$.

(ii) is left to the reader. □

9.A.12. Remark.
The proofs 9.A.11 and 9.A.10 are slightly incorrect: we have not verified the possibility of defining $f^{-1}(A)$ as in 9.A.9 (iii); for instance $[z]\,c$ can be defined, as a consequence of the remark that, if $Z \in S_f^*$, then $|[Z]|\,c \in S_f^*$ iff $c \in rg(f)$ (for all $c < U(z)$).

9.A.13. Proposition.
The *mutilation* "commutes" with sum, products and exponentials.

Proof. This means that:

$$f^{-1}(R+1+S) = f^{-1}(R) + 1 + f^{-1}(S)$$
$$f^{-1}(R \times S) = f^{-1}(R) \times f^{-1}(S)$$
$$f^{-1}(R^{\cdot S}) = f^{-1}(R)^{\cdot f^{-1}(S)} \ .$$

These properties are immediate. □

9.A.14. Remark.
The behaviour of the rungs $\underline{b}_a$ w.r.t. mutilation is more complicated: if $f \in I(a, a')$, then $f^{-1}(\underline{a}) = \underline{a}'$; but $f^{-1}(\underline{b}_a) = \underline{b}'_a$, where b' is defined by: $\forall z(z < b' \leftrightarrow f(z) < b)$, i.e. $\hat{f}(b') \leq b \leq f(b')$, when $b < a$.

9.A.15. Definition.
Let K be a finite set; we define the category $K - \mathbf{ON}$ as follows:

objects: pairs (x, d), where $x \in 0n$ and d is a function from K to x.
morphisms: the set $I(x, d\,;\, y, e)$ of those $f \in I(x, y)$ s.t. $ef = d$.

9.A.16. Definition.
Let V be a regular cardinal (in practice V will be an admissible ordinal...), and assume that (V, d) is an object of $K - \mathbf{ON}$; then we define:

(i) The category $K - \mathbf{ON} \leq (V, d)$:

objects: pairs (x, e) s.t. $I(x, e\,;\, V, d) \neq \emptyset$.

(ii) The category $K - \mathbf{ON} < (V,d)$:

objects: pairs (x,e) s.t. $I(x,e\,;\,V,d) \neq \emptyset$ and $x < V$.

In both cases the morphisms are given by $I(x,e\,;\,x',e')$, i.e. these are full subcategories.

9.A.17. Definition.

The following data define a category **RG**:

objects: rungs $(a, r, T, [\cdot]\,\cdot)$.
morphisms: the set $I(R,S)$ of all $f \in I\Big(t(R), t(S)\Big)$ such that $f^{-1}(S) = R$.

9.A.18. Definition.

(i) A **ladder** is a functor from **ON** to **RG** such that:

(1) $t\Big(L(x)\Big) = x$.

(2) $L(f) = f$.

(ii) One defines K-ladders, $K \leq (V,d)$-ladders, $K < (V,d)$-ladders, by replacing **ON** by $K - \mathbf{ON}$, $K - \mathbf{ON} \leq (V,d)$, $K - \mathbf{ON} < (V,d)$; they enjoy the conditions:

(1) $t\Big(L(x,e)\Big) = x$.

(2) $L(f) = f$.

9.A.19. Examples.

(i) $L(x) = \underline{x}$ defines a ladder.

(ii) If $k \in K$, then $L(x,d) = \underline{d(k)}_x$ defines (using 9.A.14) a K-ladder.

(iii) Sum, products, and exponentials can be used to construct new ladders and K-ladders:

$$
\begin{array}{lcll}
L(x) & = & \underline{x} + 1 + \underline{x} & \text{defines a ladder} \\
L(x,d) & = & \underline{x} \times \underline{d(k)}_x & \text{defines a } K\text{-ladder.}
\end{array}
$$

9.A.20. Definition.
Assume that $S = (a, s, U, |[\cdot]| \cdot)$ and $R = (s, r, T, |[\cdot]| \cdot)$ are rungs (the type of R is equal to the height of S); then one defines a new rung $RS = (a, r, V, \{\cdot\} \cdot)$ (the **composition** of R and S) by:

(i) $V(x) = U\Big(T(x)\Big)$ for all $x < r$.

(ii) $\{x\}\, d = [x]\, (|[T(x)]|\, d)$ for all $d \leq V(x)$.

(The verification that RS is a rung is left to the reader.)

9.A.21. Definition.
Assume that L and L' are ladders; then one defines a new ladder $L'' = L_0(\mathsf{Id} + 1 + L')$, as follows:

$$L''(x) = L(x + 1 + \|L'(x)\|)\Big(\underline{x} + 1 + L'(x)\Big) \ .$$

The composition is also defined on K-ladders by:

$$L''(x, d) = L(x + 1 + \|L'(x, d)\|, e)\Big(\underline{x} + 1 + L'(x, d)\Big) \ ,$$

where e is defined by $e(k) = d(k)$ for all $k \in K$.
(The definition can still be used for $K < (V, d)$-ladders, using the fact that $x + 1 + \|L'(x, d)\| < V$ for all (x, d) s.t. $d < V$.)

9.A.22. Theorem.
The composition maps ladders (resp. K-ladders, $K < (V, d)$-ladders into themselves.

Proof. We prove the theorem for K-ladders; the result depends on a lemma.

9.A.23. Lemma.
Let L be a K-ladder, and let $z \leq \|L(x, d)\|$; assume that $T(z) < x$, and that, for some $u < T(z)$ we have $[u, T(z)] \cap rg(d) = \emptyset$; then one can find $z', z < z' \leq L(x)$ such that: $[z']\, u \leq z < [z']\, (T(z) + 1)$.

Proof. Define (x', d') and $f, g \in I(x', d'\, ;\, x, d)$ by: $rg(f) = x - [u, T(z)[$, $rg(g) = x -]u, T(z)]$; then f and g differ only on the argument x; hence, if we define (x'', d'') and $h \in I(x'', d''\, ;\, x', d')$ by $rg(h) = x' - \{u\}$, it is clear that $fh = gh = f \wedge g$. Observe that $rg(gh) = x - [u, T(z)]$; assume now that the conclusion of the lemma is false; then obviously $z \in L(x,d)^*_{fh} = L(x,d)^*_{gh}$. Let $f^* = m_f^{L(x,d)}$, $g^* = m_g^{L(x,d)}$, $h^* = m_h^{L(x',d')}$; then $f^*h^* = g^*h^*$, and $z \in rg(f^*h^*)$, i.e. $z = f^*h^*(z'')$ for some $z'' \in \|L(x'',d'')\|$. Let $z'_0 = h^*(z'')$; then $z = f^*(z'_0) = g^*(z'_0)$. We compute the value $T'(z'_0)$ (in $L(x',d')$) by means of 9.A.9 (iii) (1) applied to f and g:

$$\hat{f}\Big(T'(z'_0)\Big) = T\Big(\hat{f}^*(z'_0)\Big) \quad \text{implies } T'(z') = u$$
$$\hat{g}\Big(T'(z'_0)\Big) = T(\hat{g}^*(z'_0)\Big) \quad \text{implies } T'(z') = u + 1\,.$$

We have obtained a contradiction. □

<u>Proof of 9.A.22</u>. Let $L'' = L \circ (\mathsf{Id} + 1 + L')$, let $S'' = L''(x,d)$, $S' = \Big(\underline{x}+1+L'(x,d)\Big)$, $S = L(y,e)$, with $y = \|S'\|$ and $e(k) = d(k)$ for all $k \in K$. Assume that $S'' = (x, s'', T'', \{\cdot\}\,\cdot)$, $S' = (x, y, T', |[\cdot]|\,\cdot)$, $S = (y, s'', T, [\cdot]\,\cdot)$; by definition $S'' = SS'$. We prove that $S''^*_f = S^*_g$, with $g = m_f^{S'}$: S''^*_f is obtained by removing all intervals $\Big[[z]\,(|[z']|\,t), [z]\,\Big(|[z']|\,(t+1)\Big)\Big[$ where $t \notin rg(f)$, whereas S^*_g is obtained by removing all intervals $\Big[[z]\,u, [z]\,(u+1)\Big[$, with $u \notin rg(g)$: the interval $\Big[[z]\,(|[z']|\,t), [z]\,\Big(|[z']|\,(t+1)\Big)\Big[$, when $t \notin rg(f)$, is obvious by a union of intervals of the form $\Big[[z]\,u_i, [z]\,(u_i+1)\Big[$ for a family (u_i), $u_i \notin rg(g)$: this proves that $S^*_g \subset S''^*_f$. Assume that the opposite inclusion is false: this means that some interval $\Big[[z]\,t, [z]\,(t+1)\Big[$, with $t \notin rg(g)$, is not included in any interval $\Big[[b]\,(|[b']|\,u), [b]\,\Big(|[b']|\,(u+1)\Big)\Big[$, for any $u \notin rg(f)$. Choose z and t such that:

– $[z]\,t$ is minimum with this property.

– z is not of the form $[z']\,T(z)$ for any $z' > z$ (9.A.7).

Now observe that:

(i) $T(z) > x$ (otherwise, take $b = z$, $b' = T(z)$, $u = t$).

(ii) $T(z) < y$ (otherwise, take $b = z$, and let b' and u be such that $|[b']|\,u \leq t < |[b']|\,(u+1)$).

(iii) $[u, T(z)] \cap rg(z) = \emptyset$: because $rg(e) \subset x$: apply (i).

By (ii) and (iii), and 9.A.23, we conclude that there is some z' such that: $[z']\,t \leq z < [z']\,(T(z)+1)$. Hence we can write $[z']\,t' < z < [z']\,(t'+1)$ for some $t' \leq T(z)$ (the equality $z = [z']\,t$ would entail $z = [z']\,T(z)...$). The interval $\big[[z']\,t', [z']\,(t'+1)\big[$ is not included in CS''^{*}_{f} (since it contains $\big[[z]\,t, [z]\,(t+1)\big[$), but is included in CS''_g: observe that: $[t, T(z)+1[\subset C\,rg(g)$.

(Proof. Let $b = z$, and choose b' and u with $T'(b') = u+1$, $u \notin rg(f)$ and $|[b']|\,u \leq t < b'$; such a choice is possible because $t \notin rg(g)$. It suffices to show that $T(z) < b'$: but, if $b' \leq T(z)$, then $\big[[z]\,t, [z]\,(t+1)\big[\subset \big[[b]\,(|[b']|\,u), [b]\,\big(|[b']|\,(u+1)\big)\big[$. □)

Finally, $[z']\,t < [z]\,t$, and this contradicts the minimality of $[z]\,t$. We have therefore proved that $S''^{*}_{f} = S^{*}_{g}$.

The end of the proof offers no special difficulty: asssume that $f \in I(x', d'\,;\, x, d)$ and let $R = (y', r'', U, [\cdot]_1\cdot) = f^{-1}(S)$, $R' = (x', y', U', |[\cdot]|_1\cdot) = f^{-1}(S')$, $R'' = (x', r'', \{\cdot\}_1\cdot) = f^{-1}(S'')$. It suffices to show that $R'' = RR'$ (since $L(y', e') = R$, with $e'(k) = d'(k)$ for all $k \in K$, $R' = \underline{x}'+1+L'(x', d')$). We get:

(1) $\hat{f}\big(U''(z)\big) = T''\big(\hat{m}^{S''}_{f}(z)\big) = T'\Big(T\big(\hat{m}^{S''}_{f}(z)\big)\Big) = T'\Big(T\big(\hat{m}^{S}_{g}(z)\big)\Big) = T'\Big(\hat{g}\big(U(z)\big)\Big) = T'\Big(\hat{m}^{S'}_{f}\big(U(z)\big)\Big) = \hat{f}\Big(U'\big(U(z)\big)\Big)$.
Hence $U''(z) = U'\big(U(z)\big)$.

(2) $\hat{m}^{S''}_{f}(\{z\}_1 u) = \{\hat{m}^{S''}_{f}(z)\}\,\hat{f}(u) = [\hat{m}^{S''}_{f}(z)]\,\Big(\Big|\Big[T\big(\hat{m}^{S''}_{f}(z)\big)\Big]\Big|\,\hat{f}(u)\Big) = [\hat{m}^{S''}_{f}(z)]\,\Big(\Big|\Big[T\big(\hat{m}^{S}_{g}(z)\big)\Big]\Big|\,\hat{f}(u)\Big) = [\hat{m}^{S}_{g}(z)]\,\Big(\Big|\Big[\hat{g}\big(U(z)\big)\Big]\Big|\,\hat{f}(u)\Big) = [\hat{m}^{S}_{g}(z)]\,\Big(\Big|\Big[\hat{m}^{S'}_{f}\big(U(z)\big)\Big]\Big|\,\hat{f}(u)\Big) = |\hat{m}^{S}_{g}(z)]\,\Big(\Big|\Big[\hat{m}^{S'}_{f}\big(U(z)\big)\Big]\Big|_1 u\Big) = \hat{m}^{S}_{g}\big([z]_1(|[U(z)]|_1 u)\big) = \hat{m}^{S''}_{f}\big([z]_1(|[U(z)]|_1 u)\big)$.
Hence $\{z\}_1 u = [z]_1(|[U(z)]|_1 u)$. □

9.A.24. Theorem.
It is possible to define a function Λ which maps $K \leq (V, d)$-ladders on $K < (V, d)$-ladders, and such that:

(1) If (W, d') is an object of $K \leq (V, d)$ (W regular cardinal) and if L' is

the restriction of L to $K-\mathbf{ON} \leq (W,d')$, then $\Lambda L'$ is the restriction of ΛL to $K_{\mathbf{ON}} < (W,d')$.

(2) If (with obvious notations) $L = L' + 1 + L''$, then $\Lambda L = \Lambda L' + 1 + M$ for some $K < (V,d)$-ladder M.

The precise definition of Λ is given during the proof.

Proof. By induction on $A = \|L(V,d)\|$.

Case 1. $A = 0$; define $(\Lambda L)(x,e) = \underline{x}$; properties (1) and (2) are trivial.

Case 2. $A \neq 0$, but $T(A) = 0$; then it is easily checked that $L = L'+1$ (i.e. $L(x,e) = L'(x,e)+1$) for a certain $K \leq (V,d)$-ladder L'. If $A' = \|L'(V,d)\|$, then clearly $A = A'+1$. We use the abbreviation $\boldsymbol{\vartheta}(M)$ (when M is a $K < (V,d)$-ladder) to denote $M + 1 + \big(M \circ (\mathsf{Id}+1)\big)$. One defines: $(\Lambda L)(x,e) = \boldsymbol{\vartheta}(\Lambda L')(x,e) + 1$. (1) and (2) are trivially fulfilled.

Case 3. $A \neq 0$; we shall use the following abisses of notations:

- T and $[\cdot]\cdot$ are used for all rungs that may occur; we assume that it is clear from the context which rung these notions are part from.
- When we use (x,c), then c is considered as a finite sequence $(x_i)_{i \in K}$ of ordinals $< x$ (and for instance it is possible to replace (x,c) by (y,c) when $y \geq x$.... .

We define $K' = K \cup \{k\}$, where k is a new point. If $z < T(A)$, then we define (V,d_z), an object of $K'-\mathbf{ON}$, as follows:

- $d_z(k') = d(k')$ when $k' \in K$.
- $d_z(k) = z$.

(More generally, if (x,e) is an element of $K-\mathbf{ON}$, we define, for all $z < x$, an element (x,e_z) of $K'-\mathbf{ON}$, by $e_z(k') = e(k')$ when $k' \in K$, $e_z(k) = z$.)

9.A.25. Lemma.
If $T(A) \neq 0$, and $z < T(A)$, then there is a (unique) $K' \leq (V,d_z)$-ladder L_z such that: $L_z(V,d_z) = L(V,d)\restriction([A]\,(z+1)-1)$.

Proof. Assume that (x,e') is an element of $K'-\mathbf{ON} \leq (V,d_z)$; then e'

can be uniquely written as $e_{z'}$, for some e and some z'. We claim, that (if $A_{x,e} = \|L(x,e)\|$) $z' < T(A_{x,e})$.

(<u>Proof</u>. In fact we have the following possibilities for the ordinal $T(A_{x,e})$:

(i) $\exists k_0 \in K\ \forall x\ \forall e\quad T(A_{x,e}) = e(k_0)$

(ii) $\exists k_0 \in K\ \forall x\ \forall y\quad T(A_{x,e}) = e(k_0) + 1$

(iii) $\forall x\ \forall e\qquad T(A_{x,e}) = 0$

(iv) $\forall x\ \forall e\qquad T(A_{x,e}) = x$

(see Exercise 9.B.5 for instance); then $z < T(A) = T(A_{V,d})$ and $h \in I(x, e_{z'}\,;\, V, d_z)$ implies $z' < T(A_{x,e})$:

(i) If $T(A) = d(k_0)$, then $z < d(k_0) = h_l(k_0)$; but $z = h(z')$ hence $z' < e(k_0) = T(A_{x,e})$.

(ii) If $T(A) = d(k_0)+1$, then $h(z') < he(k_0)+1$, hence $z' < e(k_0)+1 = T(A_{x,e})$

(iv) If $T(Z) = V$, then $z' < x = T(A_{x,e})$. □)

Then it is possible to define $R_{x,e,z'} = L(x,e)\restriction([A_{x,e}]\,(z'+1)-1)$. We claim that, when $h \in I(x, e_{z'}\,;\, V, d_z)$, then $h^{-1}\Big(L_z(V,d_z)\Big) = R_{x,e,z'}$. This is completely immediate... . From that it follows that $L_z(e,z') = R_{x,e,z'}$ defines a $K' - \mathbf{ON} \leq (V, d_z)$-ladder. □

Observe that $L_z(V,d_z) = [A]\,(z+1) - 1 < A$, hence we may use the induction hypothesis on L_z: then we must distinguish two subcases

1. $0 < T(A) < V$: if (x,e) is an object of $K - \mathbf{ON} < (V,d)$, define $(\Lambda L)(x,e)$ as follows:

$$\|(\Lambda L)(x,e)\| = \sup_{z<T(\|L(V,e)\|)} \left(\Big\|\Big(\boldsymbol{\vartheta}(\Lambda L_{e,z})\Big)(x, e_{z'}\Big\| + 1\right)$$

where $L_{e,z}$ is the unique $K' \leq (V, e_z)$-ladder such that

$$L_{e,z}(V,e_z) = L(V,e)\restriction([\|L(V,e)\|]\,(z+1)-1)$$

$$(\Lambda :)(x,e)\restriction\Big\|\boldsymbol{\vartheta}\Big(\Lambda L_{ez}\Big)\Big)(x,e_z)\Big\| = \Big(\boldsymbol{\vartheta}(\Lambda L_{ez})\Big)(x,e_z)$$

$$T(\|(\Lambda L)(x,e)\|) = T(\|L(x,e)\|) = T(\|L(V,e)\|)$$

$$[\|(\Lambda L)(x,e)\|]\,t = \sup_{z<t}\left(\Big\|\Big(\boldsymbol{\vartheta}(\Lambda L_{ez})\Big)(x,e_z)\Big\| + 1\right).$$

The fact that this defines a $K < (V,d)$-ladder, and that (1) and (2) hold, is immediate.

2. $T(A) = V$: we define

$$\|(\Lambda L)(x,e)\| = \sup_{z<x} \Big(\Big\|\Big(\boldsymbol{\vartheta}(\Lambda L_{e,z})\Big)(x,e_z)\Big\| + 1\Big)$$

$$T(\|(\Lambda L)(x,e)\|) = x$$

the rest of the definition being exactly as in subcase 1 above. 1 and 2 are immediate.... $\square$

9.A.26. <u>Definition</u>.
Let $R = (V, r, T, [\cdot]\,\cdot)$ be a rung of type V, where V is a regular cardinal; then we define the following hierarchies of functions from V to V:

(i) If $r = 0$, then

$$\boldsymbol{\gamma}_R(x) = 0 \qquad \boldsymbol{\lambda}_R(x) = x\ .$$

(ii) If $r \neq 0$, and r is a successor; then define $R' = R{\restriction} r-1$; then

$$\boldsymbol{\gamma}_R(x) = \boldsymbol{\gamma}_{R'}(x) + 1 \qquad \boldsymbol{\lambda}_R(x) = \Big(\boldsymbol{\vartheta}(\boldsymbol{\lambda}_R)\Big)(x) + 1\ .$$

(Here $\boldsymbol{\vartheta}(f)(x) = f(x) + 1 + f(x+1)$.)

(iii) If $r \neq 0$, and $T(r)$ is limit and $< V$; then define, for all $z < T(r)$: $R_z = R{\restriction}\,[r]\,z$

$$\boldsymbol{\gamma}_R(x) = \sup_{z<T(r)} \boldsymbol{\gamma}_{R_z}(x) \qquad \boldsymbol{\lambda}_R(x) = \sup_{z<T(r)} \boldsymbol{\lambda}_{R_z}(x)\ .$$

(iv) If $r \neq 0$ and $T(r) = V$, then, with R_z as in (iii):

$$\boldsymbol{\gamma}_R(x) = \boldsymbol{\gamma}_{R_x}(x) \qquad \boldsymbol{\lambda}_R(x) = \boldsymbol{\lambda}_{R_x}(x)\ .$$

9.A.27. <u>Remarks</u>.

(i) The important case is of course when $V = \omega$; when R is a rung of type ω and R is recursive enough, then the structure obtained from R by considering only the fundamental sequences of length ω, is likely to belong to O! Then we see that our definitions of $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma}$ in the context of O and in the context of rungs of type ω coincide... .

(ii) The case $V \neq \omega$ is of course less interesting; the hierarchy $\boldsymbol{\lambda}_R$ is of course (one of) the (many variants of) Bachmann hierarchy, adapted to rungs... .

9.A.28. Proposition.
Let L be a $K \leq (V,d)$-ladder, let $x > \mathsf{sup}\left(rg(d)\right)$; then:

(i) $\boldsymbol{\gamma}_{L(V,d)}(x) = \|L(x,d)\|$.

(ii) $\boldsymbol{\lambda}_{L(V,d)}(x) = \|(\Lambda L)(x,d)\|$.

Proof. (i) and (ii) are proved by induction on $A = \|(L(V,d)\|$.

(i): 1. If $A = 0$, then $L(V,d) = \underline{0}_V$,and $L(x,d) = \underline{0}_x$; hence $\boldsymbol{\gamma}_{L(V,d)}(x) = 0 = \|L(x,d)\|$.

2. If $A = B+1$, and $T(A) = 0$, then define a $K \leq (V,d)$-ladder L' by: $L = L'+1$; the induction hypothesis yields $\boldsymbol{\gamma}_{L'(V,d)}(x) = \|L'(x,d)\|$, hence

$$\boldsymbol{\gamma}_{L(V,d)}(x) = \boldsymbol{\gamma}_{L'(V,d)}(x) + 1 = \|L'(x,d)\| + 1 = \|L(x,d)\| \ .$$

3. If $A = B+1$, and $T(A) = t+1$; then define (with $K' = K \cup \{k\}$, $k \notin K$) d' from K' to V, which extends d, by $d'(k) = t$. We have already remarked (see the proof in the proof of 9.A.25) that $T(A) \leq x$, hence it follows that (x,d') is an object of $K' - \mathbf{ON} \leq (V,d')$; define a $K' \leq (V,d')$-ladder L' by:

$$L'(V,d') = L(V,d)\lceil B$$

and apply the induction hypothesis: we get

$$\boldsymbol{\gamma}_{L(V,d)}(x) = \boldsymbol{\gamma}_{L'(V,d')}(x) + 1 = \|L'(x',d')\| + 1 = \|L(x,d)\| \ .$$

4. If A is limit, and $T(A) < V$; then, given $t < T(A)$, define d_t from K' to V, extending d, by $d_t(k) = t$; define a $K' \leq (V,d_t)$-ladder L_t by:

$$L_t(V, d_t) = L(V, d)\restriction [A]\, t$$

and apply the induction hypothesis: we get

$$\boldsymbol{\gamma}_{L(V,d)}(t) = \sup_{t<T(A)} \boldsymbol{\gamma}_{L_t(V,d_t)}(x) = \sup_{t<T(A)} \|L_t(x, d)\| = \|L(x, d)\|$$

(using the fact that $T(\|L(x, d)\|) = T(\|L(V, d)\|)$).

5. If A is limit, and $T(A) = V$, then define K' and d_t exactly as above, when $t < x$; we define the $K' \leq (V, d_t)$-ladders L_t by:

$$L_t(V, d_t) = L(V, d)\restriction ([A]\,(t + 1) - 1)\ .$$

Then, using the induction hypothesis, we obtain:

$$\begin{aligned} \boldsymbol{\gamma}_{L(V,d)}(x) &= \sup_{t<x} (\boldsymbol{\gamma}_{L_t(x,d_t)}(x) + 1) = \\ & \sup_{t<x} \|L_t(x, d_t)\| + 1 = \|L(x, d)\|\ . \end{aligned}$$

(ii) is left to the reader. □

9.A.29. Corollary (theorem of comparison of hierarchies; Girard [5]).
Let L be a ladder; then

$$\boldsymbol{\lambda}_{L(V)} = \boldsymbol{\gamma}_{(\Lambda L)(V)}\ .$$

Proof. First, we must explain the meaning of "ΛL": if one restricts L to the category $\mathbf{ON} \leq W$ $(= \emptyset - \mathbf{ON} \leq (W, \emptyset))$, then ΛL is defined as a $\emptyset < (W, \emptyset)$-ladder, i.e. a $< W$-ladder. Using Property 1 of the Construction 9.A.24, it follows that the values $(\Lambda L)(x)$ are independent of the choice of $W > x \ldots$. The proof is a trivial consequence of 9.A.28:

$$\boldsymbol{\lambda}_{L(V)}(x) = \|(\Lambda L)(x)\| = \boldsymbol{\gamma}_{(\Lambda L)(V)}(x)\ .$$

9.A.30. Discussion.

The result 9.A.29 must be slightly modified in order to obtain results of the form:

$$\boldsymbol{\gamma}_{\eta_p} = \boldsymbol{\lambda}_{\eta_{p-1}}$$

for $p = 0, 1, 2, \ldots$ (recall that $\eta_{-1} = \varepsilon_0$).

Let us look more closely at the case $p = 0$; in fact, we are more interested in the equality

(1) $$\boldsymbol{\lambda}_{\omega^{(\omega^{\cdot^{\cdot^{\cdot^{\omega)}}}}}\ n\,\text{times}} = \boldsymbol{\gamma}_{[\eta_0]\,n}$$

where η_0 is an element of O, defined as in 5.A.7 (ii).

The reason for preferring this formulation is clear: the provably total recursive functions of **PA** are exactly those bounded by some $\boldsymbol{\lambda}_{\omega_n}$ (with $\omega_n = \omega^{(\omega^{\cdot^{\cdot^{\cdot^{\omega)}}}}}$ n times), hence they are exactly the functions bounded by some $\boldsymbol{\gamma}_{[\eta_0]\,n}$.

Let us now detail how one can use 9.A.29 to obtain 1:

1. Consider $L_0(x) = \underline{x}$, $L_{p+1}(x) = L_p(x) + \underline{x}^{\cdot L_p(x)}$. Then $L_{n-1}(\omega)$ is a rung of height ω_n; also, it is plain that, if one considers only the fundamental sequences of length ω, $L_{n-1}(\omega)$ can be viewed as an element ω_n of O, with $\|\omega_n\| = \omega_n$. (All one has to do is to select indices for the fundamental sequences; this is perfectly straightforward... .)

2. In the same way $(\Lambda L_n)(\omega)$ is a rung of type ω; it is not difficult to get from this an element $[\eta_0]\,n$ of O, in a way similar to 1.

3. Observe that:

$$\boldsymbol{\gamma}_{[\eta_0]\,n} = \boldsymbol{\gamma}_{(\Lambda L_n)(\omega)} = \boldsymbol{\lambda}_{L_n(\omega)} = \boldsymbol{\lambda}_{\omega_n}\ .$$

In the general case of η_p, see Exercise 9.B.9.

Annex 9.B. Exercises

9.B.1. The categories $K - \mathbf{ON}$.

(i) Define the sum of categories in the obvious way. Show that the category $K - \mathbf{ON}$ (9.A.15) is the sum of full subcategories of the form $n \to \mathbf{ON}$. ($n \to \mathbf{ON}$ is defined by:

objects: pairs (x, d) with $d \in I(n, x)$.
morphisms: $I(x, d\,;\, x', d')$ as in 9.A.15.)

(ii) Show that the categories $n \to \mathbf{ON}$ are isomorphic to the categories $\mathbf{ON}^{n+1}$.

(iii) Let D be a strongly homogeneous quasi-dendroid, and let $s = (x_0, ..., x_{n-1}) \in D^*$; let $a_0, ..., a_{p-1}$ be the underlined elements of s, listed in strictly increasing order; if $(x, d) \in p \to \mathbf{ON}$, define $F(x, d) = \{s'\,;\, s' \in D^0(x) \wedge s' \text{ extends } (x'_0, ..., x'_{n-1})\}$, where $x'_i = x_i$ when x_i is not underlined, and $x'_i = \underline{d(j)}$ when $x_i = \underline{a_j}$. Extend F into a functor from $p \to \mathbf{ON}$ to $\mathbf{QDN}$ preserving direct limits and pull-backs.

(iv) If $D = \mathbf{BCH}_q(A)$ ($A \in \mathbf{DIL}$), construct, given $s \in D^*$, F as in (iii), and let $B = \mathbf{LIN}_q(F)$. Show that B preserves direct limits and pull-backs. Using (ii) transform B into a functor B' from $\mathbf{ON}^{p+1}$ to $\mathbf{ON}$ preserving $\varinjlim$ and $\wedge$ ("dilator in $p+1$ variables"). In particular consider the cases:

1. $s = (x_0)$ (hence $p = 0$): relation of B' to the decomposition of A in sum.
2. $s = (x_0, \underline{x_1})$ when A is of kind $\mathbf{\Omega}$, and $x_0 = \max\{z\,;\, (z) \in D^*\}$: relation of B' to the functor $\mathbf{SEP}(A)$.

(*Remark. The construction of B' from A is an alternative concept of predecessor, which can be used to replace the one that we have defined here; there are two inconveniences in using this concept: 1. we must use $p+1$-variable dilators; 2. these predecessors do not define a well-founded relation as it stands…*)

9.B.2. Ladders and dilators.

(i) If L is a ladder, define a functor $\|L\|$ from **ON** to **ON** by:

$$\|L\|(x) = \|L(x)\| \qquad \|L\|(f) = \|m_f^{L(y)}\| \ \text{ if } f \in I(x,y) \ .$$

Show that $\|L\|$ is a dilator. Can all dilators be put under the form $\|L\|$ for some ladder L?

(ii) Construct a functor $\|\Lambda\|$ from **DIL** to **DIL** preserving direct limits and pull-backs, and such that $\|\Lambda\| \, \|L\| = \|\Lambda L\|$.

(*Remark.* $\|\Lambda\|$ *is the typical example of a variant of* $\mathbf{\Lambda}$.)

9.B.3. Variants of $\mathbf{\Lambda}$.

(i) If F is a dilator, define a functor G from $\mathbf{ON}^2$ to **ON** by:

$$G(x,0) = F(x) \qquad G(f,\mathbf{E}_0) = F(f)$$
$$G(x,y+1) = G(x,y) + F\big(G(x,y)\big) \qquad G(f,g+\mathbf{E}_1) = G(f,g) + F\big(G(f,g)\big)$$

If $f \in I(x,x')$, $g \in I(y,y')$, then $G(f,g+\mathbf{E}_{01}) = G(f,g) + \mathbf{E}_{0F\left(G(x',y')\right)}$, $G\big(x,\mathsf{sup}(y_i)\big) = \mathsf{sup}\big(G(x,y_i)\big)$, $G\big(f, \bigcup\limits_i g_i\big) = \bigcup\limits_i G(f, g_i)$. Show that G preserves direct limits and pull-backs, and that G is a bilator when $F \neq \underline{0}$. We set $\mathbf{IT}(F)(x) = G(x,x)$, $\mathbf{IT}(F)(f) = G(f,f)$. Define, when $T \in I(F,F')$, $\mathbf{IT}(T)$, in such a way that **IT** becomes a functor from **DIL** to **DIL** preserving $\varinjlim$ and $\wedge$. (**IT** stands for "iteration".)

(ii) Show the existence of a functor $\mathbf{\Lambda}^1$ from **DIL** to **DIL** preserving direct limits and pull-backs and s.t.:

$$\mathbf{\Lambda}^1\underline{0} = \underline{1}$$
$$\mathbf{\Lambda}^1(F+\underline{1}) = \mathbf{IT}(\mathbf{\Lambda}^1 F)$$
$$\mathbf{SEP}(\mathbf{\Lambda}^1 F)(\cdot,y) = \mathbf{\Lambda}^1\big(\mathbf{SEP}(F)(\cdot,y)\big) \ .$$

(iii) Show the existence of a functor $\mathbf{\Lambda}^{\mathbf{\Phi}}$ from **DIL** to **DIL** preserving direct limits and pull-backs, and s.t.:

$$\mathbf{\Lambda}^{\mathbf{\Phi}}\underline{0} = \mathsf{Id}$$

$$\mathbf{\Lambda}^{\mathbf{\Phi}}(F + \underline{1}) = \mathbf{\Lambda}^{\mathbf{\Phi}}F + \underline{1} + \mathbf{\Phi}(F)$$

$$\mathbf{SEP}(\mathbf{\Lambda}^{\mathbf{\Phi}}F)(\cdot, y) = \mathbf{\Lambda}^{\mathbf{\Phi}}\Big(\mathbf{SEP}(F)\Big)(\cdot, y)$$

where $\mathbf{\Phi}$ is a given functor from **DIL** to **DIL** preserving $\displaystyle\lim_{\longrightarrow}$ and $\wedge$ (ptyx of type $(\mathbf{O} \to \mathbf{O}) \to (\mathbf{O} \to \mathbf{O})$).

(*Remark. (iii) is a typical example of* primitive recursion on dilators*; one can for instance consider the particular cases:*

$$\mathbf{\Phi}(F) = G_0 \circ F \qquad G_0 \text{ fixed dilator}$$

$$\mathbf{\Phi}(F) = \mathbf{\Lambda}^{\mathbf{\Phi}}(F)\ , \qquad \text{etc.}\dots .$$

9.B.4. Variant of Λ.

(i) Assume that L is a ladder; define for all x and y rungs $M(x, y)$ by:

$$M(x, 0) = L(0) + 1\ ;$$

$$\|M(x, y)\| = \sup_{z<y} \Big(\|M(x, z)\| + 1 + \|L(\|M(x, z)\|)\| + 1\Big)$$

$$M(x, y)\restriction\Big(\|M(x, z)\| + 1 + \|L(\|M(x, z)\|)\|\Big) =$$

$$M(x, z) + 1 + L(\|M(x, z)\|)$$

$$T(\|M(x, y)\|) = y\ , \quad [\|M(x, y)\|]\, z = \|(x, z)\|\ .$$

Prove that $\mathbf{IT}(L)(x) = M(x, x)$ defines a ladder.

(ii) Prove the analogue of the hierarchy tehorem for the original Grzegorczyk hierarchy:

$$\boldsymbol{\lambda}'_0(x) = 2^x$$

$$\boldsymbol{\lambda}'_{\alpha+1}(x) = \boldsymbol{\lambda}'_\alpha(x) + 1 + it(\boldsymbol{\lambda}'_\alpha) + 1$$

$$\boldsymbol{\lambda}'_\alpha(x) = \boldsymbol{\lambda}'_{[\alpha]\,x}(x)$$

where $it(f)$ stands for a function obtained by iterating f:

$$it(f)(0) = f(0) + 1 \;;$$
$$it(f)(x+1) = it(f)(x) + 1 + f\big(it(f)(x)\big) + 1 \;.$$

9.B.5. More about rungs.

(i) Using 9.A.23, show that, when L is a K-ladder, and $z = \|L(x,d)\|$, then we have only the following possibilities:

1. $T(z) = 0$ and $T\big(m_f^{L(y,e)}(z)\big) = 0$ for all $f \in I(x,d\,;\, y,e)$ and all y, e.
2. $T(z) = x$ and $T\big(m_f^{L(y,e)}(z)\big) = y \ldots$.
3. $T(z) = d(i)$ for some $i \in K$, and $T\big(m_f^{L(y,e)}(z)\big) = f\big(T(z)\big)\ldots$.
4. $T(z) = d(i) + 1$ for some $i \in K$, and $T\big(m_f^{L(y,e)}(z)\big) = f\big(T(z)\big)\ldots$.

(ii) If L is a ladder and $z \leq \|L(x)\|$ show that one of the following holds:

1. $[z']\,T(z) \leq z < [z']\,(T(z)+1)$ for some $z' > z$; in that case show that $T\big(m_f^{L(y)}(z)\big) = f\big(T(z)\big)$ for all y and $f \in I(x,y)$.
2. $T(z) = x$; in that case show that $T\big(m_f^{L(y)}(z)\big) = y \ldots$.
3. If 1 and 2 fail, show that $T(z)$ is either 0 or a successor, and that in both cases $T\big(m_f^{L(y)}(z)\big) = \hat{f}\big(T(z)\big)\ldots$.

(*Hint.* *Use 9.A.23.*)

(iii) In general, if L is a K-ladder, show that all points $z < \|L(x,d)\|$ can be said to be of type I or of type II; this means that, for all y, e and $f \in I(x,d\,;\,y,e)$:

- If z is of type I, then $z' = m_f^{L(y,e)}(z)$ is of type I and $T(z') = (f + \mathbf{E}_1)\big(T(z)\big)$.
- If z is of type II, then $z' = m_f^{L(y,e)}(z)$ is of type II and $T(z') = \hat{f}\big(T(z)\big)$.

(iv) Assume that L is a K-ladder; define a K-ladder L' as follows: in $L(x,d)$, if z is of type II, let $T'(z) = 0$, $[z']0 = z$, T and $[\ ]$ being unchanged for type I points. Check that L' is a K-ladder.
(*Hint. Use 9.A.23.*)

9.B.6. Nice flowers.
Assume that F is a flower; show the existence of an integer k such that $F \circ (\underline{k} + \mathsf{Id})$ is a nice flower.

9.B.7. Products without nicety.
Consider $F = \underline{2} \cdot \mathsf{Id}$, and $T = (\mathbf{E}'_{\underline{01}} + \mathbf{E}'_{\underline{1}})\mathbf{E}'_{\mathsf{Id}}$, i.e. $T \in I^1(\mathsf{Id}, F)$, $T(x)(z) = 2z+1$. Define $F_n = F \circ \ldots \circ F$ (n times), and $T_{nn+1} \in I^1(F_n, F_{n+1})$ by $T_{nn+1} = \mathbf{E}_{F_n} \circ T$. Define from that T_{nm} when $n \leq m$. Show that the direct limit of (F_n, T_{nm}) in **PIL** is the predilator $\underline{\tilde{\omega}} \cdot \mathsf{Id}$ ($\tilde{\omega}$ is the order opposite to ω).

9.B.8. Regularity (Boquin [94]).
We consider the category $\overline{\mathbf{ON}}$ defined by:

objects: ordinals.
morphisms from x to y: the set $\bar{I}(x,y)$ of all $f \in I(x,y)$ s.t. $f + \mathbf{E}_1$ is a *continuous* function from $x+1$ to $y+1$.

A **regular** dilator is a dilator which sends $\overline{\mathbf{ON}}$ into $\overline{\mathbf{ON}}$. If D and D' are regular dilators, then $\bar{I}^1(D,D')$ is the set of all $T \in I^1(D,D')$ s.t. $T(x) \in \bar{I}\big(D(x), D'(x)\big)$ for all $x \in 0n$.

(i) Let F be a regular dilator, and let $D = \mathbf{BCH}_q(\underline{1} + F + \underline{1})$; if $s = \langle x_0, \ldots, x_{2n-1}\rangle \in D^* - D$, we consider the sets $T(s) = \{u\,;\, \langle x_0, \ldots, x_{2n-1}, u\rangle \in D^*\}$ and $U(s) = \{u\,;\, \langle x_0, \ldots, x_{2n-1}, u\rangle \in D\}$. Prove that

1. $0 \in U(s)$; from that conclude that $\forall s' \in D^*$ of the form $\langle x'_0, \ldots, x'_{2m-1}\rangle$, $T(s')$ is an ordinal.
2. $T(s)$ is not a limit ordinal; furthermore, its greatest element is in $U(s)$.
3. If $u \in T(s) - U(s)$, then $u = u' + 1$, for some $u' \in U(s)$ (3 implies 1).

(*Hint. The proof consists in exhibiting limit points in some* $F(x)$, *and looking at their images under* $F(f)$, *for an appropriate* f. *The replacement of* F *by* $\underline{1}+F+\underline{1}$ *is needed to ensure 1, 2 for* $s = \langle\ \rangle$... .
1 is proved by considering first the case x_{2n-1} *limit, and the smallest point in* $D^0(x)$ *of the form* $s * t \dots t$ *must be* $\langle 0 \rangle$.
2 is proved by considering $\langle x_0, ..., x_{2n-2}, x_{2n-1}+1, 0\rangle \in D$, *and showing that this point cannot be a limit...! The same thing must hold for all points* $D^0(f)(s)$... .
3. Replace s *by* $D^0(f)(s) = \langle x'_0, ..., x'_{2n-1}\rangle$, *and show that the smallest* t *s.t.* $\langle x'_0, ..., x'_{2n-1}\rangle * t \in D^0(x)$ *cannot be a limit*... .)

(ii) Show conversely that, if $D = \mathbf{BCH}_q(\underline{1}+F+\underline{1})$ enjoys 1–3, then F is a regular dilator.

(iii) Assume that $T \in I^1(F, F')$, and consider the function $f = \mathbf{BCH}_q(\mathbf{E}_{\underline{1}}+T+\mathbf{E}_{\underline{1}})$ from $\mathbf{BCH}_q(\underline{1}+F+\underline{1})$ $(= D)$ to $\mathbf{BCH}_1(\underline{1}+F'+\underline{1})$ $(= D')$. Show that $T \in \bar{I}^1(F, F')$ iff:

1. $f(\langle x_0, ..., x_{2n-1}, 0\rangle) = f^*(\langle x_0, ..., x_{2n-1}\rangle) * \langle 0\rangle$.
2. If $u \notin U(s)$ (in D), $u < T(s)$, then
$f(s * \langle u+1\rangle) = f^*(s) * \langle v+1\rangle$ (if $f^*(s * \langle u\rangle) = f^*(s) * \langle v\rangle$).
3. If $u \in T(s)$, u limit (in D), then
$f(s * \langle u\rangle) = \sup_{u'<u} f^*(s) * \langle v'\rangle$ (if $f^*(s * \langle u'\rangle) = f^*(s) * \langle v'\rangle$).

(iv) If F is a regular flower, show that F is nice; same question when $T \in \bar{I}^1(F, F')$ and F, F' are regular flowers. When F is a regular flower, what are the points in $\bar{I}^1(\mathsf{Id}, F)$? Show that a flower F is regular iff it is of the form $\underline{a} + \int(\underline{1}+G+\underline{1})$ for some regular G.

(v) Show that the functor length $\mathbf{LH}$, when restricted to regular bilators and regular morphisms of dilators, preserves pull-backs.

(vi) Consider a sum $F = \sum_{i<x} F_i$ of regular dilators; when is such a sum regular? Similar question for a sum $\sum_{i<f} T_i$ of regular morphisms.

(vii) There is an obvious concept of regular bilator; regular bilators are always nice; show that the functors **SEP** and **UN** can still be defined on regular objects.
(*Hint. Define a natural transformation T from* $\mathbf{SEP}(F)$ *to* $F \circ A$, *with* $A(x,y) = y + 1 + x$, *and show that* $rg\big(T(x,y)\big)$ *is a closed subset of* $F \circ A(x,y)$*....*)

Remark. Regular dilators are certainly a more effective, more regular (!) concept than dilators. For instance the disturbing features of dilators such as non preservation of pull-backs disappear with regular dilators; from this, for instance, the kind of a dilator becomes more easily computable (see (viii) below).... Moreover, since regular bilators are nice, it is likely that a simplified version of $\mathbf{\Lambda}$ *can be defined on regular dilators, taking its value among regular bilators: the essential simplification would be:*

$$(\mathbf{\Lambda}F)(x,y) = \big(\mathbf{\Lambda}\,\mathbf{SEP}(F)(\cdot,y)\big)(x,0)$$
$$(\mathbf{\Lambda}F)(f,g) = \big(\mathbf{\Lambda}\,\mathbf{SEP}(F)(\cdot,g)\big)(f,\mathbf{E}_0)$$
$$(\mathbf{\Lambda}T)(x,y) = \big(\mathbf{\Lambda}\,\mathbf{SEP}(T)(\cdot,y)\big)(x,0)\ .$$

However, the structure of regular dilators is slightly less "algebraic" than the structure of dilators, and for this reason, I so not think that dilators must be replaced by regular dilators everywhere.... of course the replacement is made possible by the property:

(viii) If F is a dilator, construct a regular dilator F' together with $T \in I^1(F,F')$.

(ix) If F is a regular dilator; show that

- $F(1) = 0 \to F$ of kind $\mathbf{0}$.
- $F(1)$ limit $\to F$ of kind $\boldsymbol{\omega}$.
- $F(1)$ successor $\to F$ of kind $\mathbf{1}$ or $\mathbf{\Omega}$; the distinction between these two subcases is obtained by looking at $F(\mathbf{E}_{01})$.

9.B.9. About the η_p's.

(i) From the definition of η_p, conclude that $\eta_p = \underbrace{\Lambda(...(\Lambda}_{p+1 \text{ times}} L)...)$ where L is a ladder s.t. $\|L(\omega)\| = \varepsilon_0$, that the reader will explicitly define.

(ii) Conclude that $\boldsymbol{\gamma}_{\eta_p} = \boldsymbol{\lambda}_{\eta_{p-1}}$.

(iii) Prove a similar result involving on one hand ordinals $< \eta_p$, and ordinals $< \eta_{p-1}$ on the other hand, on the model of what we did for $p = 0$ in 9.A.30.

CHAPTER 10
THE $\boldsymbol{\beta}$-RULE

Mostowski introduced the concept of $\boldsymbol{\beta}$-model, ([95]), a generalization of the concept of ω-model: an ω-model of $\mathbf{PRA}^2$ is a $\boldsymbol{\beta}$-model (in Mostowski's sense) iff the following holds: as soon as $\boldsymbol{m} \models WO(\bar{f})$, then f is (the characteristic function of) a well-order. Mostowski raised the question of finding a syntactic characterization of truth in all $\boldsymbol{\beta}$-models of a given theory. What rendered the question delicate is the fact that truth in all $\boldsymbol{\beta}$-models is Π^1_2 complete, hence Mostowski's problem could not be solved by means of ω-logic, i.e. Π^1_1 methods.

I solved this question in 1978; the syntactic characterization makes use of a *functorial* concept of $\boldsymbol{\beta}$-proof; our notion of $\boldsymbol{\beta}$-model is slightly different from Mostowski's original formulation, but one can easily show that our concept solves Mostowski's problem in the original formulation.

We shall first prove the $\boldsymbol{\beta}$-completeness theorem for a rather simple formulation of $\boldsymbol{\beta}$-logic, and then we shall consider more general situations and the corresponding completeness theorems... .

10.1. The $\boldsymbol{\beta}$-completeness theorem

10.1.1. Definition.

(i) A **$\boldsymbol{\beta}$-language** is a language $\boldsymbol{L}$ with a distinguished type $\boldsymbol{o}$ (the type of ordinals), together with a distinguished predicate $\leq^{\boldsymbol{o}}$ (with two places of type $\boldsymbol{o}$) and such that the only terms of $\boldsymbol{L}$ of type $\boldsymbol{o}$ are variables. When we speak of a formula of $\boldsymbol{L}$, we always assume that it has no free variables of type $\boldsymbol{o}$.

(ii) Assume that L is a $\boldsymbol{\beta}$-language; a **$\boldsymbol{\beta}$-theory**(in the language $\boldsymbol{L}$) is simply a theory whose underlying language is $\boldsymbol{L}$.

(iii) If $\boldsymbol{L}$ is a $\boldsymbol{\beta}$-language, then a **$\boldsymbol{\beta}$-model** of $\boldsymbol{L}$ is a model of $\boldsymbol{L}$ in the usual sense, such that $\boldsymbol{m}(\boldsymbol{o})$ is equal to some ordinal α, and $\boldsymbol{m} \models \bar{\zeta} \ \leq^{\boldsymbol{o}} \ \bar{\xi} \leftrightarrow \zeta \leq \xi$ for all $\zeta, \xi < \alpha$. $\alpha = 0$ is allowed; this is in contradiction with the familiar definition of "model", where one assumes that $\boldsymbol{m}(\boldsymbol{\tau}) \neq \emptyset$ for all $\boldsymbol{\tau}$. A **$\boldsymbol{\beta}$-model** of T is a $\boldsymbol{\beta}$-model of $\boldsymbol{L}$ in which (the closures of) all axioms of $\boldsymbol{T}$ are true.

10.1.2. Definition.

Let $\boldsymbol{T}$ be $\boldsymbol{\beta}$-theory; then a formula A of $\boldsymbol{L}$ is **$\boldsymbol{\beta}$-valid** in $\boldsymbol{T}$ iff every closed instantiation A' of A in $\boldsymbol{L}[\boldsymbol{m}]$ is true in $\boldsymbol{m}$, where $\boldsymbol{m}$ is an arbitrary $\boldsymbol{\beta}$-model of $\boldsymbol{T}$. $\boldsymbol{T} \overset{\boldsymbol{\beta}}{\vdash} A$ means "A is $\boldsymbol{\beta}$-valid in $\boldsymbol{T}$".

10.1.3. Example.

The example we are giving is not particularly elegant, but it has the advantage of bridging our concept of $\boldsymbol{\beta}$-model with Mostowski's original definition. (The most interesting examples of $\boldsymbol{\beta}$-theories will be found in the next chapter: they are connected with the treatment of *theories of inductive definitions* by means of $\boldsymbol{\beta}$-logic.) Let us add the type $\boldsymbol{o}$ to $\boldsymbol{L}^2_{pr}$; we also add the predicate letter $\leq^{\boldsymbol{o}}$, and a predicate letter R with three places of respective types (2),
bi and $\boldsymbol{o}$, and we consider the axioms:

$$\forall x^{\boldsymbol{o}} \forall x^{\boldsymbol{o}} (R(f, y, x) \wedge R(f, y, x') \rightarrow x = x')$$

$$\exists x^{\boldsymbol{o}} \ R(f, y, x)$$

$$\forall x^{\boldsymbol{o}} \forall x^{\boldsymbol{o}} (WO(f) \ \wedge \ f(y, y') = 0 \wedge y \neq y' \wedge R(f, y, x) \wedge \\ R(f, y', x') \rightarrow \neg(x' \ \leq^{\boldsymbol{o}} \ x) \ .$$

(These axioms express the existence, when f is a well-order, of a strictly increasing function from f to $\boldsymbol{o}$.) Now remark that:

(i) Every model of this theory yields, when restricted to the language $\boldsymbol{L}^2_{pr}$, a $\boldsymbol{\beta}$-model in the sense of Mostowski: if $\boldsymbol{m} \models WO(\bar{f})$, then one can define a strictly increasing function h from f to $\boldsymbol{m}(\boldsymbol{o})$ by $h(y) = x \leftrightarrow \boldsymbol{m} \models R(\bar{f}, \bar{y}, \bar{x})$; since $\boldsymbol{m}(\boldsymbol{o})$ is an ordinal, f is a well-order. $\boldsymbol{m}$ is an ω-model: if f is the characteristic function of the usual ordering of the integers, then the fact that $WO(f)$ is provable in $\mathbf{PRA}^2$ entails the existence of a strictly increasing function from $\boldsymbol{m}(\boldsymbol{\pi})$ to $\boldsymbol{m}(\boldsymbol{o})$, and this forces $\boldsymbol{m}(\boldsymbol{\iota})$ to be order isomorphic to ω.

(ii) Conversely, given a $\boldsymbol{\beta}$-model of $\mathbf{PRA}^2$ in the sense of Mostowski, consider the ordinal $\alpha = \mathsf{sup}\,\|R\|$, where R varies through the relations of the form $\boldsymbol{m} \models \bar{f}(\bar{x}, \bar{y}) = \bar{0}$, for some $\bar{f}^2 \in |\boldsymbol{m}|_{(2)}$ such that $\boldsymbol{m} \models WO(f)$. Define $R(\bar{f}, \bar{y}, \bar{x})$ to be true iff:

- either $\neg WO(\bar{f})$ and x is 0.
- or $WO(f)$ and x is the order type of the set of predecessors of y, w.r.t. f.

(iii) Remarks (i) and (ii) clearly show that A is true in all $\boldsymbol{\beta}$-models of $\mathbf{PRA}^2$ in the sense of Mostowski iff A is $\boldsymbol{\beta}$-valid in the theory we just constructed... . Hence any syntactic characterization of $\boldsymbol{\beta}$-validity will induce a solution of Mostowski's problem.

10.1.4. Remark.
There are many similarities between ω-logic and $\boldsymbol{\beta}$-logic. One can surely mix these concepts, and obtain a notion of "$\boldsymbol{\beta}\omega$-logic"; the question is "Is it good taste to do that?"

(i) In most situations, the combination $\boldsymbol{\beta}$-logic + ω-logic is fairly good: it enables us to concentrate on questions concerning $\boldsymbol{\beta}$-logic, without bothering about integers.

(ii) However, ω-logic can be eliminated by an adequate choce of the formalism. For the most subtle applications, e.g. cut-elimination, the choice of a specific expression of the ω-rule by means of the $\boldsymbol{\beta}$-rule is a delicate subject; in fact the results of the next chapter, on inductive definitions, will enable us to express the ω-rule by means of the $\boldsymbol{\beta}$-rule in a completely satisfactory way.

(iii) Hence the combination $\boldsymbol{\beta}$-logic + ω-logic can be used for rough work; but something is gained by the elimination of ω-logic... .

10.1.5. Theorem.

(i) Assume that $\boldsymbol{T}$ is a prim. rec. $\boldsymbol{\beta}$-theory; then the set $\{\lceil A \rceil ; \boldsymbol{T} \overset{\boldsymbol{\beta}}{\vdash} A\}$ is Π^1_2.

(ii) It is possible to choose a prim. rec. $\boldsymbol{\beta}$-theory $\boldsymbol{T}$ in such a way that the set $\{\lceil A \rceil ; \boldsymbol{T} \overset{\boldsymbol{\beta}}{\vdash} A\}$ is Π^1_2-complete (see Remark 6.1.5).

Proof. (i) $\boldsymbol{T} \overset{\boldsymbol{\beta}}{\vdash} A$ can be written

$$\forall \boldsymbol{m}(\boldsymbol{m} \text{ is a } \boldsymbol{\beta}\text{-model of } \boldsymbol{T} \to \boldsymbol{m} \models A) \ .$$

(As in 6.1.4, we assume for simplicity that A is closed, and that all axioms of $\boldsymbol{T}$ are closed.)
It suffices to restrict our attention to denumerable $\boldsymbol{\beta}$-models.

(Proof. If $\boldsymbol{m}$ is a $\boldsymbol{\beta}$-model of $\boldsymbol{T}+\neg A$, then by the Löwenheim-Skolem theorem, one can find a denumerable submodel $\boldsymbol{n}$ of $\boldsymbol{m}$, which is still a model of $\boldsymbol{T}+\neg A$. $\boldsymbol{n}(\boldsymbol{o})$, which is a subset of the ordinal $\boldsymbol{m}(\boldsymbol{o})$ is isomorphic to an ordinal, hence $\boldsymbol{n}$ is isomorphic to a $\boldsymbol{\beta}$-model $\boldsymbol{n}'$, hence we have found a denumerable $\boldsymbol{\beta}$-model of $\boldsymbol{T}+\neg A$. $\square$)

On the model of 6.1.4, we introduce a formula $C(f)$: $\forall B(B$ is an axiom of $\boldsymbol{T} \to f(B) = 0) \wedge \forall B\big(f(\neg B) = 1 - f(B)\big) \wedge \forall B \forall C\big(f(B \wedge C) = \mathsf{sup}\big(f(B), f(C)\big) \wedge f(B \vee C) = \mathsf{inf}\big(f(B), f(C)\big) \wedge f(B \to C) = \mathsf{inf}\big(1 - f(B), f(C)\big)\big) \wedge \forall B \forall \boldsymbol{\tau}\Big(f\big(\forall x^{\boldsymbol{\tau}} B(x^{\boldsymbol{\tau}})\big)\Big) = \mathsf{sup}\Big\{f\big(B(x^{\boldsymbol{\tau}}_n)\big) ; n \in I\!N\Big\} \wedge f\big(\exists x^{\boldsymbol{\tau}} B(x^{\boldsymbol{\tau}})\big) = \mathsf{inf}\Big\{f\big(B(x^{\boldsymbol{\tau}}_n)\big) ; n \in I\!N\Big\} \wedge \forall g \exists n(f(x^{\boldsymbol{\tau}}_{g(n+1)} \leq^{\boldsymbol{o}} x^{\boldsymbol{\tau}}_{g(n+1)}) = 1) \wedge LO(\leq^{\boldsymbol{o}})$. (The last conjunct is a formula expressing that $\leq^{\boldsymbol{o}}$ is a linear oder, whereas the last but one conjunct expresses that there is no s.d.s. for $\leq^{\boldsymbol{O}}$, i.e. that $\leq^{\boldsymbol{O}}$ is a well-order.) Clearly $T \overset{\boldsymbol{\beta}}{\vdash} A$ iff $\forall f(C(f) \to f(A) = 0)$; now, $C(f)$ (more precisely its precise formulation by means of Gödel numberings) is Π^1_1, and it follows that $\boldsymbol{T} \overset{\boldsymbol{\beta}}{\vdash} A$ is Π^1_2.

(ii) Let $\boldsymbol{T}$ be the theory of Example 10.1.3 plus the axiom $(\Sigma^0_1 - CA^*)$. Assume that $A = \forall f\ B(f)$ is a closed Π^1_2 formula; then by the results of Chapter 5, one can find a term $t(f)$ such that the equivalence $WO\big(t(f)\big) \leftrightarrow \neg B(f)$ is provable in $\mathbf{PRA}^2 + (\Sigma^0_1 - CA^*)$, hence $\boldsymbol{\beta}$-valid in $\boldsymbol{T}$.

Assume that A is true; if $\boldsymbol{m}$ is a $\boldsymbol{\beta}$-model of $\boldsymbol{T}$, and f is an element of $\boldsymbol{m}$ of type (1), then $\boldsymbol{m} \models WO\big(t(\bar{f})\big) \to t(f)$ is the characteristic function of a well-order, i.e. $\neg B(f)$. Hence, if $\boldsymbol{m} \models \exists f\ \neg B(f)$, $\neg A$ holds: hence $A \to \boldsymbol{m} \models A$.

Conversely, if A is true in all $\boldsymbol{\beta}$-models of $\boldsymbol{T}$, then A is true in all $\boldsymbol{\beta}$-models of $\boldsymbol{T}$ in Mostowski's sense, hence A is true in the standard model, i.e. A is true.

Hence we have shown that, when A is a closed Π^1_2 formula, $A \leftrightarrow \boldsymbol{T} \overset{\boldsymbol{\beta}}{\vdash} A$. Now, if X is a Π^1_2 subset of $I\!N$, i.e. $n \in X \leftrightarrow A(\bar{n})$ for a suitable Π^1_2 formula A, then

$$n \in X \leftrightarrow T \overset{\boldsymbol{\beta}}{\vdash} A(\bar{n}) \ .$$

Hence the set $X_0 = \{\ulcorner A \urcorner\, ; T \overset{\boldsymbol{\beta}}{\vdash} A\}$ is Π^1_2-complete: X_0 is Π^1_2 by (i), and if X is any Π^1_2 subset of $I\!N$, one can find a prim. rec. function f such that

$$n \in X \leftrightarrow f(n) \in X_0$$

(take $f(n) = \mathsf{Sub}(\ulcorner A \urcorner, \ulcorner x_0 \urcorner, \ulcorner \bar{n} \urcorner)$). □

10.1.6. <u>Definition</u>.

(i) Assume that $\boldsymbol{L}$ is a $\boldsymbol{\beta}$-language, and let α be an ordinal; one defines a language $\boldsymbol{L}[\alpha]$ by adding to $\boldsymbol{L}$ constants $\bar{\zeta}$ for all $\zeta < \alpha$. (Since $\boldsymbol{L}[\alpha]$ contains constants of type $\boldsymbol{o}$, this language is not a $\boldsymbol{\beta}$-language. But free variables of type $\boldsymbol{o}$ are not allowed in $\boldsymbol{L}[\alpha]$.)

(ii) The sequent calculus $\boldsymbol{L}K_\alpha$ is defined as follows: the sequents are made of formulas of $\boldsymbol{L}[\alpha]$; $\boldsymbol{L}K_\alpha$ uses the following specific rules:

+ Axioms: $\vdash \bar{\zeta} \leq^{\boldsymbol{O}} \bar{\xi}$ when $\zeta \leq \xi < \alpha$
  $\bar{\xi} \leq^{\boldsymbol{O}} \bar{\zeta} \vdash$ when $\xi < \zeta < \alpha$

+ Logical rules for $\forall\boldsymbol{O}$ and $\exists\boldsymbol{O}$: these rules are akin to the rules for $\forall\boldsymbol{L}$ and $\exists\boldsymbol{L}$ in ω-logic:

$$\frac{\ldots\ \Gamma \vdash A(\bar{\zeta}), \Delta\ \ldots\ (\text{all } \zeta > \alpha)}{\Gamma \vdash \forall x^{\boldsymbol{O}}\ A(x^{\boldsymbol{O}}), \Delta}\ r\forall\boldsymbol{O} \qquad \frac{\Gamma, A(\bar{\zeta}) \vdash \Delta}{\Gamma, \forall x^{\boldsymbol{O}}\ A(x^{\boldsymbol{O}}) \vdash \Delta}\ l\forall\boldsymbol{O}$$

$$\frac{\Gamma \vdash A(\bar{\zeta}), \Delta}{\Gamma \vdash \exists x^{\boldsymbol{O}}\ A(x^{\boldsymbol{O}}), \Delta}\ r\exists\boldsymbol{O} \qquad \frac{\ldots\ \Gamma, A(\bar{\zeta}) \vdash \Delta\ \ldots\ (\text{all } \zeta < \alpha)}{\Gamma, \exists x^{\boldsymbol{O}}\ A(x^{\boldsymbol{O}}) \vdash \Delta}\ l\exists\boldsymbol{O}$$

10.1.7. Remark.
A trivial solution to the problem of characterizing $\boldsymbol{\beta}$-validity is the following: A is $\boldsymbol{\beta}$-valid in $\boldsymbol{T}$ iff for all α, at first sight: "for all denumerable α", A is provable in $\boldsymbol{T} + \boldsymbol{L}K_\alpha$. With this solution, a $\boldsymbol{\beta}$-proof is a family $\boldsymbol{\pi}_\alpha)_{\alpha\in 0n}$ of proofs, such that $\boldsymbol{\pi}_\alpha$ is a proof of $\vdash A$ in $\boldsymbol{T} + \boldsymbol{L}K_\alpha$.

This solution cannot be accepted, because such a family $(\boldsymbol{\pi}_\alpha)$ is not a syntactic object, even with a very liberal acceptation of "syntactic":

(i) When α is an ordinal, $\boldsymbol{\pi}_\alpha$ must be "α-recursive" or "recursive in α"; the concepts available in the literature on α-recursion are not precisely effective ... i.e. we have a problem as to the sense in which $\boldsymbol{\pi}_\alpha$ is "syntactic".

(ii) But the idea of a family of proofs indexed by the class of all ordinals is, syntactically speaking, a monstrosity: the family of all $\boldsymbol{\beta}$-models of $\boldsymbol{T}$ is an acceptable syntactic object as well, if one is ready to accept such families of proofs as syntactic objects!

(iii) However, this vulgar solution to the question of $\boldsymbol{\beta}$-completeness can be used as a starting point: we must generate the family $(\boldsymbol{\pi}_\alpha)$ in an effective way; in fact observe that objection (i) does not hold for *finite* α's, since for $n < \omega$ $\boldsymbol{\pi}_n$ is a finite proof, and is therefore purely syntactic. Furthermore the family $(\boldsymbol{\pi}_n)_{n<\omega}$ is certainly enumerable by a prim. rec. function. Hence all our task will be to "extend" the family $(\boldsymbol{\pi}_n)_{n<\omega}$ by an effective procedure to a family $(\boldsymbol{\pi}_\alpha)_{\alpha \in 0n}$. The construction uses direct limits... .

10.1.8. Definition.

Let us restate the concept of a proof in $\boldsymbol{L}K_\alpha$ in a way akin to 6.1.8:
A proof in $\boldsymbol{L}K_\alpha$ is a pair (D, φ), where:

(i) D is a quasi-dendroid of type α.

(ii) $s * (n) \in D^* \wedge n' \leq n \to s * (n') \in D^* \wedge n < 2$.

(iii) φ is a function whose domain is D^* and such that, for all $t \in D^*$, one of the following holds:

(1) $\varphi(t) = \langle \ulcorner Ax \urcorner, \ulcorner \Gamma \vdash \Delta \urcorner \rangle$ and $\Gamma \vdash \Delta$ is $A \vdash A$, or $\vdash \bar{\xi} \leq \bar{\zeta}$ for some ξ and ζ such that $\xi \leq \zeta < \alpha$ or $\bar{\xi} \leq \bar{\zeta} \vdash$ for some ξ and ζ such that $\zeta < \xi < \alpha$; furthermore $t \in D$.

(2) $\varphi(t) = \langle \ulcorner r\wedge \urcorner, \ulcorner \Gamma, \Lambda \vdash A \wedge B, \Delta, \Pi \urcorner \rangle$ and $t * (0) \in D^*$ and $t * (1) \in D^*$ and $t * (2) \notin D^*$, and $\Big(\varphi\big(t * (0)\big)\Big)_1 = \ulcorner \Gamma \vdash A, \Delta \urcorner$ and $\Big(\varphi\big(t * (1)\big)\Big)_1 = \ulcorner \Lambda \vdash B, \Pi \urcorner$ (for some sequents $\Gamma \vdash A, \Delta$ and $\Lambda \vdash B, \Pi$ of $\boldsymbol{L}[\alpha]$; we shall not mention this any longer...).

(3) $\varphi(t) = \langle \ulcorner l1\wedge \urcorner, \ulcorner \Gamma, A \wedge B \vdash \Delta \urcorner \rangle$ and $t * (0) \in D^*$ and $t * (1) \notin D^*$ and $\Big(\varphi\big(t * (0)\big)\Big)_1 = \ulcorner \Gamma, A \vdash \Delta \urcorner$... .

(4) $\varphi(t) = \langle \ulcorner l2\wedge \urcorner, \ulcorner \Gamma, A \wedge B \vdash \Delta \urcorner \rangle$ and $t * (0) \in D^*$ and $t * (1) \notin D^*$ and $\Big(\varphi\big(t * (0)\big)\Big)_1 = \ulcorner \Gamma, B \vdash \Delta \urcorner$... .

(5) $\varphi(t) = \langle \ulcorner r1\vee \urcorner, \ulcorner \Gamma \vdash A \vee B, \Delta \urcorner \rangle$ and $t * (0) \in D^*$ and $t * (1) \notin D^*$ and $\Big(\varphi\big(t * (0)\big)\Big)_1 = \ulcorner \Gamma \vdash A, \Delta \urcorner$... .

(6) $\varphi(t) = \langle \ulcorner r2\vee \urcorner, \ulcorner \Gamma \vdash A \vee B, \Delta \urcorner \rangle$ and $t * (0) \in D^*$ and $t * (1) \notin D^*$ and $\Big(\varphi\big(t * (0)\big)\Big)_1 = \ulcorner \Gamma \vdash B, \Delta \urcorner$... .

(7) $\varphi(t) = \langle \ulcorner l\vee \urcorner, \ulcorner \Gamma, \Lambda, A \vee B \vdash \Delta, \Pi \urcorner \rangle$ and $t * (0) \in D^*$ and $t * (1) \in D^*$ and $t * (2) \notin D^*$ $\Big(\varphi\big(t * (0)\big)\Big)_1 = \ulcorner \Gamma, A \vdash \Delta \urcorner$ and $\Big(\varphi\big(t * (1)\big)\Big)_1 = \urcorner \Lambda, B \vdash \Pi \urcorner$... .

(8) $\varphi(t) = \langle\lceil r\neg\rceil, \lceil\Gamma \vdash \neg A, \Delta\rceil\rangle$ and $t * (0) \in D^*$ and $t * (1) \notin D^*$ and $\Big(\varphi\Big(t * (0)\Big)\Big)_1 = \lceil\Gamma, A \vdash \Delta\rceil\ldots$.

(9) $\varphi(t) = \langle\lceil l\neg\rceil, \lceil\Gamma\neg A \vdash \Delta\rceil\rangle$ and $t * (0) \in D^*$ and $t * (1) \notin D^*$ and $\Big(\varphi\Big(t * (0)\Big)\Big)_1 = \lceil\Gamma, \vdash A, \Delta\rceil\ldots$.

(10) $\varphi(t) = \langle\lceil r \rightarrow\rceil, \lceil\Gamma \vdash A \rightarrow B, \Delta\rceil\rangle$ and $t * (0) \in D^*$ and $t * (1) \notin D^*$ and $\Big(\varphi\Big(t * (0)\Big)\Big)_1 = \lceil\Gamma, A \vdash B, \Delta\rceil\ldots$.

(11) $\varphi(t) = \langle\lceil l \rightarrow\rceil, \lceil\Gamma\Lambda, A \rightarrow B \vdash \Delta, \Pi\rceil\rangle$ and $t * (0) \in D^*$ and $t * (1) \notin D^*$ and $t * (2) \notin D^*$ and $\Big(\varphi\Big(t * (0)\Big)\Big)_1 = \lceil\Gamma \vdash A, \Delta\rceil$ and $\Big(\varphi\Big(t * (0)\Big)\Big)_1 = \lceil\Lambda, B \vdash \Pi\rceil\ldots$.

(12) $\varphi(t) = \langle\lceil r\forall\boldsymbol{o}\rceil, \lceil\Gamma \vdash \forall x^{\boldsymbol{o}}\ A(x^{\boldsymbol{o}}), \Delta\rceil\rangle$ and $t * (\underline{\xi}) \in D^*$ and $\Big(\varphi\Big(t * (\underline{x})\Big)\Big)_1 = \lceil\Gamma \vdash A(\underline{x}), \Delta\rceil$ for all $\xi < \alpha\ldots$.

(13) $\varphi(t) = \langle\lceil r\forall\boldsymbol{\tau}\rceil, \lceil\Gamma \vdash \forall x^{\boldsymbol{\tau}}\ A(x^{\boldsymbol{\tau}}), \Delta\rceil\rangle$ and $t * (0) \in D^*$ and $t * (1) \notin D^*$ and $\Big(\varphi\Big(t * (0)\Big)\Big)_1 = \lceil\Gamma \vdash A(y), \Delta\rceil$ for some y such that $\forall x\ A(x) \sim \forall y\ A(y)$, and y not free in $\Gamma \vdash \Delta$ and $\boldsymbol{\tau} \neq \boldsymbol{o}$.

(14) $\varphi(t) = \langle\lceil l\forall\boldsymbol{\tau}\rceil, \lceil\Gamma, \forall x^{\boldsymbol{\tau}}\ A(x^{\boldsymbol{\tau}}) \vdash \Delta\rceil\rangle$ and $t * (0) \in D^*$ and $t * (1) \notin D^*$ and $\Big(\varphi\Big(t * (0)\Big)\Big)_1 = \lceil\Gamma, A(u) \vdash \Delta\rceil$ for some term u of tupe $\boldsymbol{\tau}$, substituable for $x^{\boldsymbol{\tau}}$ in $A\ldots$. (If $\boldsymbol{\tau} = \boldsymbol{o}$, u must be equal to $\bar{\xi}$, for some $\xi < \alpha$.)

(15) $\varphi(t) = \langle\lceil r\exists\boldsymbol{\tau}\rceil, \lceil\Gamma \vdash \exists x^{\boldsymbol{\tau}}\ A(x^{\boldsymbol{\tau}}), \Delta\rceil\rangle$ and $t, *(0) \in D^*$ and $t * (1) \notin D^*$ and $\Big(\varphi\Big(t * (0)\Big)\Big)_1 = \lceil\Gamma \vdash A(u), \Delta\rceil$ for some term u substitutable for $x^{\boldsymbol{\tau}}$ in $A\ldots$. (If $\boldsymbol{\tau} = \boldsymbol{o}$, u must be equal to $\bar{\xi}$, for some $\xi < \alpha$.)

(16) $\varphi(t) = \langle\lceil l\exists\boldsymbol{o}\rceil, \lceil\Gamma, \exists x^{\boldsymbol{o}}\ A(c^{\boldsymbol{o}}) \vdash \Delta\rceil\rangle$ and $t * (\underline{\xi}) \in D^*$ and $\Big(\varphi\Big(t * (\underline{\xi})\Big)\Big)_1 = \lceil\Gamma, A(\underline{\xi}) \vdash \Delta\rceil$ for all $\xi < \alpha\ldots$.

(17) $\varphi(t) = \langle\lceil l\exists\boldsymbol{\tau}\rceil, \lceil\Gamma, \exists x^{\boldsymbol{\tau}}\ A(x^{\boldsymbol{\tau}}) \vdash \Delta\rceil\rangle$ and $t * (0) \in D^*$ and $t * (1) \notin D^*$ and $\Big(\varphi\Big(t * (0)\Big)\Big)_1 = \lceil\Gamma, A(y) \vdash \Delta\rceil$ for some y such that $\forall x\ A(x) \sim \forall y\ A(y)$, and y not free in $\Gamma \vdash \Delta$ and $\boldsymbol{\tau} \neq \boldsymbol{o}\ldots$.

(18) $\varphi(t) = \langle\lceil rW\rceil, \lceil\Gamma \vdash A, \Delta\rceil\rangle$ and $t * (0) \in D^*$ and $t * (1) \notin D^*$ and $\Big(\varphi\Big(t * (0)\Big)\Big)_1 = \lceil\Gamma \vdash \Delta\rceil\ldots$.

(19) $\varphi(t) = \langle\lceil lW\rceil, \lceil\Gamma, A \vdash \Delta\rceil\rangle$ and $t * (0) \in D^*$ and $t * (1) \notin D^*$ and $\Big(\varphi\Big(t * (0)\Big)\Big)_1 = \lceil\Gamma \vdash \Delta\rceil\ldots$.

(20) $\varphi(t) = \langle \ulcorner rE \urcorner, \ulcorner \Gamma \vdash \Delta', B, A, \Delta'' \urcorner \rangle$ and $t * (0) \in D^*$ and $t * (1) \notin D^*$ and $\Big(\varphi\big(t * (0)\big)\Big)_1 = \ulcorner \Gamma \vdash \Delta', A, B, \Delta'' \urcorner \ldots$.

(21) $\varphi(t) = \langle \ulcorner lE \urcorner, \ulcorner \Gamma', B, A, \Gamma'' \vdash \Delta \urcorner \rangle$ and $t * (0) \in D^*$ and $t * (1) \notin D^*$ and $\Big(\varphi\big(t * (1)\big)\Big)_1 = \ulcorner \Gamma', A, B, \Gamma'' \vdash \Delta \urcorner \ldots$.

(22) $\varphi(t) = \langle \ulcorner rC \urcorner, \ulcorner \Gamma \vdash A, \Delta \urcorner \rangle$ and $t * (0) \in D^*$ and $t * (1) \notin D^*$ and $\Big(\varphi\big(t * (0)\big)\Big)_1 = \ulcorner \Gamma \vdash A, A, \Delta \urcorner \ldots$.

(23) $\varphi(t) = \langle \ulcorner lC \urcorner, \ulcorner \Gamma, A \vdash \Delta \urcorner \rangle$ and $t * (0) \in D^*$ and $t * (1) \notin D^*$ and $\Big(\varphi\big(t * (0)\big)\Big)_1 = \ulcorner \Gamma, A, A \vdash \Delta \urcorner \ldots$.

(24) $\varphi(t) = \langle \ulcorner Cut \urcorner, \ulcorner \Gamma, \Delta \vdash \Delta, \Pi \urcorner \rangle$ and $t * (0) \in D^*$ and $t * (1) \in D^*$ and $t * (2) \notin D^*$ and $\Big(\varphi\big(t * (0)\big)\Big)_1 = \ulcorner \Gamma \vdash A, \Delta \urcorner$ and $\Big(\varphi\big(t * (1)\big)\Big)_1 = \ulcorner \Lambda, A \vdash \Pi \urcorner \ldots$.

10.1.9. <u>Remarks</u>.

(i) We use the symbols $\ulcorner\ \urcorner$ to denote some kind of Gödel numbering for the rules ($\ulcorner Ax \urcorner, \ldots, \ulcorner Cut \urcorner$) and for the formulas and sequents of $\boldsymbol{L}[\alpha]$; this Gödel numbering has not yet been defined; of course the difficulty comes from the parameters $\bar{\xi} < \alpha$, which make impossible any attempt to Gödel number formulas of $\boldsymbol{L}[\alpha]$ by means of integers... . We use ordinals $< \omega^{1+\alpha}$ to Gödel number expressions of $\boldsymbol{L}[\alpha]$:

- Observe that finite sequences of ordinals $< \omega^{1+\alpha}$ can be encoded by means of ordinals $< \omega^{1+\alpha}$: let $y_0, \ldots, y_{n-1}$ be such a sequence and write $y_i = \omega^{x_1} \cdot a_i^1 + \ldots + \omega^{x_p} \cdot a_i^p + b_i$, with $a_i^1, \ldots, a_i^p$, b_i integers, and $x_1 > \ldots > x_p > 0$; then we define $\langle y_0, \ldots, y_{n-1} \rangle$ to be $\omega^{x_1} \cdot A^1 + \ldots + \omega^{x_p} \cdot A^p + B$, with $B = \langle b_0, \ldots, b_{n-1} \rangle = p_0^{b_0+1} \cdot p_1^{b_1+1} \cdot \ldots \cdot p_{n-1}^{b_{n-1}+1}$ and $A^k = p_0^{a_0^k} \cdot p_1^{a_1^k} \cdot \ldots \cdot p_{n-1}^{a_{n-1}^k} - 1$.
- The Gödel numbers of the symbols of $\boldsymbol{L}$ are defined as usual: they are integers; the Gödel number of the constant $\bar{\xi} < \alpha$ is defined to be $\omega^{1+\xi}$.
- Gödel numbers for expressions of $\boldsymbol{L}[\alpha]$ are obtained by means of the function $\langle \cdot, \ldots, \cdot \rangle$ defined above.

(ii) It is certainly necessary to be more precise when naming our rules in 10.1.8; especially the rules $(l\forall \boldsymbol{o})$ and $(r\exists \boldsymbol{o})$ must be carefully Gödel numbered: for instance the Gödel number assigned to an instance of $(l\forall \boldsymbol{o})$ acting on ξ could be $\langle 14, \omega^{1+\xi}\rangle \ldots$.

(iii) Our Gödel numbering and the function $\langle \quad \rangle$ are "functorial": this will clearly appear later on ... (10.1.12).

10.1.11. <u>Definition</u>.
Let $\boldsymbol{\pi} = (D, \varphi)$ be a α'-proof an assume that $f \in I(\alpha, \alpha')$; then we eventually define an α-proof ${}^f\boldsymbol{\pi} = ({}^fD, {}^f\varphi)$ as follows:

– ${}^fD$ is defined as in 8.D.6.

– ${}^f\varphi$ is the function making the following diagram commutative,

$$\begin{tikzcd} {}^fD \arrow[r, "\mu_f^D"] \arrow[d, "{}^f\varphi"'] & D \arrow[d, "\varphi"] \\ \omega^{1+\alpha} \arrow[r, "\omega^{1+f}"'] & \omega^{1+\alpha'} \end{tikzcd}$$

if such a function exists; ${}^f\varphi$ is not defined otherwise.

10.1.12. <u>Remark</u>.
The Gödel numbering is functorial in the following sense: assume that $f \in I(\alpha, \alpha')$ and that $A \in \boldsymbol{L}[\alpha]$; let B be the formula obtained from A by replacing all parameters $\bar{\xi}$ (with $\xi < \alpha$ by $\overline{f(\xi)}$; then

$$\lceil B \rceil = \omega^{1+f}(\lceil A \rceil) \ .$$

Now, it is not hard to restate 10.1.11 in a more familiar language: we start with an α'-proof and

(i) first we cut all premises of index $\xi \notin rg(f)$ (and what is above those premises) in all rules $(r\forall \boldsymbol{o})$ and $(l\exists \boldsymbol{o})$: this is a process of mutilation, familiar from dendroids (in fact dendroids, quasi-dendroids, homogeneous trees ... were built up on the model of α-proofs...).

(ii) in the mutilated proof, if there remains parameters $\bar{\xi}$, with $\xi \notin rg(f)$, then ${}^f\boldsymbol{\pi}$ cannot be defined; otherwise, replace systematicaly all parameters $\overline{f(\xi)}$ by $\bar{\xi}$: the resulting proof is ${}^f\boldsymbol{\pi}$. See examples below.

10.1.13. Examples.

(i) Consider the ω-proof (there is an ambiguity between "ω-proof" in the sense of Chapter 6 and in the sense of "α-proof" with $\alpha = \omega$...):

$$
\begin{array}{ccccc}
\bar{2} \leq \bar{0} \vdash & \bar{3} \leq \bar{1} \vdash & \bar{4} \leq \bar{2} \vdash & \bar{5} \leq \bar{3} \vdash & \bar{6} \leq \bar{4} \vdash \\
\quad r\neg & \quad r\neg & \quad r\neg & \quad r\neg & \quad r\neg \\
\vdash \neg(\bar{2} \leq \bar{0}) & \vdash \neg(\bar{3} \leq \bar{1}) & \vdash \neg(\bar{4} \leq \bar{2}) & \vdash \neg(\bar{5} \leq \bar{3}) & \vdash \neg(\bar{6} \leq \bar{4}) \\
\quad r\exists\boldsymbol{o} & \quad r\exists\boldsymbol{o} & \quad r\exists\boldsymbol{o} & \quad r\exists\boldsymbol{o} & \quad r\exists\boldsymbol{o} \\
\vdash \exists y^{\boldsymbol{o}}\, \neg(y \leq \bar{0}) & \vdash \exists y^{\boldsymbol{o}}\, \neg(y \leq \bar{1}) & \vdash \exists y^{\boldsymbol{o}}\, \neg(y \leq \bar{2}) & \vdash \exists y^{\boldsymbol{o}}\, \neg(y \leq \bar{3}) & \vdash \exists y^{\boldsymbol{o}}\, \neg(y \leq \bar{4}) \ \ldots \\
 & & & & \quad r\forall\boldsymbol{o} \\
 & & \forall x^{\boldsymbol{o}}\, \exists y^{\boldsymbol{o}}\, \neg(y \leq x) & &
\end{array}
$$

Consider the function $f \in I(\omega, \omega)$ defined by $f(n) = 2n$; then what remains after mutilation w.r.t. f is:

$$
\begin{array}{ccc}
\bar{2} \leq \bar{0} \vdash & \bar{4} \leq \bar{2} \vdash & \bar{6} \leq \bar{4} \vdash \\
\quad r\neg & \quad r\neg & \quad r\neg \\
\vdash \neg(\bar{2} \leq \bar{0}) & \vdash \neg(\bar{4} \leq \bar{2}) & \vdash \neg(\bar{6} \leq \bar{4}) \\
\quad r\exists\boldsymbol{o} & \quad r\exists\boldsymbol{o} & \quad r\exists\boldsymbol{o} \\
\vdash \exists y^{\boldsymbol{o}}\, \neg(y \leq \bar{0}) & \vdash \exists y^{\boldsymbol{o}}\, \neg(y \leq \bar{2}) & \vdash \exists y^{\boldsymbol{o}}\, \neg(y \leq \bar{4}) \ \ldots \\
 & & \quad r\forall\boldsymbol{o} \\
 & \forall x^{\boldsymbol{o}}\, \exists y^{\boldsymbol{o}}\, \neg(y \leq x) &
\end{array}
$$

All parameters that remain in the mutilated proof are in $rg(f)$; hence it is possible to define ${}^{f}\boldsymbol{\pi}$ by replacing everywhere $\overline{2n}$ by $\bar{n}$:

$$
\begin{array}{ccc}
\bar{1} \leq \bar{0} \vdash & \bar{2} \leq \bar{1} \vdash & \bar{3} \leq \bar{2} \vdash \\
\quad r\neg & \quad r\neg & \quad r\neg \\
\vdash \neg(\bar{1} \leq \bar{0}) & \vdash \neg(\bar{2} \leq \bar{1}) & \vdash \neg(\bar{3} \leq \bar{2}) \\
\quad r\exists\boldsymbol{o} & \quad r\exists\boldsymbol{o} & \quad r\exists\boldsymbol{o} \\
\vdash \exists y^{\boldsymbol{o}}\, \neg(y \leq \bar{0}) & \vdash \exists y^{\boldsymbol{o}}\, \neg(y \leq \bar{1}) & \vdash \exists y^{\boldsymbol{o}}\, \neg(y \leq \bar{2}) \ \ldots \\
 & & \quad r\forall\boldsymbol{o} \\
 & \forall x^{\boldsymbol{o}}\, \exists y^{\boldsymbol{o}}\, \neg(y \leq x) &
\end{array}
$$

Consider now the function $g \in I(\omega, \omega)$ defined by $f(n) = 3n$; then what remains after mutilation w.r.t. g is:

$$\dfrac{\dfrac{\dfrac{\bar{2} \leq \bar{0} \vdash}{\vdash \neg(\bar{2} \leq \bar{0})}\, r\neg}{\vdash \exists y^{\boldsymbol{o}}\, \neg(y \leq \bar{0})}\, r\exists\boldsymbol{o} \qquad \dfrac{\dfrac{\bar{5} \leq \bar{3} \vdash}{\vdash \neg(\bar{5} \leq \bar{3})}\, r\neg}{\vdash \exists y^{\boldsymbol{o}}\, \neg(y \leq \bar{3})}\, r\exists\boldsymbol{o} \quad \ldots}{\forall x^{\boldsymbol{o}}\, \exists y^{\boldsymbol{o}}\, \neg(y \leq x)}\, r\forall\boldsymbol{o}$$

In this mutilated proof, parameters $\bar{2}, \bar{5}, \ldots$ which are not in $rg(g)$ still occur, hence ${}^{g}\boldsymbol{\pi}$ does not exist.

If $h \in I(n, \omega)$, then the parameter $\overline{h(n-1)+2}$ occurs in $\boldsymbol{\pi}$ after mutilation w.r.t. h, and so ${}^{h}\boldsymbol{\pi}$ is immediate from a semantic viewpoint: there is no n-proof, when $n \neq 0$, of $\forall x^{\boldsymbol{o}} \exists y^{\boldsymbol{o}}\ \neg(y \leq x)$ since this formula expresses that $\boldsymbol{o}$ is void or a limit ordinal... .

(ii) But the most important example is given by the *principle of transfinite induction*; it is not an exaggeration to say that $\boldsymbol{\beta}$-logic is introduced in the purpose of giving a purely logical proof of this principle! Let A be a formula of $\boldsymbol{L}$ with a free variable of tyep $\boldsymbol{o}$. We consider the formula Prog(A):

$$\forall x^{\boldsymbol{o}} \Big(\forall y^{\boldsymbol{o}} \big(x \leq y \vee A(y) \big) \rightarrow A(x) \Big) \ .$$

Then we define by induction on $\xi < \alpha$ (induction on ξ is excessive! see 10.1.21 below) an α-proof $\boldsymbol{\pi}_{\alpha,\xi}$ of the sequent Prog$(A) \vdash A(\bar{\xi})$: assume that $\boldsymbol{\pi}_{\alpha,\zeta}$ has been defined for all $\zeta < \xi$, then $\boldsymbol{\pi}_{\alpha,\xi}$ is:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\boldsymbol{\pi}_{\alpha\zeta}\ \ (\text{if } \zeta<\xi)}{\mathrm{Prog}(A)\vdash A(\bar{\zeta})}}{\mathrm{Prog}(A)\vdash \bar{\xi}\le\bar{\zeta}\vee A(\bar{\zeta})}\,r2\vee \ \ldots\ \dfrac{\dfrac{(\text{if } \xi\le\zeta<\alpha)}{\vdash \bar{\xi}\le\bar{\zeta}}}{\mathrm{Prog}(A)\vdash \bar{\xi}\le\bar{\zeta}\vee A(\bar{\zeta})}\,r1\vee\ \ \ldots}{\mathrm{Prog}(A)\vdash\forall y^{\boldsymbol{O}}\big(\bar{\xi}\le y\vee A(y)\big)}\,r\forall\boldsymbol{O} \qquad A(\bar{\xi})\vdash A(\bar{\xi})}{\mathrm{Prog}(A),\forall y^{\boldsymbol{O}}\big(\bar{\xi}\le y\vee A(y)\big)\to A(\bar{\xi})\vdash A(\bar{\xi})}\,l\to}{\mathrm{Prog}(A)\vdash A(\bar{\xi})}\,l\forall\boldsymbol{O}
$$

We prove by induction (induction on ξ is excessive! see 10.1.21 below) on $\xi<\alpha$ that, if $f\in I(\alpha,\alpha')$ ${}^{f}\boldsymbol{\pi}_{\alpha',f(\xi)}\simeq\boldsymbol{\pi}_{\alpha,\xi}$, i.e. ${}^{f}\boldsymbol{\pi}_{\alpha',f(\xi)}$ exists and equals $\boldsymbol{\pi}_{\alpha,\xi}$: assume that the property holds for all $\zeta<\xi$; then it is plain that ${}^{f}\boldsymbol{\pi}_{\alpha',f(\xi)}$ equals

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{f\boldsymbol{\pi}_{\alpha' f(\zeta)}\ \ (\text{if } f(\zeta)<f(\xi))}{\mathrm{Prog}(A)\vdash A(\bar{\zeta})}}{\mathrm{Prog}(A)\vdash \bar{\xi}\le\bar{\zeta}\vee A(\bar{\zeta})}\,r2\vee \ \ldots\ \dfrac{\dfrac{(\text{if } f(\xi)\le f(\zeta)<\alpha')}{\vdash \bar{\xi}\le\bar{\zeta}}}{\mathrm{Prog}(A)\vdash \bar{\xi}\le\bar{\zeta}\vee A(\bar{\zeta})}\,r1\vee\ \ \ldots}{\mathrm{Prog}(A)\vdash\forall y^{\boldsymbol{O}}\big(\bar{\xi}\le y\vee A(y)\big)}\,r\forall\boldsymbol{O} \qquad A(\bar{\xi})\vdash A(\bar{\xi})}{\mathrm{Prog}(A),\forall y^{\boldsymbol{O}}\big(\bar{\xi}\le y\vee A(y)\big)\to A(\bar{\xi})\vdash A(\bar{\xi})}\,l\to}{\mathrm{Prog}(A)\vdash A(\bar{\xi})}\,l\forall\boldsymbol{O}
$$

and, using the induction hypothesis, this equals $\boldsymbol{\pi}_{\alpha,\xi}$.

Now consider the proof $\boldsymbol{\pi}_\alpha$:

$$\dfrac{\dfrac{\begin{array}{c}\boldsymbol{\pi}_{\alpha,\xi} \quad (\text{if } \xi < \alpha)\\ \mathrm{Prog}(A) \vdash A(\bar{\xi})\end{array}}{\mathrm{Prog}(A) \vdash \forall x^{\boldsymbol{O}} A(x)}\, r\forall\boldsymbol{O}}{\vdash\ \mathrm{Prog}(A) \to \forall x^{\boldsymbol{O}} A(x)}\, r\to$$

$\boldsymbol{\pi}_\alpha$ establishes the principle of transfinite induction (on A); Assume that $f \in I(\alpha, \alpha')$; then ${}^f\boldsymbol{\pi}_{\alpha'}$ exists and is equal to

$$\dfrac{\dfrac{\begin{array}{c}f\boldsymbol{\pi}_{\alpha',f(\xi)} \quad (\text{if } f(\xi) < \alpha')\\ \mathrm{Prog}(A) \vdash A(\bar{\xi})\end{array}}{\mathrm{Prog}(A) \vdash \forall x^{\boldsymbol{O}} A(x)}\, r\forall\boldsymbol{O}}{\vdash\ \mathrm{Prog}(A) \to \forall x^{\boldsymbol{O}} A(x)}\, r\to$$

i.e. to $\boldsymbol{\pi}_\alpha$: ${}^f\boldsymbol{\pi}_{\alpha'} \simeq \boldsymbol{\pi}_\alpha$ if $f \in I(\alpha, \alpha')$.

10.1.14. Definition.

Let $\boldsymbol{L}$ be a $\boldsymbol{\beta}$-language, let $\Gamma \vdash \Delta$ be a sequent of $\boldsymbol{L}$, and let $\boldsymbol{T}$ be a $\boldsymbol{\beta}$-theory in the language $\boldsymbol{L}$; a $\boldsymbol{\beta}$-proof of $\Gamma \vdash \Delta$ in $\boldsymbol{T}$ is a family $(\boldsymbol{\pi}_\alpha)_{\alpha \in 0n}$ such that, for all α:

$\boldsymbol{\pi}_\alpha$ is a proof of $\Gamma \vdash \Delta$ in $\boldsymbol{T}[\alpha] = \boldsymbol{L}K_\alpha + \boldsymbol{T}$ and enjoying the **homogeneity condition**:

$$\forall\alpha\forall\alpha'\forall f \in I(\alpha, \alpha')\ {}^f\boldsymbol{\pi}_{\alpha'} \simeq \boldsymbol{\pi}_\alpha\ .$$

To the $\boldsymbol{\beta}$-proof $(\boldsymbol{\pi}_\alpha)$ associate the function f: $f(n) = \ulcorner\boldsymbol{\pi}_n\urcorner$ (since $\boldsymbol{\pi}_n$ is a finite proof, there is an obvious way of defining a Gödel number $\ulcorner\boldsymbol{\pi}_n\urcorner$...). $(\boldsymbol{\pi}_\alpha)$ is said to be **recursive** (resp. **prim. rec.**) iff the function f is recursive (resp. prim. rec.). If $(\boldsymbol{\pi}_\alpha)$ is recursive, then an index of f is called a a **code** of $(\boldsymbol{\pi}_\alpha)$.

10.1.15. Example.
The family $(\boldsymbol{\pi}_\alpha)$ of 10.1.13 is the typical example of a $\boldsymbol{\beta}$-proof: this family if a $\boldsymbol{\beta}$-proof of the sequent $\vdash \mathrm{Prog}(A) \to \forall x^{\boldsymbol{O}}\ A(x)$. Furthermore, this $\boldsymbol{\beta}$-proof is prim. rec.

10.1.16. Definition.
The following data define, when $\boldsymbol{T}$ is a $\boldsymbol{\beta}$-theory in the language $\boldsymbol{L}$, a category $\mathbf{DEM}_{\boldsymbol{T}}$:

objects: pairs $(a, \boldsymbol{\pi})$ where $\boldsymbol{\pi}$ is an α-proof.
morphisms from $(\alpha, \boldsymbol{\pi})$ *to* $(\beta, \boldsymbol{\lambda})$: the set $I(\alpha, \boldsymbol{\pi}\,;\, \beta, \boldsymbol{\lambda})$ of all $f \in I(\alpha, \beta)$ s.t. ${}^f\boldsymbol{\lambda}$ is defined and ${}^f\boldsymbol{\lambda} = \boldsymbol{\pi}$.

10.1.17. Theorem.
Consider the following functor from $\mathbf{DEM}_{\boldsymbol{T}}$ to $\mathbf{ON}$:

$$\mathsf{Tr}\Big((\alpha, \boldsymbol{\pi})\Big) = \alpha \qquad \mathsf{Tr}(f) = f\ .$$

(i) Let $\Big((\alpha_i, \boldsymbol{\pi}_i), f_{ij}\Big)$ be a direct system in $\mathbf{DEM}_{\boldsymbol{T}}$, and assume that $\Big((\alpha, \boldsymbol{\pi}), f_i\Big)$ enjoys 8.1.11 (i)–(iii) w.r.t. $\Big((\alpha_i, \boldsymbol{\pi}_i), f_{ij}\Big)$; then the direct system has a direct limit $\Big((\beta, \boldsymbol{\lambda}), g_i\Big)$ in $\mathbf{DEM}_{\boldsymbol{T}}$ and $(\beta, g_i) = \varinjlim\ (\alpha_i, f_{ij})$.

(ii) Assume that $f_i \in I\Big((\alpha_i, \boldsymbol{\pi}_i), (\beta, \boldsymbol{\lambda})\Big)$ $(i = 1, 2)$; then the pull-back of f_1 and f_2, considered as morphisms in the category $\mathbf{DEM}_{\boldsymbol{T}}$ exists, and is equal to their pull-back when f_1 and f_2 are considered as morphisms in $\mathbf{ON}$.

Proof. The proof essentially rests upon the following lemma:

10.1.18. Lemma.
If $\boldsymbol{\pi}$ is an α-proof, one can construct a function $x_{\boldsymbol{\pi}}$ from the set $P_f(\alpha)$ of finite subsets of α to $P(\alpha)$ such that:

(i) $X \subset Y \to x_{\boldsymbol{\pi}}(X) \subset x_{\boldsymbol{\pi}}(Y)$.

(ii) If $f \in I(\beta, \alpha)$, then ${}^f\boldsymbol{\pi}$ exists iff $\forall X\Big(X$ finite and $X \subset rg(f) \to x_{\boldsymbol{\pi}}(x) \subset rg(f)\Big)$.

Proof. We use the definition of ${}^f\boldsymbol{\pi}$ that is given in 10.1.12; if X is a finite subset of α, then we mutilate w.r.t. f such that $rg(f) = X$, as in 10.1.12 (i); more precisely we consider the result of suppressing in $\boldsymbol{\pi}$, all branches above premises of index $\xi \notin X$ in the rules $(r\forall\boldsymbol{o})$ and $(l\exists\boldsymbol{o})$; we define $x_{\boldsymbol{\pi}}(X)$ to consist of all ordinals ζ such that $\bar{\zeta}$ occurs in the remaining (mutilated) proof. Observe that $x_{\boldsymbol{\pi}}$ is obviously increasing w.r.t. inclusion; furthermore, the hypothesis "X finite" was of no use, and $x_{\boldsymbol{\pi}}(X)$ is defined for arbitrary subsets X of α: we obviously have $x_{\boldsymbol{\pi}}(X) = \bigcup\limits_{\substack{Y \subset X \\ Y \text{ finite}}} x_{\boldsymbol{\pi}}(Y)$.

(**Proof**. If $\bar{\zeta}$ occurs somewhere (at stage s) in $\boldsymbol{\pi}$, then there are only finitely many rules $(r\forall\boldsymbol{o})$ and $(l\exists\boldsymbol{o})$ below this occurrence; if $\beta_1, ..., \beta_n$ are the underlined ordinals occurring in s, then $\zeta \in x_{\boldsymbol{\pi}}(X) \leftrightarrow \beta_1, ..., \beta_n \in X$. □)

Now, ${}^f\boldsymbol{\pi}$ is defined iff $x_{\boldsymbol{\pi}}\Big(rg(f)\Big) \subset rg(f)$: this proves (ii). □

10.1.17 is proved as follows:

(i) Consider $X = \bigcup\limits_i rg(f_i)$; since I is directed, and ${}^{f_i}\boldsymbol{\pi}$ exists for all $i \in I$, it follows that ${}^h\boldsymbol{\pi}$ is defined (with h such that $rg(h) = X$); the result is a consequence of 10.1.18 (ii); assume that $h \in I(\beta, \alpha)$, and define y_i by $f_i = hg_i$. One easily checks that $\Big((\alpha, {}^h\boldsymbol{\pi}), g_i\Big)$ enjoys 8.1.11 (i)–(iii) w.r.t. $\Big((\alpha_i, \boldsymbol{\pi}_i), f_{ij}\Big)$. h is the only solution of $f_i = hg_i$ for all i. Furthermore, ${}^h\boldsymbol{\pi}$ is uniquely determined by the data $(\alpha_i, \boldsymbol{\pi}_i)$, f_{ij}: assume that $\boldsymbol{\pi}_i = (D_i, \varphi_i)$ and define

$$D = \bigcup_i \mu_{g_i}(D_i) \qquad (\mu_f \text{ is defined in 8.D.6})$$

$$\varphi \circ \mu^D_{g_i} = \omega^{1+g_i} \circ \varphi_i \;.$$

Then ${}^h\boldsymbol{\pi} = (D, \varphi)$. All these results show that 8.1.11 (iv) holds.

(ii) Using 10.1.18 (ii), we observe that, if

$$\forall X \text{ finite, } X \subset rg(f_1) \rightarrow x_{\boldsymbol{\pi}}(X) \subset rg(f_1)$$

and $$\forall X \text{ finite, } X \subset rg(f_2) \rightarrow x_{\boldsymbol{\pi}}(X) \subset rg(f_2) \ ,$$

then

$$\forall X \text{ finite, } X \subset rg(f_1) \cap rg(f_2) \rightarrow x_{\boldsymbol{\pi}}(X) \subset rg(f_1) \cap rg(f_2) \ .$$

So if f_3 is defined by $rg(f_3) = rg(f_1) \cap rg(f_2)$, it follows that, the hypothesis: ${}^{f_1}\boldsymbol{\pi}$ and ${}^{f_2}\boldsymbol{\pi}$ defined entail ${}^{f_3}\boldsymbol{\pi}$ defined. The fact that $f_3 = f_1 \wedge f_2$ in the category $\mathbf{DEM}_{\boldsymbol{T}}$ is rather trivial, and therefore left to the reader. □

10.1.19. Remark.
In defining the category $\mathbf{DEM}_{\boldsymbol{T}}$, the following immediate property has been used implicitly: if $f \in I(\alpha, \beta)$ and $g \in I(\beta, \gamma)$ are such that ${}^{g}\boldsymbol{\pi}$ and ${}^{f}({}^{g}\boldsymbol{\pi})$ are defined, then ${}^{(gf)}\boldsymbol{\pi}$ is defined and equal to ${}^{f}({}^{g}\boldsymbol{\pi})$.

10.1.20. Theorem.
$(\boldsymbol{\pi}_\alpha)$ is a $\boldsymbol{\beta}$-proof iff

$$\boldsymbol{\pi}(\alpha) = (\alpha, \boldsymbol{\pi}_\alpha) \qquad \boldsymbol{\pi}(f) = f$$

defines a functor from $\mathbf{ON}$ to $\mathbf{DEM}_{\boldsymbol{T}}$. $\boldsymbol{\pi}$ preserves direct limits and pull-backs.

Proof. The first half of the theorem is a mere triviality. If $(\alpha, f_i) = \varinjlim (\alpha_i, f_{ij})$, then by 10.1.17 (i) $(\boldsymbol{\pi}_\alpha, f_i) = \varinjlim (\boldsymbol{\pi}_{\alpha_i}, f_{ij})$; hence $\boldsymbol{\pi}$ preserves direct limits. Preservation of pull-backs is shown in a similar way. □

10.1.21. Remarks.

(i) It is therefore possible to identify $\boldsymbol{\beta}$-proofs with those functors $\boldsymbol{\pi}$ from $\mathbf{ON}$ to $\mathbf{DEM}_{\boldsymbol{T}}$ such that $\mathsf{Tr} \circ \boldsymbol{\pi} = \mathsf{ID}_{\mathbf{ON}}$.

(ii) Are $\boldsymbol{\beta}$-proofs acceptable "syntactic" objects ? In our discussion 10.1.7, we said that a reasonable condition to ask for, would be that the family $(\boldsymbol{\pi}_\alpha)$ is determined by the subfamily $(\boldsymbol{\pi}_n)_{n<\omega}$. But if $\boldsymbol{\pi}$ is a $\boldsymbol{\beta}$-proof, we have $\big((\alpha, \boldsymbol{\pi}_\alpha), f_i\big) = \varinjlim \big((\alpha_i, \boldsymbol{\pi}_{\alpha_i}), f_{ij}\big)$, when $(\alpha, f_i) =$

$\lim\limits_{\longrightarrow} (\alpha_i, f_{ij})$ with all α_i's $< \omega$. Hence $\boldsymbol{\pi}$ is determined by its restriction to the category $\mathbf{ON} < \omega$. But the restriction of $\boldsymbol{\pi}$ to $\mathbf{ON} < \omega$ is completely determined by the family $(\boldsymbol{\pi}_n)$, since $\boldsymbol{\pi}(f) = f$ for all f. (Moreover, $\boldsymbol{\pi}_\alpha$ can be constructed in an effective way from the family $(\boldsymbol{\pi}_n)$ and α: given a direct system of integers (α_i, f_{ij}) such that $(\alpha, f_i) = \lim\limits_{\longrightarrow} (\alpha_i, f_{ij})$, then observe that (with $\boldsymbol{\pi}_\alpha = (D_i, \varphi_i)$, $= (D, \varphi)$) $D = \bigcup\limits_i \mu_{f_i}(D_i)$ and $\varphi \circ \mu_{f_i} = \omega^{1+f_i} \circ \varphi_i$. This gives an effective way to compute $\boldsymbol{\pi}_\alpha$.

10.1.22. Theorem ($\boldsymbol{\beta}$-completeness; Girard, 1978).
A is $\boldsymbol{\beta}$-valid in $\boldsymbol{T}$ iff there is a $\boldsymbol{\beta}$-proof of the sequent $\vdash A$ in $\boldsymbol{T}$. If $\boldsymbol{T}$ is a recursive theory, then this $\boldsymbol{\beta}$-proof can be chosen recursive.

Proof. (i) One side of the proof is immediate: let $\boldsymbol{m}$ be a $\boldsymbol{\beta}$-model of $\boldsymbol{T}$; then if $\boldsymbol{m}(\boldsymbol{o}) = \alpha$, and $\boldsymbol{\pi}$ is a $\boldsymbol{\beta}$-proof of A, then we consider the α-proof $\boldsymbol{\pi}_\alpha$; then it is immediately shown that all closed instantiations (in $\boldsymbol{m}$) of sequents of $\boldsymbol{\pi}_\alpha$ are true.

(ii) Conversely, we shall prove a more precise result; we use "pre" as usual, i.e. pre α-proof means α-proof without well-foundedness conditions, pre $\boldsymbol{\beta}$-proof means a family $(\boldsymbol{\pi}_\alpha)$ of pre α-proofs... .

10.1.23. Theorem.
It is possible to construct a pre $\boldsymbol{\beta}$-proof $(\boldsymbol{\pi}_\alpha)$ of $\vdash A$ in $\boldsymbol{T}$ with the following property:

$$\begin{aligned}\forall \alpha \in 0n \quad (&\boldsymbol{\pi}_\alpha \text{ is an } \alpha\text{-proof} \leftrightarrow \text{in all models } \boldsymbol{m} \text{ s.t. } \boldsymbol{m}(\boldsymbol{o}) \leq \alpha \\ &\text{all closed instantiations of } A \text{ are true}) .\end{aligned}$$

Furthermore, if $\boldsymbol{T}$ is recursive, and $\boldsymbol{\pi}_n$ is an n-proof for all n, then $\boldsymbol{\pi}$ is recursive.

Proof. The proof essentially follows the argument of Theorem 6.1.13; the modifications are not very big, but rather subtle... .

1. Instead of constructing a pre ω-proof, we shall construct, for each α, a pre α-proof $\boldsymbol{\pi}_\alpha$.

2. We must manage to satisfy the requirements ${}^f\boldsymbol{\pi}_\alpha = \boldsymbol{\pi}_\beta$ for all $f \in I(\alpha, \beta)$.

3. It is not possible to require that an infinite branch in $\boldsymbol{\pi}_\alpha$ contains all formulas of $\boldsymbol{L}[\alpha]$ (first think of the case α not denumerable; more seriously, there is no way of establishing (even for denumerable α) a bijection between ω and $\boldsymbol{L}[\alpha]$ which should be "functorial").

4. Concretely, if we perform our construction in stages, then at each stage, we shall have in all "hypotheses":

   – a sequent $\Gamma \vdash \Delta$ of $\boldsymbol{L}[\alpha]$.

   – a distinguished formula B in $\Gamma \vdash \Delta$.

   – a finite set $\{\alpha_1, ..., \alpha_q\}$ of ordinals, containing all ordinal parameters of $\Gamma \vdash \Delta$; this set will change only when the rule used is $(r\forall\boldsymbol{o})$ or $(l\exists\boldsymbol{o})$... .

   We proceed exactly as in 6.1.13:

(i) Assume that $\Gamma \vdash \Delta$ is a weakening of a sequent $\Gamma' \vdash \Delta'$ of $\boldsymbol{L}[\alpha]$ of one of the following forms:

$$\begin{array}{lll} B \vdash B & & \\ \vdash \bar{\alpha}_i < \bar{\alpha}_j & \text{for some } \alpha_i < \alpha_j & (1 \le i,\ j \le q) \\ \bar{\alpha}_i < \bar{\alpha}_j \vdash & \text{for some } \alpha_j \le \alpha_i & (1 \le i,\ j \le q) \\ \vdash B & \text{where } B \text{ is a proper axiom of } \boldsymbol{T}\ ; & \end{array}$$

(we assume that these axioms are closed...).
Then the portion of proof is:

$$\begin{array}{ccc} \Gamma' & \vdash & \Delta' \\ & \vdots & \qquad \text{weakenings} \\ \Gamma & \vdash & \Delta \end{array}$$

This portion of proof has no hypothesis; we have clearly obtained an α-proof of $\Gamma \vdash \Delta$.

(ii) Assume that we are not in case (i) and that B occurs in Δ; then we have the following possibilities:

1. $B^\dagger$ is atomic; let C be the smallest formula (in the sense of the Gödel numbering 10.1.9 (i)) which does not occur in $\Gamma \vdash \Delta$, *and whose parameters are among* $\alpha_1, ..., \alpha_q$; then our portion of proof is

$$\dfrac{\Gamma \vdash C, \Delta \qquad \Gamma, C \vdash \Delta}{\Gamma \vdash \Delta}\ \mathsf{CUT}$$

The distinguished formulas and parameters will be described later on... .

2. $B^\dagger$ is a conjunction $B_1 \wedge B_2$; the portion of proof is

$$\dfrac{\Gamma \vdash B_1, \Delta \qquad \Gamma \vdash B_2, \Delta}{\Gamma \vdash \Delta}\ r\wedge$$

3. $B^\dagger$ is a disjunction $B_1 \vee B_2$; the portion of proof is

$$\dfrac{\dfrac{\Gamma \vdash B_1, B_2, \Delta}{\Gamma \vdash B_1 \vee B_2, B_2, \Delta}\ r1\vee}{\Gamma \vdash \Delta}\ r2\vee$$

4. $B^\dagger$ is a negation $\neg B_1$; the portion of proof is

$$\dfrac{\Gamma, B_1 \vdash \Delta}{\Gamma \vdash \Delta}\ r\neg$$

5. $B^\dagger$ is an implication $B_1 \rightarrow B_2$; the portion of proof is

$$\frac{\Gamma, B_1 \vdash B_2, \Delta}{\Gamma \vdash \Delta}\ r \rightarrow$$

6. $B^\dagger$ is $\forall x^{\boldsymbol{\tau}}\ B_1(x_p^{\boldsymbol{\tau}})$, with $\boldsymbol{\tau} \neq \boldsymbol{o}$; the portion is

$$\frac{\Gamma \vdash B_1(x_p^{\boldsymbol{\tau}}), \Delta}{\Gamma \vdash \Delta}\ r\forall\boldsymbol{\tau}$$

   where $x_p^{\boldsymbol{\tau}}$ is a variable which does not occur in $\Gamma \vdash \Delta$.

7. $B^\dagger$ is $\exists x^{\boldsymbol{\tau}}\ B_1(x_p^{\boldsymbol{\tau}})$, with $\boldsymbol{\tau} \neq \boldsymbol{o}$; the portion is

$$\begin{array}{l} \Gamma \vdash B_1(t_0^{\boldsymbol{\tau}}), ..., B_1(t_n^{\boldsymbol{\tau}}), \Delta \\ \hline \Gamma \vdash B_1(t_1^{\boldsymbol{\tau}}), ..., B_1(t_n^{\boldsymbol{\tau}}), \Delta \\ \hline \vdots \\ \Gamma \vdash B_1(t_n^{\boldsymbol{\tau}}), \Delta \\ \hline \Gamma \vdash \Delta \end{array} \begin{array}{l} \\ r\exists\boldsymbol{\tau} \\ \\ r\exists\boldsymbol{\tau} \\ \\ \\ r\exists\boldsymbol{\tau} \end{array}$$

   if n is the "stage" and $t_0, ..., t_n$ are the first $n+1$ terms of type $\boldsymbol{\tau}$ according to the Gödel numbering, *whose only ordinal parameters are among* $\alpha_1, ..., \alpha_q$.

8. $B^\dagger$ is $\forall x^{\boldsymbol{o}}\ B_1(x_p^{\boldsymbol{o}})$; the portion is

$$\frac{...\ \Gamma \vdash B_1(\bar{\xi}), \Delta\ ...\ \textsf{all}\ \xi < \alpha}{\Gamma \vdash \Delta}\ 4\forall\boldsymbol{o}$$

9. $B^\dagger$ is $\exists x^{\boldsymbol{o}}\ B_1(x_p^{\boldsymbol{o}})$; the portion is

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash B_1(\bar{\alpha}_1), ..., B_1(\bar{\alpha}_q), \Delta}{\Gamma \vdash B_1(\bar{\alpha}_2), ..., B_1(\bar{\alpha}_q), \Delta}\, r\exists\boldsymbol{o}}{\vdots}\, r\exists\boldsymbol{o}}{\dfrac{\Gamma \vdash B_1(\bar{\alpha}_q), \Delta}{\Gamma \vdash \Delta}\, r\exists\boldsymbol{o}}$$

($\alpha_1, ..., \alpha_1$ are the distinguished parameters associated with the hypothesis $\Gamma \vdash \Delta$).

(iii) Assume that we are not in case (i) and that B occurs in Γ; then we have the following possibilities:

1. If $B^\dagger$ is atomic; the portion is

$$\dfrac{\Gamma \vdash C, \Delta \qquad \Gamma, C \vdash \Delta}{\Gamma \vdash \Delta}\,\mathsf{CUT}$$

where C is defined as in (ii) 1.

2. $B^\dagger$ is a conjunction $B_1 \wedge B_2$; the portion of proof is

$$\dfrac{\dfrac{\Gamma, B_1, B_2 \vdash \Delta}{\Gamma, B_1, B_1 \wedge B_2 \vdash \Delta}\, l2\wedge}{\Gamma \vdash \Delta}\, l1\wedge$$

3. $B^\dagger$ is a disjunction $B_1 \vee B_2$; the portion of proof is

$$\dfrac{\Gamma, B_1 \vdash \Delta \qquad \Gamma, B_2 \vdash \Delta}{\Gamma \vdash \Delta}\, l\vee$$

4. $B^\dagger$ is a negation $\neg B_1$; the portion of proof is

$$\frac{\Gamma \vdash B_1, \Delta}{\Gamma \vdash \Delta}\ l\neg$$

5. $B^\dagger$ is an implication $B_1 \to B_2$; the portion of proof is

$$\frac{\Gamma \vdash B_2, \Delta \qquad \Gamma, B_2 \vdash \Delta}{\Gamma, B_1 \to B_2 \vdash \Delta}\ l\to$$

6. $B^\dagger$ is $\forall x_p^{\boldsymbol{\tau}}\ B_1(x_p^{\boldsymbol{\tau}})$, with $\boldsymbol{\tau} \neq \boldsymbol{o}$; the portion of proof is

$$\begin{array}{l} \Gamma, B_1(t_0), ..., B_1(t_n) \vdash \Delta \\ \hline \Gamma, B_1(t_1), ..., B_1(t_n) \vdash \Delta \\ \hline \vdots \\ \Gamma, B_1(t_n) \vdash \\ \hline \Gamma \vdash \Delta \end{array} \begin{array}{l} \\ l\forall\boldsymbol{\tau} \\ \\ l\forall\boldsymbol{\tau} \\ \\ \\ l\forall\boldsymbol{\tau} \end{array}$$

where $t_0, ..., t_n$ are the first $n+1$ terms of type $\boldsymbol{\tau}$ of $\boldsymbol{L}$, according to the Gödel number *whose only ordinal parameters are among* $\alpha_1, ..., \alpha_q$.

7. $B^\dagger$ is $\exists x_p^{\boldsymbol{\tau}}\ B_1(x_p^{\boldsymbol{\tau}})$, with $\boldsymbol{\tau} \neq \boldsymbol{o}$; assume that $x_p^{\boldsymbol{\tau}}$ is not free in $\Gamma \vdash \Delta$; then our portion of proof is

$$\frac{\Gamma, B_1(x_p^{\boldsymbol{\tau}}) \vdash \Delta}{\Gamma \vdash \Delta}\ l\exists\boldsymbol{\tau}$$

8. $B^\dagger$ is $\forall x_p^{\boldsymbol{o}}\ B_1(x_p^{\boldsymbol{o}})$; our portion of proof is

$$\dfrac{\dfrac{\dfrac{\Gamma, B_1(\bar{\alpha}_1), ..., B_1(\bar{\alpha}_q) \vdash \Delta}{\Gamma, B_1(\bar{\alpha}_2), ..., B_1(\bar{\alpha}_q) \vdash \Delta}\, l\forall\boldsymbol{o}}{\vdots}\, l\forall\boldsymbol{o}}{\dfrac{\Gamma, B_1(\bar{\alpha}_q) \vdash \Delta}{\Gamma \vdash \Delta}\, l\forall\boldsymbol{o}}$$

if $\alpha_1, ..., \alpha_q$ are the distinguished parameters associated with the "hypothesis" $\Gamma \vdash \Delta$.

9. $B^\dagger$ is $\exists x_p^{\boldsymbol{O}}\ B_1(x_p^{\boldsymbol{O}})$; our portion of proof is

$$\dfrac{...\ \Gamma, B_1(\bar{\xi}) \vdash \Delta\ ...\ \mathsf{all}\ \xi < \alpha}{\Gamma \vdash \Delta}\, l\exists\boldsymbol{o}$$

The distinguished formulas in the hypotheses of the portion of proofs constructed in (ii) or (iii) are determined exactly as in 6.1.13; we must also define the distinguished set of parameters: let us say that, at stage 0, in the sequent A, this set is void (A contains no ordinal parameters); if at stage n, the distinguished parameter of $\Gamma \vdash \Delta$ are $\alpha_1, ..., \alpha_q$, and $\Gamma' \vdash \Delta'$ is one of the "hypotheses" constructed above $\Gamma \vdash \Delta$ by means of (ii) or (iii), then the distinguished parameters associated with $\Gamma' \vdash \Delta'$ are

- $\alpha_1, ..., \alpha_q$ in (ii) 1–7 and 9, and (iii) 1–8.
- $\alpha_1, ..., \alpha_q$, ξ in (ii) 8 and (iii) 9, when $\Gamma' \vdash \Delta'$ is the ξ^{th} "hypothesis" $\Gamma \vdash B_1(\bar{\xi}), \Delta$ or $\Gamma, B_1(\bar{\xi}) \vdash \Delta$.

The next step is to prove that the definition of the pre proofs $\boldsymbol{\pi}_\alpha$ is functorial, i.e. that, when $f \in I(\alpha, \beta)$, that ${}^f\boldsymbol{\pi}_\beta = \boldsymbol{\pi}_\alpha$.

We prove by induction on n, that the truncated proofs $\boldsymbol{\pi}^n_\alpha$, obtained at stage n enjoy ${}^f\boldsymbol{\pi}^n_\beta = \boldsymbol{\pi}^n_\alpha$; more precisely if $\Gamma \vdash \Delta$ is an hypothesis of $\boldsymbol{\pi}^n_\beta$, corresponding to $\Gamma' \vdash \Delta'$ in $\boldsymbol{\pi}^n_\alpha$ w.r.t. the mutilation process w.r.t. f, and if $B^\dagger$, $B'^\dagger$, $\alpha_1, ..., \alpha_q$, $\alpha'_1, ..., \alpha'_q$, are the respective distinguished formulas and parameters, then:

– $B^\dagger$ and $B'^\dagger$ correspond one to another (e.g. if $B^\dagger$ is the i^{th} formula of Γ, $B'^\dagger$ is the i^{th} formula of Γ'...).

– $q` = q$ and $\alpha_1 = f(\alpha'_1), ..., \alpha_q = f(\alpha'_q)$.

The details are left to the reader; but observe that in step (ii) 1, the fact that $\lceil C \rceil$ is "functorial" is absolutely essential; furthermore, since the parameters of C are among $\alpha_1, ..., \alpha_q$, and $\alpha_1, ..., \alpha_q \in rg(f)$, then ${}^f C$ is defined... .

Hence, we have clearly defined a pre $\boldsymbol{\beta}$ proof $\boldsymbol{\pi}$; if $\boldsymbol{\pi}(\alpha)$ is an α-proof, i.e. is well-founded, then so is $\boldsymbol{\pi}(\beta)$ for all $\beta \leq \alpha$ (because $\boldsymbol{\pi}(\beta) = {}^f\boldsymbol{\pi}(\alpha)$ for some f). In particular, if $\boldsymbol{m}$ is any $\boldsymbol{\beta}$-model of $\boldsymbol{T}$ such that $\boldsymbol{m}(\boldsymbol{o}) \leq \alpha$, it will be immediate that all closed instantiations of A are valid in $\boldsymbol{m}$. On the other hand, if $\boldsymbol{\pi}(\alpha)$ is not an α-proof, consider a s.d.s. $\Gamma_n \vdash \Delta_n$ in this pre proof; more precisely, we can require that, for all n, $\Gamma_n \vdash \Delta_n$ is an "hypothesis" at stage n, and that $\Gamma_{n+1} \vdash \Delta_{n+1}$ is one of the hypotheses of the portion of proof above $\Gamma_n \vdash \Delta_n$.

Assume that $\alpha_1^n, ..., \alpha_{q_n}^n$ are the distinguished parameters associated with $\Gamma_n \vdash \Delta_n$; if $\boldsymbol{\tau}$ is a type and m is any integer, then one can find $m' \geq m$ s.t. the distinguished formula of $\Gamma_{m'} \vdash \Delta_{m'}$ is $\exists x^{\boldsymbol{\tau}}\ B(x^{\boldsymbol{\tau}})$ and belongs to $\Delta_{m'}$, for some B in which x occurs at least once; then the first $m+1$ terms t of type $\boldsymbol{\tau}$ (according to their Gödel number), whose ordinal parameters are among $\alpha_1^n, ..., \alpha_{q_n}^n$, occur in $\Gamma_{m'+1} \vdash \Delta_{m'+1}$. This plainly shows that a term t occurs in some $\Gamma_n \vdash \Delta_n$ iff all its ordinal parameters belong to X, where X is the set $\{\alpha_i^n\,;\ 1 \leq i \leq q_n\}$.

We define a model $\boldsymbol{m}$ for $\boldsymbol{L}$ as follows:

– $|\boldsymbol{m}|_{\boldsymbol{\tau}}$ is the set of all terms of type $\boldsymbol{\tau}$ occurring in some $\Gamma_n \vdash \Delta_n$.

– $\boldsymbol{m}(f)(t_1, ..., t_k) = ft_1...t_k$.

– $\boldsymbol{m}(p)(t_1, ..., t_k) = \boldsymbol{\dagger}$ if for some n, $pt_1...t_k$ belongs to Γ_n.

– $\boldsymbol{m}(p)(t_1, ..., t_k) = \boldsymbol{f}$ if for some n, $pt_1...t_k$ belongs to Δ_n.

We prove the following lemma:

10.1.24. Lemma.
If B is a formula of $\boldsymbol{L}$ whose ordinal parameters belong to X, there is an integer n such that b belongs to $\Gamma_n \vdash \Delta_n$, but not to both sides of the sequent.

Proof. Similar to 6.1.14. $\square$

Hence the definition just given is sound. We now establish that

10.1.25. Lemma.

(i) $\boldsymbol{m} \models B \leftrightarrow \exists n$ (B occurs in Γ_n).

(ii) $\boldsymbol{m} \models \neg B \leftrightarrow \exists n$ (B occurs in Δ_n) when all ordinal parameters of B are in X.

Proof. We prove (i) and (ii) by induction on the degree of B; the proof is exactly 6.1.15, except that must we must consider two additional cases:

(8) If B is $\forall x^{\boldsymbol{O}}\ B_1(x^{\boldsymbol{O}})$: (i) assume that B occurs in Γ_n, and let $\xi \in X$, and choose k such that $\xi = \alpha_i^k$ for some i s.t. $1 \leq i \leq q_k$; now choose $m \geq n, k$ such that B is the distinguished formula of $\Gamma_m \vdash \Delta_m$; then $\Gamma_{m+1} = \Gamma_m,\ B_1(\overline{\alpha_1^m}), ..., B_1(\overline{\alpha_{q_m}^m})$, hence, by the induction hypothesis $B_1(\bar{\xi})$ is true in $\boldsymbol{m}$; since ξ was arbitrary in X, and $|\boldsymbol{m}|_{\boldsymbol{O}} = \{\bar{\xi}\,;\, \xi \in X\}$, it follows that B is true in $\boldsymbol{m}$.

(ii) Assume that B occurs in Δ_n; then B is the distinguished formula of $\Gamma_p \vdash \Delta_p$ for some $p \geq n$, and $\Delta_{m+1} = B_1(\bar{\xi}), \Delta_m$ for some $\xi \in X$; using the induction hypothesis, $B_1(\bar{\xi})$ is false in $\boldsymbol{m}$, and it follows that B is false in $\boldsymbol{m}$ as well.

(9) If B is $\exists x^{\boldsymbol{O}}\ B_1(x^{\boldsymbol{O}})$: symmetric to (8). $\square$

Now, $\boldsymbol{m}$ is not a $\boldsymbol{\beta}$-model, since $|\boldsymbol{m}|_{\boldsymbol{O}}$ is not an ordinal; but if $f \in I(\beta, \alpha)$ is such that $rg(f) = X$, it is plain that we can construct a $\boldsymbol{\beta}$-model $\boldsymbol{m}'$ which is isomorphic to $\boldsymbol{m}$. It is immediate that the axioms of $\boldsymbol{m}$ and $\neg A$ are true in $\boldsymbol{m}$ (and in $\boldsymbol{m}$), hence we have found a $\boldsymbol{\beta}$-model $\boldsymbol{m}'$ of $\boldsymbol{T}$ in which (some closed instantiation of) A fails; furthermore, $|\boldsymbol{m}|_{\boldsymbol{O}} \leq \alpha$. The essential property of 10.1.23 is therefore established... .

Now, everything done is clearly prim. rec. in the data; if $\boldsymbol{T}$ is a recursive theory, then the trees $\boldsymbol{\pi}_n$ will clearly be recursive.... If $\boldsymbol{\pi}_n$ is an n-proof for all n, then the function which associates to n the height of the tree $\boldsymbol{\pi}_n$ is recursive, and from that it follows that it will be possible to define a recursive function $\{e\}\, n \simeq \ulcorner \boldsymbol{\pi}_n \urcorner$ ... details are left to the reader. $\square$

Proof of 10.1.22. If A is $\boldsymbol{\beta}$-valid in $\boldsymbol{T}$ and $\boldsymbol{\pi}$ is the pre $\boldsymbol{\beta}$-proof constructed in 10.1.23, then $\boldsymbol{\pi}(\alpha)$ must be an α-proof for all α, hence $\boldsymbol{\pi}$ is a $\boldsymbol{\beta}$-proof; moreover, if $\boldsymbol{T}$ is recursive, so is $\boldsymbol{\pi}$. $\square$

10.1.26. Theorem.
The following sets are Π^1_2-complete:

(i) the set of all codes of recursive $\boldsymbol{\beta}$-proofs (for a suitable $\boldsymbol{T}$).

(ii) the set of all codes of recursive dilators.

(iii) the set of all codes of recursive flowers.

Moreover, "recursive" can be strenghtened into "primitive recursive" in (i), (ii), (iii)....

Proof. Recall that $X \subset \mathbb{N}$ is Π^1_2-complete iff:

– X is a Π^1_2 subset of $\mathbb{N}$.

– Any Π^1_2 subset Y of $\mathbb{N}$ can be expressed as $\{z\,;\, f(z) \in X\}$ for some prim. rec. function f.

We first observe that the sets considered in (i)–(iii) are Π^1_2: in the three cases, this is for the same reason: each of these sets can be expressed by

$$e \in X \leftrightarrow \forall f\Big(WO(f) \to WF\big(\pi_e(f)\big)\Big) \wedge A$$

(A arithmetic, WO is the predicate "is a well-order", WF is the predicate "is well-founded"); such a formula of $\boldsymbol{L}^2_{pr}$ is plainly (equivalent to a) Π^1_2 (formula):

(i) A is the formula expressing that e is the index of a recursive pre $\boldsymbol{\beta}$-proof: $\{e\}\, n$ is defined for all n, and is the index of an n-proof. Hence we only need to express that the pre α-proofs $\boldsymbol{\pi}_\alpha$ are well-founded for all α; by an argument akin to 8.2.7, it suffices to look for $\alpha < \aleph_1$, But denumerable ordinals can be encoded by means of functions, and $WO(f)$ expresses that f is the code of an ordinal $< \aleph_1$. Now, if $\{e\}\, n = \ulcorner\boldsymbol{\pi}_n\urcorner = \ulcorner(D_n, \varphi_n)$, we define a tree T_f by: $\langle ..., x_i, ..., \underline{x}_j, ...\rangle \in T_f \leftrightarrow$ all underlined $\underline{x}_j$'s are such that $f(x_j, x_j) = 0$, and if one replaces all underlined elements $\underline{x}_{j_1}, ..., \underline{x}_{j_k}$ by $\underline{x}'_{j_1}, ..., \underline{x}'_{j_k}$ in such a way that: $f(x_{j_p}, x_{j_q}) = 0 \leftrightarrow x'_{j_p} \leq x'_{j_q}$ (all p, q), then the sequence $\langle ..., x_i, ..., \underline{x}'_j, ...\rangle$ belongs to D_n, with $n = \sup\limits_j\ (x'_j + 1)$.

The requirement $\forall\alpha \in 0n\ \boldsymbol{\pi}_\alpha$ is an α-proof can be rewritten as $\forall f\Big(WO(f) \rightarrow\ WRT(T_f)\Big)$. (In fact T_f cannot be expressed by a term $t(f,e)$ of $\boldsymbol{L}^2_{pr}$, because of the use of the partial function $\{e\}\cdot$ in it; but $z \in T_f$ can be expressed by means of an arithmetic formula $T(f,e,z)$, and $WTR(T_f)$ can be written as $\forall g \exists n\ \neg T\Big(f, e, g^*(n)\Big)$ in $\boldsymbol{L}^2_{pr} \ldots$.)

(ii) A is the formula expressing that e is the index of a recursive predilator P_e: $\{e\}\,\langle z_0, ..., z_n\rangle$ is defined for all strictly increasing sequences $z_0 < ... < z_n$ and is equal to $\langle z'_0, ..., z'_m\rangle$ for some m and $z'_0, ..., z'_m$; moreover

1. $\{e\}\,\langle 0, ..., n\rangle$ is equal to a sequence of the form $\langle 0, ..., m\rangle$.

2. If $\langle z_0, ..., z_n\rangle$, $\langle y_0, ..., y_{z_n}\rangle$ are two sequences, consider $\{e\}\,\langle z_0, ..., z_n\rangle = \langle z'_0, ..., z'_m\rangle$, $\{e\}\,\langle y_0, ..., y_{z_n}\rangle = \langle y'_0, ..., y'_p\rangle$ and $\{e\}\,\langle y_{z_0}, ..., y_{z_n}\rangle = \langle y''_0, ..., y''_q\rangle$; then we have $p = z'_m$ and $y''_0 = y'_{z'_0}, ..., y''_q = y'_{z'_m}$.

When f is the code of a denumerable well-order, we define the value of our predilator on f by means of the *normal form theorem*:

- The trace of P_e consists of those pairs $\langle z, n\rangle$ such that $z < lh(\{e\}\,\langle 0, ..., n\rangle) - 1$ and if $\langle z_0, ..., z_m\rangle$ is such that $z_0 < ... < z_m = n$, and $z = (\{e\}\,\langle z_0, ..., z_m\rangle)_i$ for some $i \leq z$, then necessarily $m = n$.

- $P_e(f)$ consists of all finite sequences $\langle z, x_0, ..., x_{n-1}\rangle$ such that $\langle z, n\rangle$ is in the trace of P, and $x_0, ..., x_{n-1}$ are pairwise distinct, and $f(x_0, x_1) = f(x_1, x_2) = ... = f(x_{n-2}, x_{n-1}) = 0$.

- $P_e(f)$ is ordered as follows: $\langle z, x_0, ..., x_{n-1}\rangle < \langle z', x'_0, ..., x'_{n'-1}\rangle$ is defined as follows: choose integers $i_0, ..., i_{n-1}, j_0, ..., j_{n'-1} < n + n'$ such that

$i_k < j_k \leftrightarrow f(x_k, x'_e) = 0$ and $x_k \neq x'_e$ $(k < n,\ l < n')$ and consider the sequences $s = \{e\}\langle i_0, ..., i_{n-1}, n+n'\rangle$ and $s' = \{e\}\langle j_0, ..., j_{n'-1}, n+n'\rangle$; then $(s)_z < (s')_{z'}$.

The linear order $P_e(f)$ is obviously expressed by a formula which is arithmetical, and P_e is a dilator iff:

$$A(e) \wedge Af\Big(WO(f) \to WO\big(P_e(f)\big)\Big) .$$

(iii) Exactly as in (ii) except that a third condition is required in the formula $A(e)$:

3. If $n \leq m$, then $\{e\}\langle 0, ..., n-1, m\rangle$ is of the form $\langle 0, ..., p-1, q\rangle$ for some q and $p \leq q$.

Hence the three sets considered are Π^1_2 sets; now we establish the essential part of Π^1_2-completeness:

(i) By 10.1.5, for a suitable prim. rec. $\boldsymbol{\beta}$-theory $\boldsymbol{T}$ the set $\{A\,;\, \boldsymbol{T} \vdash^{\boldsymbol{\beta}} A\}$ is Π^1_2-complete. In fact, it is immediate that all $\boldsymbol{\beta}$-models of $\boldsymbol{T}$ are such that $\boldsymbol{m}(\boldsymbol{o})$ is infinite, (equivalently $\models \mathsf{Inf}$, where Inf is the formula $\exists x^{\boldsymbol{O}} \forall y^{\boldsymbol{O}} \exists z^{\boldsymbol{O}}\ \neg(z \leq^{\boldsymbol{O}} y)$ hence the set $X = \{A\,;\, \boldsymbol{T} \vdash^{\boldsymbol{\beta}} \mathsf{Inf} \to A\}$ is Π^1_2-complete. Now we apply 10.1.23; we consider the following prim. rec. function

– If n is the Gödel number of a formula A of $\boldsymbol{L}$, then $f(n)$ is the index of the pre $\boldsymbol{\beta}$-proof of $\mathsf{Inf} \to A$ constructed in 10.1.23 (since $\mathsf{Inf} \to A$ is valid in all p-models, with $p < \omega$, it follows that the pre $\boldsymbol{\beta}$-proof of $\mathsf{Inf} \to A$ is recursive; its index is a prim. rec. function of $\ulcorner A \urcorner$...).

– Otherwise $f(n) = 0$.

Then,, if Y is any Π^1_2 set, we can find a prim. rec. function g such that $Y = \{n\,;\, g(n) \in X\}$, and then $Y = \Big\{n\,;\, f\big(g(n)\big)$ is the code of a recursive $\boldsymbol{\beta}$-proof$\Big\}$.

(ii) We shall construct a prim. rec. function h with the following properties:

– If e is the code of a recursive pre $\boldsymbol{\beta}$-proof, then $h(e)$ is the code of a recursive predilator.

– e is the code of a recursive $\boldsymbol{\beta}$-proof iff $h(e)$ is the code of a recursive dilator.

(Using such an h, the Π^1_2 set Y of (i) can be expressed as $Y = \Big\{n\,;\, h\Big(f\Big(g(n)\Big)\Big)$ is the code of a recursive dilator$\Big\}$.) We proceed as follows: If e is the code of the pre $\boldsymbol{\beta}$-proof $(\boldsymbol{\pi}_\alpha)$, then $h(e)$ is a code of the predilator $\mathbf{LIN}_q(D)$, where D is the pre *shqd* s.t. $\boldsymbol{\pi}_\omega = (D, \varphi)$; the details are straightforward, and left to the reader... .

(iii) We construct a prim. rec. function h' with the following properties:

– If e is a code for a recursive predialtor, then $h'(e)$ is a code for a recursive preflower.

– e is a code for a recursive dilator iff $h'(e)$ is a code for a recursive flower.

If e is a code for the recursive predilator F, then $h'(e)$ is defined to be the code of the recursive predilator $\int F(y)dy$, which is a preflower... .

The remaining properties of the theorem are left as exercises for the reader (see 10.1.27). □

10.1.27. Exercise.

(i) Show that the set of codes of recursive sh dendroids is Π^1_2-complete.

(ii) Using (i), show that the set of codes of prim. rec. sh dendroids is Π^1_2-complete.
(*Hint. If D is a sh dendroid, and $s \in D \leftrightarrow \{e\}\,s \simeq 0$, define D' by: $s \in D' \leftrightarrow lh(s) - 1 = \langle (lh(s)-1)_0, (lh(s)-1)_1 \rangle \wedge \exists m \le lh(s)$ $(T_1(e, (s)_{(lh(s)-1)_0}, m) \wedge U(m) = 0)$.*)

(iii) Using (ii) show that the set of all codes of prim. rec. dilators (resp. flowers) is Π^1_2-complete.

(iv) Find a prim. rec. $\boldsymbol{\beta}$-theory $\boldsymbol{T}$ such that the set of all indices of prim. rec. $\boldsymbol{\beta}$-proofs in $\boldsymbol{T}$ is Π^1_2-complete.
(*Hint. Observe that we have a prim. rec. bound on the height of the n-proof $\boldsymbol{\pi}_n$ of $\mathsf{Inf} \to A$ constructed as in 10.1.23, and conclude that $\boldsymbol{\pi}$ is prim. rec.*)

10.1.28. Exercise.
Prove the analogue of the $\boldsymbol{\beta}$-completeness theorem for "$\boldsymbol{\beta}\omega$-logic", i.e. theories in languages with distinguished types $\boldsymbol{L}$ and $\boldsymbol{o}$, $\boldsymbol{L}$ being treated as in Chapter 6, and $\boldsymbol{o}$ as in Chapter 10. Is it possible to derive this result from the $\boldsymbol{\beta}$-completeness theorem?

10.1.29. Exercise.

(i) Prove the analogue of the $\boldsymbol{\beta}$-completeness theorem when ordinals are replaced by linear orders. Call the resulting proofs "**OL**-proofs".

(ii) Show that **OL**-proofs are of height uniformly bounded by some integer N.

(iii) Using (ii) find an effective way of replacing **OL**-proofs by finitary proofs (in usual Σ^0_1 logic) in a system containing an additional axiom "$\leq^{\boldsymbol{o}}$ is a linear order".

10.1.30. Exercise.
Assume that D is a dilator; a D-model is a $\boldsymbol{\beta}$-model whose ordinal part $\boldsymbol{m}(\boldsymbol{o})$ is of the form $D(\alpha)$, for some $\alpha \in 0n$; a D-proof is a family $(\boldsymbol{\pi}_\alpha)$ such that, for all $\alpha \in 0n$, $\boldsymbol{\pi}_\alpha$ is a $D(\alpha)$-proof, and for all f, α, β, s.t. $f \in I(\alpha, \beta)$, $\boldsymbol{\pi}_\alpha = {}^{D(f)}\boldsymbol{\pi}_\beta$.

Prove the analogue of the $\boldsymbol{\beta}$-completeness theorem. What can you say concerning the following particular cases:

(i) $D = \underline{\omega}$.

(ii) $D = \mathsf{Id}$.

(iii) $D = \underline{1} + \mathsf{Id}$.

10.1.31. Exercise.
Assume that $\alpha \in 0n$ and $X \subset \alpha$; we define the concept of (α, X)-proof as follows:

– Axioms whose ordinal parameters are in X are (α, X)-proofs.

– All rules (except $(r\forall\boldsymbol{o})$ and $(l\exists\boldsymbol{o})$) preserve the concept of (α, X)-proof.

– If $(\boldsymbol{\pi}_z)_{z<\alpha}$ are $(\alpha, X \cup \{z\})$-proofs of $\Gamma \vdash A(\bar{z}), \Delta$ (resp. $\Gamma, A\bar{z} \vdash \Delta$), then one can construct an (α, X)-proof of $\Gamma \vdash \forall x^{\boldsymbol{O}}\, A(x), \Delta$ (resp. $\Gamma, \exists x^{\boldsymbol{O}}\, A(x) \vdash \Delta$).

(i) Give a definition of the concept of pre (α, X)- proof.[(*)]

(ii) Assume that $f \in I(\alpha, \beta)$ and that $f(X) = Y$; if $\boldsymbol{\pi}$ is a (β, Y)-proof, show that ${}^f\boldsymbol{\pi}$ always exists and is an (α, X)-proof. Prove the same result for preproofs.[(*)] What does happen when $f(X) \supset Y$? When $f(X) \subset Y$?

(iii) Show that, if $\boldsymbol{\pi}$ is an (α, X_i)-proof for all $i \in I$ $(I \neq \emptyset)$, then $\boldsymbol{\pi}$ is an $(\alpha, \bigcap\limits_i X_i)$-proof; conclude that there is a smallest subset $X_0 \subset \alpha$ s.t. ${}^f\boldsymbol{\pi}$ exists for all f s.t. $rg(f) \supset X_0$. If $(\boldsymbol{\pi}_\alpha)$ is a $\boldsymbol{\beta}$-proof, show that $\boldsymbol{\pi}_\alpha$ is an $(\alpha, \emptyset)$-proof for all α. Extend these results to preproofs.[(*)]

(iv) Show that $\boldsymbol{\pi}$ is an $(\alpha, \emptyset)$-proof, iff given any sequent $\Gamma \vdash \Delta$ occurring in $\boldsymbol{\pi}$, whose ordinal parameters are $z_1, \ldots, z_n$, it is possible to find rules $(R_1), \ldots, (R_n)$ "below" $\Gamma \vdash \Delta$ such that for all i (R_i) is $(r\forall\boldsymbol{o})$ or $(l\exists\boldsymbol{o})$ and $\Gamma \vdash \Delta$ is above the z_i^{th} premise of R_i. Similar question for preproofs. Find a characterization of (α, X)-proofs; conclude the exercise by finding a direct construction of the set X_0 of (iii).

10.1.32. <u>Exercise</u>.
Assume that $(\boldsymbol{\pi}_\alpha)$ is a family of $(\alpha, \emptyset)$-proofs in sequent calculus; using the large cardinal axiom $\exists x(x \to (\omega_1)^{<\omega}_{\omega})$, construct a $\boldsymbol{\beta}$-proof of the same sequent.

(*Hint. If $f \in I(n, x)$, define ${}^f\boldsymbol{\pi}$; ${}^f\boldsymbol{\pi}$ varies through a denumerable set D; define a partition $(X_i)_{i \in D}$ of $\bigcup\limits_i I(n, x)$ by $f \in X_i \leftrightarrow {}^f\boldsymbol{\pi} = i$. The large cardinal axiom says that there is a homogeneous set $Y \subset \alpha$ of order type ω_1 $(= \aleph_1)$; in other terms if $f, g \in I(n, \alpha)$ are such that $rg(f), rg(g) \subset Y$,*

[1(*)] The question of the extension to preproofs is an interesting way of testing one's understanding of the technique of "preobjects".

then ${}^f\boldsymbol{\pi} = {}^g\boldsymbol{\pi}$*. Then consider* $\boldsymbol{\pi}'_{\omega_1} = {}^h\boldsymbol{\pi}$*, where* $h \in I(\omega_1, x)$ *is defined by* $rg(h) = Y$*. Prove that there is a unique* $\boldsymbol{\beta}$*-proof* $(\boldsymbol{\lambda}_\alpha)$ *such that* $\boldsymbol{\lambda}_{\omega_1} = \boldsymbol{\pi}'_{\omega_1}$*.)* (Remark. In some situations, this result is the only way of obtaining $\boldsymbol{\beta}$-completeness, for instance for the negative fragment of intuitionistic $\boldsymbol{\beta}$-logic, w.r.t. an ad hoc concept of $\boldsymbol{\beta}$-model; see the work of Vauzeilles [96] for more details; an open question is whether a general completeness theorem of that kind implies some large cardinal axiom of the kind used here... .)

10.2. Predecessors of a $\boldsymbol{\beta}$-proof

The problem is the following: *functorial* proofs ($\boldsymbol{\beta}$-proofs,...) appear as functors from categories like **ON** into categories of proofs (in the more familiar sense of: infinitary proofs). The most obvious way of dealing with such *functorial* proofs is to work on their values for such and such argument (i.e. object of the initial category); for instacne, when $\boldsymbol{\pi} = (\boldsymbol{\pi}_\alpha)$ is a $\boldsymbol{\beta}$-proof, then we work on the α-proofs $\boldsymbol{\pi}_\alpha$. (A typical example can be found in the cut-elimination theorem of 10.A.12: for each α we define a cut-free α-proof $N(\boldsymbol{\pi}_\alpha)$, then we observe that N is a functor, hence $\big(N(\boldsymbol{\pi}_\alpha)\big)$ defines a $\boldsymbol{\beta}$-proof.) Roughly speaking, the new functorial viewpoint does not bring much when $\boldsymbol{\beta}$-proofs are used in such a way: the constructions already exist for α-proofs, and what we do is only to verify that they are compatible with the new functorial framework. (Of course, for the "philosophy" of the subject, the fact that "infinitary" operations can be controlled by means of finitary approximations, makes an essential change of viewpoint!) But anyway, the functorial side is rather external to the constructions, and by no means affects the "deep structure" of what is going on. Let us label this inessential use of functoriality as the *pointwise* techniques. Non pointwise techniques, which make an essential use of functoriality are called *global*. A typical example is the cut-elimination theorem of Chapter 11 (11.4.1); let us explain why there is no pointwise cut-elimination theorem in that case: we are working with a $\boldsymbol{\beta}$-theory $\boldsymbol{T}$ which is such that $\boldsymbol{T}[\alpha]$ is inconsistent for many values α (for instance, for all $\alpha < \omega_1^{CK}$); we prove a cut-elimination theoremm for $\boldsymbol{T}$; but this theorem cannot be obtained through a cut-elimination theorem for the $\boldsymbol{T}_\alpha$'s, since they can be inconsistent, whereas cut-elimination implies inconsistency: a pointwise cut-elimination is therefore impossible.

The natural question to ask is the following: which tool can we use when we work directly on proofs-as-functors (i.e. globally)? The natural answer is the following: given a functorial proof, make it appear as the succession of application of *specific rules applying to functorial proofs* without passing through "pointwise" proofs.

The situation here is very close to the results of Chapter 9: predecessors of dilators and predecessors of $\boldsymbol{\beta}$-proofs are indeed closely related.

Our concept of $\boldsymbol{\beta}$-proof is too restrictive and we shall consider a more general concept of F-proof:

10.2.1. Definition.
Assume that F is a dilator; then

(i) An F**-model** of a $\boldsymbol{\beta}$-theory is a $\boldsymbol{\beta}$-model of $\boldsymbol{T}$ whose ordinal part $\boldsymbol{m}(\boldsymbol{o})$ is of the form $F(\alpha)$, for some $\alpha \in 0n$.

(ii) An F**-proof** in $\boldsymbol{T}$ is a family $(\boldsymbol{\pi}_\alpha)$ such that, for all α, $\boldsymbol{\pi}_\alpha$ is a $F(\alpha)$-proof, and if $f \in I(\alpha, \beta)$, ${}^{F(f)}\boldsymbol{\pi}_\beta = \boldsymbol{\pi}_\alpha$.

10.2.2. Remarks.

(i) This concept was considered in Exercise 10.1.30, where we proved completeness of F-proofs w.r.t. F-models.

(ii) In the sequel, we shall be mainly concerned with $\underline{a} + \mathsf{Id}$-proofs, where $a \in 0n$. When $a = 0$, this notion corresponds to the concept of $\boldsymbol{\beta}$-proof; the predecessors of Id-proofs are in general not Id-proofs, but $\underline{a} + \mathsf{Id}$-proofs, for some $a \neq 0$.

10.2.3. Definition.
Assume that $\boldsymbol{\pi} = (\boldsymbol{\pi}_\alpha)$ is an $\underline{a} + \mathsf{Id}$-proof; then we define

(i) The **last rule of** $\boldsymbol{\pi}$: consider the $a + \alpha$-proof $\boldsymbol{\pi}_\alpha$; and let (R_α) be the *name* of the last rule of $\boldsymbol{\pi}_\alpha$ (e.g. $(R_\alpha) = (Ax)$, $(l\exists)$, $(r1\vee)$, $(l\exists\boldsymbol{o})$, $(\mathsf{CUT}), ...$); an immediate property is that $(R_\alpha) = (R_\beta)$ for all $\alpha, beta \in 0n$; the last rule of $\boldsymbol{\pi}$ is by definition this (R_α).

(ii) If (R) is the last rule of $\boldsymbol{\pi}$, the **conclusion** of (R): assume that $\Gamma_\alpha \vdash \Delta_\alpha$ is the conclusion of $\boldsymbol{\pi}_\alpha$; then $\Gamma_\alpha \vdash \Delta_\alpha$ is a sequent of $\boldsymbol{L}\,[a+\alpha]$, and, if $f \in I(\alpha, \beta)$, we have ${}^{(\mathbf{E}_a+f)}(\Gamma_\beta \vdash \Delta_\beta) = (\Gamma_\alpha \vdash \Delta_\alpha)$: this means that the ordinal parameters of $\Gamma_\beta \vdash \Delta_\beta$ are obtained from those of $\Gamma_\alpha \vdash \Delta_\alpha$ by an application of the function $\mathbf{E}_a + f$. In particular, if ξ is an ordinal parameter of $\Gamma_\alpha \vdash \Delta_\alpha$, then the ordinal

$(\mathbf{E}_a + f)(\xi)$ depends only on β, and this is only possible if $\xi < a$: we have shown that $\Gamma_\alpha \vdash \Delta_\alpha$ is a sequent of $\boldsymbol{L}\,[\alpha]$, which is independent of α; this sequent is by definition the conclusion of $\boldsymbol{\pi}$.

(iii) If (R) is the last rule, and $\Gamma \vdash \Delta$ is the conclusion of $\boldsymbol{\pi}$, the **premises** of (R), and their proofs:

1. If (R) is (Ax), (R) has no premise; we can figure it by:

$$\boldsymbol{\pi} = \left\{ \begin{array}{ll} & Ax \\ \Gamma \vdash \Delta & \end{array} \right. .$$

2. If (R) is (rW), (lW), (rC), (lC), (rE), (lE), $(l1\wedge)$, $(l2\wedge)$, $(r1\vee)$, $(r2v)$, $(r\neg)$, $(l\neg)$, $(r \rightarrow)$, $(r\forall\boldsymbol{\tau})$, $l\forall\boldsymbol{\tau})$, $(r\exists\boldsymbol{\tau})$, $l\exists\boldsymbol{\tau})$ with $\boldsymbol{\tau} \neq \boldsymbol{o}$, $(l\forall\boldsymbol{o})$ or $(r\exists\boldsymbol{o})$, then one can write:

$$\boldsymbol{\pi}_\alpha = \left\{ \begin{array}{ll} \boldsymbol{\lambda}_\alpha \ \vdots & \\ \Lambda_\alpha \vdash \Pi_\alpha & \\ & R \\ \Gamma \vdash \Delta & \end{array} \right. .$$

Observe that $(\boldsymbol{\lambda}_\alpha)$ defines an $\underline{a} + \mathsf{Id}$-proof $\boldsymbol{\lambda}$ (since ${}^{(\mathbf{E}_a+f)}\boldsymbol{\lambda}_\beta = \boldsymbol{\lambda}_\alpha$ when $f \in I(\alpha, \beta)$); if $\Lambda \vdash \Pi$ is the conclusion of this proof, we say that the premise of (R) is $\Lambda \vdash \Pi$, and its proof is $\boldsymbol{\lambda}$:

$$\boldsymbol{\pi} = \left\{ \begin{array}{ll} \boldsymbol{\lambda} \ \vdots & \\ \Lambda \vdash \Pi & \\ & R \\ \Gamma \vdash \Delta & \end{array} \right. .$$

3. If (R) is $(r\wedge)$, $(l\vee)$, $(l \rightarrow)$, (CUT), then write:

$$\boldsymbol{\pi}_\alpha = \left\{ \begin{array}{ccc} \boldsymbol{\lambda}'_\alpha \ \vdots & \boldsymbol{\lambda}''_\alpha \ \vdots & \\ \Lambda'_\alpha \vdash \Pi'_\alpha & \Lambda''_\alpha \vdash \Pi''_\alpha & \\ & & R \\ & \Gamma \vdash \Delta & \end{array} \right.$$

and observe that $(\boldsymbol{\lambda}'_\alpha)$ and $(\boldsymbol{\lambda}''_\alpha)$ define $\underline{a} + \mathsf{Id}$-proofs $\boldsymbol{\lambda}'$ and $\boldsymbol{\lambda}''$, the conclusions of which are $\Lambda' \vdash \Pi'$ and $\Lambda'' \vdash \Pi''$; the premises of (R) are $\Lambda' \vdash \Pi'$ and $\Lambda'' \vdash \Pi''$, respectively proved by $\boldsymbol{\lambda}'$ and $\boldsymbol{\lambda}''$:

$$\boldsymbol{\pi} \;=\; \left\{ \dfrac{\begin{array}{cc} \boldsymbol{\lambda}' \;\vdots & \boldsymbol{\lambda}'' \;\vdots \\ \Lambda' \vdash \Pi' & \Lambda'' \vdash \Pi'' \end{array}}{\Gamma \vdash \Delta} R \right. .$$

4. If (R) is $(r\forall\boldsymbol{o})$ or $(l\exists\boldsymbol{o})$, then write:

$$\boldsymbol{\pi}_\alpha \;=\; \left\{ \dfrac{\begin{array}{c} \boldsymbol{\lambda}_{\alpha,\beta} \;\vdots \\ \ldots\; \Lambda_{\alpha,\beta} \vdash \Pi_{\alpha\beta} \;\ldots\; \mathsf{all}\ \beta < \alpha \end{array}}{\Gamma \vdash \Delta} R \right. .$$

Here the situation is more delicate, because: the number of R is variable, and the sequents $\Lambda_{\alpha,\beta} \vdash \Pi_{\alpha,\beta}$ do not belong in general to $\boldsymbol{L}\,[a]$. But observe that, if $f \in I(\alpha, \alpha')$, then ${}^{(\mathbf{E}_a+f)}\boldsymbol{\lambda}_{\alpha',(\mathbf{E}_a+f)(\beta)} = \boldsymbol{\lambda}_{\alpha\beta}$. Hence, if $\beta \in 0n$, consider the ordinal $b_\beta = \mathsf{sup}\,(a, \beta+1)$, and let $c_\beta = b_\beta - a$; the family $(\boldsymbol{\chi}^\beta_\alpha)_{\alpha\in 0n}$ defined by: $\boldsymbol{\chi}^\beta_\alpha = \boldsymbol{\lambda}_{c_\beta+\alpha,\beta}$ defines a $b_\beta + \mathsf{Id}$-proof $\boldsymbol{\chi}^\beta$:

$${}^{(\mathbf{E}_{b_\beta}+f)}\boldsymbol{\chi}^\beta_{\alpha'} = {}^{(\mathbf{E}_a+\mathbf{E}_{c_\beta}+f)}\boldsymbol{\lambda}_{c_\beta+\alpha',\beta} = \boldsymbol{\lambda}_{c_\beta+\alpha,\beta} = \boldsymbol{\chi}^\beta_\alpha \;.$$

The sequent $\Lambda_{\alpha,\beta} \vdash \Pi_{\alpha\beta}$ therefore belongs to $\boldsymbol{L}\,[b_\beta]$, say $\Lambda_{\alpha,\beta} \vdash \Pi_{\alpha,\beta}$ is $\Theta_\beta \vdash \Xi_\beta$; the premises of (R) are the sequents $\Theta_\beta \vdash \Xi_\beta$, and their proofs are the $b_\beta + \mathsf{Id}$- proofs $\boldsymbol{\chi}^\beta$:

$$\boldsymbol{\pi} \;=\; \left\{ \dfrac{\begin{array}{c} \boldsymbol{\chi}^\beta \;\vdots \\ \ldots\; \Theta_\beta \vdash \Xi_\beta \;\ldots\; \mathsf{all}\ \beta \in 0n \end{array}}{\Gamma \vdash \Delta} R \right. .$$

This rule is very special, since it requires a proper class of premises; however, the premises are not generated in an arbitrary way; for instance the family $(\boldsymbol{\chi}^\beta)$ is itself functorial! (See 10.2.7.)

When (R) is $(r\forall\boldsymbol{o})$, then one can write:

$$\boldsymbol{\pi} \;=\; \left\{ \begin{array}{l} \quad \boldsymbol{\chi}^{\beta} \;\vdots \\ \dots\ \Gamma \vdash A(\bar{\beta}), \Delta' \ \dots \\ \hline \quad \Gamma \vdash \forall x^{\boldsymbol{o}}\ A(x^{\boldsymbol{o}}), \Delta' \end{array} r\forall\boldsymbol{o} \right.$$

and when (R) is $(l\exists\boldsymbol{o})$, we have:

$$\boldsymbol{\pi} \;=\; \left\{ \begin{array}{c} \boldsymbol{\chi}^{\beta} \;\vdots \\ \dots\ \Gamma', A(\bar{\beta}) \vdash \Delta \\ \hline \Gamma', \exists x^{\boldsymbol{o}}\ A(x^{\boldsymbol{o}}) \vdash \Delta \end{array} r\exists\boldsymbol{o} \right. .$$

10.2.4. Definition.

(i) Assume that $\boldsymbol{\pi} = (\boldsymbol{\pi}_{\alpha})$ is a G-proof, and that $T \in I^1(F, G)$; then we eventually define an F-proof ${}^{T}\boldsymbol{\pi} = (\boldsymbol{\lambda}_{\alpha})$ by $\boldsymbol{\lambda}_{\alpha} = {}^{T(\alpha)}\boldsymbol{\pi}$.

(ii) A particular case is $G = \underline{b} + \mathsf{Id}$, $F = \underline{a} + \mathsf{Id}$, and $f \in I(a, b)$, $T = \underline{f} + \mathbf{E}^1_{\mathsf{Id}}$; then we use the notation ${}^{f}\boldsymbol{\pi}$ instead of ${}^{T}\boldsymbol{\pi}$.

10.2.5. Remark.
${}^{T}\boldsymbol{\pi}$ is defined exactly when ${}^{T(\alpha)}\boldsymbol{\pi}_{\alpha}$ is defined for all α; if $f \in I(\alpha, \beta)$, and $T\boldsymbol{\pi}$ is defined, then $T(\beta)F(f) = G(f)T(\alpha)$, and so ${}^{G(f)T(\alpha)}\boldsymbol{\pi}_{\beta}$ is defined and equal to ${}^{T(\alpha)}({}^{G(f)}\boldsymbol{\pi}_{\beta}) = {}^{T(\alpha)}\boldsymbol{\pi}_{\alpha} = \boldsymbol{\lambda}_{\alpha}$, and ${}^{F(f)}\boldsymbol{\lambda}_{\beta} = {}^{F(f)}({}^{T(\beta)}\boldsymbol{\lambda}_{\beta})$ is defined and equal to $\boldsymbol{\lambda}_{\alpha}$. (This establishes that $(\boldsymbol{\lambda}_{\alpha})$ is an F-proof.) The fact that ${}^{F(f)}\boldsymbol{\lambda}_{\beta}$ is defined is a consequence of:

10.2.6. Lemma.
Assume that $g \in I(\gamma, \delta)$, and that $g = g'g''$; assume that the δ-proof $\boldsymbol{\chi}$ is such that ${}^{g}\boldsymbol{\chi}$ and ${}^{g'}\boldsymbol{\chi}$ are defined; then ${}^{g''}({}^{g'}\boldsymbol{\chi})$ is defined and is equal to ${}^{g}\boldsymbol{\chi}$.

Proof. Left to the reader. □

Applied to $g = G(f)T(\alpha)$ and $g' = T(\beta)$, $g'' = F(f)$.

10.2.7. Proposition.
With the notations of 10.2.3 (iii) 4, we have, when $f \in I(\gamma, \gamma')$:

$$^{(\mathbf{E}_a + f + \mathbf{E}_1)}\boldsymbol{\chi}^{a+\gamma'} = \boldsymbol{\chi}^{a+\gamma} \ .$$

Proof. Easy exercise for the reader. (See the proof 10.2.14 (iv) for a close argument.) □

10.2.8. Definition.
We define the relation $\ll$: "is a **predecessor** of" between $\underline{a} + \mathsf{Id}$-proofs, as follows:

(i) The immediate predecessors of $\boldsymbol{\pi}$ are (with the notations of 10.2.3):

- if (R) is 1-ary, the proof $\boldsymbol{\lambda}$ of 10.2.3 (iii) 2.
- if (R) is 2-ary, the proofs $\boldsymbol{\lambda}'$, $\boldsymbol{\lambda}''$ of 10.2.3 (iii) 3.
- if (R) is $0n$-ary, the proofs of $\boldsymbol{\chi}^{\beta}$ of 10.2.3 (iii) 4.

(ii) $\boldsymbol{\pi} \ll \boldsymbol{\pi}'$ means that there is a sequence $\boldsymbol{\pi} = \boldsymbol{\pi}_0, ..., \boldsymbol{\pi}_n = \boldsymbol{\pi}'$ such that $n \neq 0$, and for all $i < n$ $\boldsymbol{\pi}_i$ is an immediate predecessor of $\boldsymbol{\pi}_{i+1}$.

10.2.9. Theorem.
The relation $\ll$ is well-founded.

Proof. Otherwise, there is a sequence $\boldsymbol{\pi} = \boldsymbol{\pi}^0, ..., \boldsymbol{\pi}^n, ...$ such that, for all n, $\boldsymbol{\pi}^{n+1}$ is an immediate predecessor of $\boldsymbol{\pi}^n$; if $\boldsymbol{\pi}^n$ is an $a_n + \mathsf{Id}$-proof, then $a_n \leq a_{n+1}$ for all n; then let $a = \sup\limits_n a_n$, and let $a' = \omega^a$; then $a_n + a' = a'$ for all n, and, if $\boldsymbol{\pi}^n = (\boldsymbol{\pi}^n_\alpha)$, then we can consider the a'-proofs $\boldsymbol{\pi}^n_{a'}$: if (R^n) is the last rule of $\boldsymbol{\pi}^n$, then

- if (R^n) is unary, then $\boldsymbol{\pi}^{n+1}_{a'}$ is a strict subproof of $\boldsymbol{\pi}^n_{a'}$.
- if (R^n) is binary, then $\boldsymbol{\pi}^{n+1}_{a'}$ is a strict subproof of $\boldsymbol{\pi}^n_{a'}$.
- if (R^n) is $0n$-ary, and $\boldsymbol{\pi}^{n+1}$ is of the form $\boldsymbol{\chi}^{\beta}$, then $\beta < a_{n+1} \leq a \leq a'$, and $\boldsymbol{\chi}^{\beta}_{a'}$ is a strict subproof of $\boldsymbol{\pi}^n_{a_{n+1}+a'} = \boldsymbol{\pi}^n_{a'}$.

We have found a s.d.s. for the relation "is a strict subproof", starting with $\boldsymbol{\pi}^0_{a'} = \boldsymbol{\pi}_{a'}$; this contradicts the well-foundedness of $\boldsymbol{\pi}_{a'}$. □

10.2.10. Remark.

We shall now compare the relations $\ll$ that have been defined between dilators, and between proofs; however, the relation $\ll$ between dilators must be modified into $\ll'$: in 9.4.2 replace (i) by (i)$'$: $H \neq \underline{0} \rightarrow G \ll' F+G+H$.

This is a variant of the concept of predecessor of a dilator, which has the following features:

- $\ll$ is still well-founded.
- but the class of predecessors of a given element is not necessarily linearly ordered.

10.2.11. Definition.

(i) Assume that $\boldsymbol{\pi}$ is an α-proof; then we define an ordinal $\mathsf{Lin}(\boldsymbol{\pi})$ as follows:

- If $\boldsymbol{\pi}$ consists of an axiom, $\mathsf{Lin}(\boldsymbol{\pi}) = 0$.
- If the last rule of $\boldsymbol{\pi}$ is unary, and is applied to $\boldsymbol{\lambda}$, then $\mathsf{Lin}(\boldsymbol{\pi}) = \mathsf{Lin}(\boldsymbol{\lambda}) + 1$.
- If the last rule of $\boldsymbol{\pi}$ is binary, and is applied to $\boldsymbol{\lambda}'$ and $\boldsymbol{\lambda}''$ then $\mathsf{Lin}(\boldsymbol{\pi}) = \mathsf{Lin}(\boldsymbol{\lambda}') + 1 + \mathsf{Lin}(\boldsymbol{\lambda}'') + 1$ (of course $\boldsymbol{\lambda}'$ is the "leftmost" predecessor...).
- If the last rule of $\boldsymbol{\pi}$ is α-ary and is applied to the $\boldsymbol{\lambda}^\xi$'s, then $\mathsf{Lin}(\boldsymbol{\pi}) = \Big(\sum\limits_{\xi<\alpha} \mathsf{Lin}(\boldsymbol{\lambda}^\xi) + 1\Big)$.

(ii) Assume that $\boldsymbol{\pi}$ is a β-proof, that $f \in I(\alpha, \beta)$, and that ${}^f\boldsymbol{\pi}$ is defined; then we define a function $\mathsf{Lin}(f, \boldsymbol{\pi}) \in I\big(\mathsf{LIN}({}^f\boldsymbol{\pi}), \mathsf{Lin}(\boldsymbol{\pi})\big)$ as follows:

- If $\boldsymbol{\pi}$ consists of an axiom, $\mathsf{Lin}(f, \boldsymbol{\pi}) = \mathbf{E}_0$.
- If $\boldsymbol{\pi}$ comes from $\boldsymbol{\lambda}$ by means of a unary rule (the rules $(r\forall\boldsymbol{o})$ and $(l\exists\boldsymbol{o})$ are not considered as unary, even when $\beta = 1\,!$), then $\mathsf{Lin}(f, \boldsymbol{\pi}) = \mathsf{Lin}(f, \boldsymbol{\lambda}) + \mathbf{E}_1$.

- If $\boldsymbol{\pi}$ comes from $\boldsymbol{\lambda}'$ and $\boldsymbol{\lambda}''$ by means of a binary rule (if $\beta = 2$, $(r\forall\boldsymbol{o})$ and $(l\exists\boldsymbol{o})$ are not considered as binary!), then $\mathsf{Lin}(f,\boldsymbol{\pi}) = \mathsf{Lin}(f,\boldsymbol{\lambda}') + \mathbf{E}_1 + \mathsf{Lin}(f,\boldsymbol{\lambda}'') + \mathbf{E}_1$.
- If $\boldsymbol{\pi}$ comes from $(\boldsymbol{\lambda}^\xi)^{\xi<\beta}$ by means of $(r\forall\boldsymbol{o})$ or $(l\exists\boldsymbol{o})$, then $\mathsf{Lin}(f,\boldsymbol{\pi}) = \sum\limits_{\xi<f} (\mathsf{Lin}(f,\boldsymbol{\lambda}_{f(\xi)}) + \mathbf{E}_1)$.

(iii) Assume that $(\boldsymbol{\pi}_\alpha) = \boldsymbol{\pi}$ is an $\underline{a}+\mathsf{Id}$-proof; then we define $\mathbf{LIN}(\boldsymbol{\pi})$ by:

$$\mathbf{LIN}(\boldsymbol{\pi})(\alpha) = \mathsf{Lin}(\boldsymbol{\pi}_\alpha)$$

$$\mathbf{LIN}(\boldsymbol{\pi})(f) = \mathsf{Lin}(f,\boldsymbol{\pi}_\beta) \qquad \text{when } f \in I(\alpha,\beta)\ .$$

10.2.12. <u>Theorem</u>.
If $\boldsymbol{\pi}$ is an $\underline{a}+\mathsf{Id}$-proof, then $\mathbf{LIN}(\boldsymbol{\pi})$ is a dilator; more precisely, let (R) be the last rule of $\boldsymbol{\pi}$:

(i) If $(R) = (Ax)$, then $\mathbf{LIN}(\boldsymbol{\pi}) = \underline{0}$.

(ii) If (R) is unary, and the immediate predecessor of $\boldsymbol{\pi}$ is $\boldsymbol{\lambda}$, then $\mathbf{LIN}(\boldsymbol{\pi}) = \mathbf{LIN}(\boldsymbol{\lambda}) + \underline{1}$.

(iii) If (R) is binary, and the immediate predecessors of $\boldsymbol{\pi}$ are $\boldsymbol{\lambda}'$ and $\boldsymbol{\lambda}''$, then $\mathbf{LIN}(\boldsymbol{\pi}) = \mathbf{LIN}(\boldsymbol{\lambda}') + \underline{1} + \mathbf{LIN}(\boldsymbol{\lambda}'') + \underline{1}$.

(iv) If (R) is $0n$-ary, and the immediate predecessors of $\boldsymbol{\pi}$ are $(\boldsymbol{\chi}^\beta)_{\beta\in 0n}$, consider the two-variable dilator

$$F(x,y) = \mathsf{Lin}(\boldsymbol{\chi}_x^{a+y}) \qquad F(f,g) = \mathsf{Lin}(g+\mathbf{E}_1+f\,;\,\boldsymbol{\chi}_{x'}^{a+y'})$$

(when $f \in I(x,x')$, $g \in I(y,y')$), and let $G = \int (F+\underline{1})dy$; then $\mathbf{LIN}(\boldsymbol{\pi}) = \sum\limits_{\xi<\alpha} (\mathbf{LIN}(\boldsymbol{\chi}^\xi)+\underline{1}) + \mathbf{UN}(G)$.

10.2.13. <u>Corollary</u>.
If $\boldsymbol{\pi} \ll \boldsymbol{\pi}'$, then $\mathbf{LIN}(\boldsymbol{\pi}) \ll' \mathbf{LIN}(\boldsymbol{\pi}')$.

<u>Proof</u>. Since $\ll$ and $\ll'$ are transitive, it suffices to investigate the case "$\boldsymbol{\pi}$ immediate predecessor of $\boldsymbol{\pi}'$";

1. If the last rule (R) of $\boldsymbol{\pi}'$ is unary, then $\mathbf{LIN}(\boldsymbol{\pi}') = \mathbf{LIN}(\boldsymbol{\pi}) + \underline{1}$ and $\mathbf{LIN}(\boldsymbol{\pi})$ is a predecessor of $\mathbf{LIN}(\boldsymbol{\pi}')$.

2. If the last rule (R) of $\boldsymbol{\pi}'$ is binary, and if $\boldsymbol{\pi}''$ is the other immediate predecessor of $\boldsymbol{\pi}'$, then

   either $\mathbf{LIN}(\boldsymbol{\pi}') = \mathbf{LIN}(\boldsymbol{\pi}) + \underline{1} + \mathbf{LIN}(\boldsymbol{\pi}'') + \underline{1}$
   or $\quad\mathbf{LIN}(\boldsymbol{\pi}') = \mathbf{LIN}(\boldsymbol{\pi}'') + \underline{1} + \mathbf{LIN}(\boldsymbol{\pi}) + \underline{1}$

   in both cases $\mathbf{LIN}(\boldsymbol{\pi}) \ll' \mathbf{LIN}(\boldsymbol{\pi}')$.

3. If the last rule (R) of $\boldsymbol{\pi}'$ is $0n0$ary, and if $\boldsymbol{\pi}$ is $\boldsymbol{\chi}^{\xi}$, then observe that the partial sums $H^{\eta} = \sum\limits_{\zeta<\eta} (\mathbf{LIN}(\boldsymbol{\chi}^{\zeta}) + \underline{1})$ are predecessors (for $\ll$) of $\mathbf{LIN}(\boldsymbol{\pi})$.

   (Proof. If $\eta \leq a$, then $\mathbf{LIN}(\boldsymbol{\pi}) = H^{\eta} + H'$ for some $H' \neq \underline{0}$; if $\eta \geq a$, say $\eta = a + \eta'$, then the dilator $H^{a} + \mathbf{SEP}\,\mathbf{UN}(G)(\cdot, \eta') = H^{a} + G(\cdot, \eta')$ is a predecessor of $\mathbf{LIN}(\boldsymbol{\pi})$; but $G(\cdot, \eta') = \sum\limits_{\zeta<\eta'} (\mathbf{LIN}(\boldsymbol{\chi}^{a+\zeta}) + \underline{1})$, hence $H^{a} + G(\cdot, \eta') = H^{\eta}$ is a predecessor of $\mathbf{LIN}(\boldsymbol{\pi})$. $H^{\xi+1} = H^{\xi} + \mathbf{LIN}(\boldsymbol{\chi}^{\xi}) + \underline{1}$ shows that $\mathbf{LIN}(\boldsymbol{\chi}^{\xi})$ is a predecessor of $\mathbf{LIN}(\boldsymbol{\pi})$ (for $\ll'$). $\square$)

Proof of 10.2.12. The fact that $\mathbf{LIN}(\boldsymbol{\pi})$ is a dilator (and that F in (iv) preserves direct limits and pull-backs) is left to the reader; we verify (i)–(iv): (i)–(iii) are completely trivial, hence it remains to check (iv): let $H = \mathbf{LIN}(\boldsymbol{\pi})$; then $H(\alpha) = \sum\limits_{\alpha<\beta} \mathsf{Lin}(\boldsymbol{\lambda}_{\alpha\beta}) + 1$ (with the notations of 12.2.3). But $\boldsymbol{\lambda}_{\alpha\beta}$ is equal to $\boldsymbol{\chi}_{\alpha-c_{\beta}}$, i.e.

$$H(\alpha) = \sum_{\beta<\alpha} \mathbf{LIN}(\boldsymbol{\chi}^{\beta})(a - c_{\beta}) \ =$$

$$\sum_{\xi<\alpha} (\mathbf{LIN}(\boldsymbol{\chi}^{\xi})(\alpha) + 1) + \sum_{\beta<\alpha} \Big(\mathbf{LIN}(\boldsymbol{\chi}^{a+\beta})\big(\alpha - (\beta+1)\big) + 1\Big) \ .$$

But $\sum\limits_{\beta<\alpha} \Big(\mathbf{LIN}(\boldsymbol{\chi}^{a+\beta})\big(\alpha - (\beta+1)\big) + 1\Big) = \sum\limits_{\beta<\alpha} (F(\alpha - (\beta+1), \beta) + 1) =$ $\mathbf{UN}(G)(\alpha)$ by 9.3.18. This shows that $\mathbf{LIN}(\boldsymbol{\pi})$ and $\sum\limits_{\xi<\alpha} (\mathbf{LIN}(\boldsymbol{\chi}^{\xi}) + \underline{1}) +$ $\mathbf{UN}(G)$ take the same values on ordinals; the case of morphisms is similar.

□

10.2.14. Theorem.
The "rules" of inference defined in 10.2.3 are valid for $\boldsymbol{\beta}$-proofs; more precisely, to each application of one of these rules corresponds a transformation of $\underline{a}+\mathsf{Id}$-proofs.

Proof. Concretely this means that, if we are given a family of functorial proofs which is such that rule (R) can be applied, then the result of applying the rule can be directly obtained by a functorial proof:

(i) If (R) is 0-ary (axiom), this means that $\Gamma \vdash \Delta$ is an axiom of $\boldsymbol{T}[a]$, hence of $\boldsymbol{T}[\beta]$ for all $\beta \geq a$. The proofs:

$$\boldsymbol{\pi}_\alpha = \begin{cases} & \quad Ax \\ \Gamma \vdash \Delta & \end{cases}$$

define an $a+\mathsf{Id}$-proof of $\Gamma \vdash \Delta$.

(ii) If (R) is 1-ary, and is applied to $\boldsymbol{\lambda}$, for instance, R is $(r\exists\boldsymbol{o})$ and $\boldsymbol{\lambda}$ is an $\underline{a}+\mathsf{Id}$-proof of $\Gamma \vdash A(\bar{\xi}), \Delta$, with $\xi < a$, then

$$\boldsymbol{\pi}_\alpha = \begin{cases} \boldsymbol{\lambda}_\alpha \; \vdots & \\ \ldots\; \Gamma \vdash A(\bar{\xi}), \Delta & \\ & r\exists\boldsymbol{o} \\ \Gamma \vdash \exists x^{\boldsymbol{o}}\, A(x^{\boldsymbol{o}}), \Delta & \end{cases}$$

defines an $\underline{a}+\mathsf{Id}$-proof of $\Gamma \vdash \exists x\, A, \Delta$.

(iii) If (R) is binary, and is applied to $\boldsymbol{\lambda}'$ and $\boldsymbol{\lambda}''$, for instance $(R) = (\mathsf{CUT})$, then

$$\boldsymbol{\pi}_\alpha = \begin{cases} \boldsymbol{\lambda}'_\alpha \; \vdots \qquad\qquad \boldsymbol{\lambda}''_\alpha \; \vdots & \\ \Gamma \vdash A, \Delta \qquad \Lambda, A \vdash \Pi & \\ & \mathsf{CUT} \\ \Gamma, \Lambda \vdash \Delta, \Pi & \end{cases}$$

defines an $\underline{a} + \mathsf{Id}$-proof.

(iv) The interesting case is when (R) is $0n$-ary, typically $(R) = (r\forall\boldsymbol{o})$: assume that we are given a family $\boldsymbol{\chi}^\beta)_{\beta \in 0n}$, such that, for all β, $\boldsymbol{\chi}^\beta$ is a $b_\beta + \mathsf{Id}$-proof of $\Gamma \vdash A(\bar{\beta}), \Delta$, with $b_\beta = \mathsf{sup}\,(\alpha, \beta + 1)$: (R) can be formally written as:

$$\dfrac{\begin{array}{c}\boldsymbol{\chi}^\beta \ \vdots \\ \ldots\ \Gamma \vdash A(\bar{\beta}), \Delta\ \ldots\ \mathsf{all}\ \beta \in 0n\end{array}}{\Gamma \vdash \forall x^{\boldsymbol{o}}\ A(x^{\boldsymbol{o}}), \Delta}\ r\forall\boldsymbol{o}\ .$$

But we make an extra assumption on the family $(\boldsymbol{\chi}^\beta)$: the family must enjoy 10.2.7, i.e. if $f \in I(\gamma, \gamma')$:

$$^{(\mathbf{E}_a + f + \mathbf{E}_1)}\boldsymbol{\chi}^{a+\gamma'} = \boldsymbol{\chi}^{a+\gamma}\ .$$

We consider now the proofs

$$\boldsymbol{\pi}_\alpha \quad = \quad \left\{ \dfrac{\begin{array}{c}\boldsymbol{\lambda}_{\alpha\beta} = \boldsymbol{\chi}^\beta_{\alpha - c_\beta}\ \vdots \\ \ldots\ \Lambda \vdash A(\bar{\beta}), \Delta\ \ldots\ \mathsf{all}\ \beta < \alpha\end{array}}{\Gamma \vdash \forall x^{\boldsymbol{o}}\ A(x^{\boldsymbol{o}}), \Delta} \right.\ .$$

If $f \in I(\alpha, \alpha')$, then we show that $^{(\mathbf{E}_a + f)}\boldsymbol{\lambda}_{\alpha', (\mathbf{E}_a + f)(\beta)} = \boldsymbol{\lambda}_{\alpha, \beta}$: this will prove that $^{(\mathbf{E}_a + f)}\boldsymbol{\pi}_{\alpha'} = \boldsymbol{\pi}_\alpha$, i.e. that $(\boldsymbol{\pi}_\alpha)$ defines an $\underline{a} + \mathsf{Id}$-proof:

1. If $\beta < a$, then $(\mathbf{E}_a + f)(\beta) = \beta$, and $\boldsymbol{\lambda}_{\alpha', \beta} = \boldsymbol{\chi}^\beta_{\alpha' - a}$; $^{(\mathbf{E}_a + f)}\boldsymbol{\chi}^\beta_{\alpha' - a} = \boldsymbol{\chi}^\beta_{\alpha - a} = \boldsymbol{\lambda}_{\alpha, \beta}$.

2. If $\beta = a + \gamma$, then $(\mathbf{E}_a + f)(\beta) = a + f(\gamma) = \beta'$; if $a + \alpha' = \beta' + \delta'$, define $h \in I(\beta + \delta', \alpha')$ by:

$$h(x) = (\mathbf{E}_a + f)(x) \qquad \mathsf{if}\ x < \beta$$
$$h(\beta + x) = \beta' + x$$

and $g \in I(\alpha, \beta + \delta')$ by:

$$g(z) = z \qquad \text{if } z < \beta$$
$$g(\beta + z) = \beta + \Big((\mathbf{E}_a + f)(\beta + z) - \beta'\Big)\ .$$

Clearly $\mathbf{E}_a + f = hg$, hence ${}^{(\mathbf{E}_a+f)}\boldsymbol{\lambda}_{\alpha',\beta'} = {}^{g}({}^{h}\boldsymbol{\lambda}_{\alpha'\beta'})$; but h can be written as : $h = \mathbf{E}_a + h' + \mathbf{E}_1 + \mathbf{E}_{\delta'-1}$, hence ${}^{h}(\boldsymbol{\lambda}_{\alpha'\beta'}) = ({}^{(\mathbf{E}_a+h'+\mathbf{E}_1)}\boldsymbol{\chi}^{\beta'})_{\delta'-1} = \boldsymbol{\chi}^{\beta}_{\delta'-1}$; now observe that g can be written as: $g = \mathbf{E}_a + \mathbf{E}_\gamma + \mathbf{E}_1 + g'$ for some g' and δ such that $g' \in I(\delta - 1, \delta' - 1)$; then ${}^{g}\boldsymbol{\chi}^{\beta}_{\delta'-1} = \boldsymbol{\chi}^{\beta}_{\delta-1} = \boldsymbol{\lambda}_{\alpha,a+\gamma+1+\delta-1} = \boldsymbol{\lambda}_{\alpha,\beta}$ (a more complete proof would contain a justification of the existence of ${}^{(\mathbf{E}_a+f}\boldsymbol{\pi}_\alpha ...$). $\square$

10.2.15. Remarks.

(i) In fact 10.2.3 and 10.2.14 are reciprocal transformations, as one can easily verify... .

(ii) The problem of the $\boldsymbol{\beta}$-completeness raised by Mostowski was: "find a "$\boldsymbol{\beta}$-rule", analogous to the ω-rule, complete w.r.t. validity in all $\boldsymbol{\beta}$-models". The answer given in 12.1 does not give a *specific rule* (in the case of the ω-rule:

$$\frac{A(\bar{0}), ..., A(\bar{n})\ ...}{\forall x\ A(x)}\ ).$$

Now, 10.2.3 and 10.2.14 answer this question satisfactorily: the $\boldsymbol{\beta}$-rule is exactly the rule (stated here only for $\forall \boldsymbol{o}$): from $b_\beta + \mathsf{Id}$-proofs $\boldsymbol{\chi}^\beta$ of $\Gamma \vdash A(\bar{\beta}), \Delta$ such that $f \in I(\gamma, \gamma') \rightarrow {}^{\mathbf{E}_a+f}\boldsymbol{\chi}^{a+\gamma'} = \boldsymbol{\chi}^{a+\gamma}$, we can conclude that $\Gamma \vdash \forall x^{\boldsymbol{O}}\ A(x^{\boldsymbol{O}}), \Delta$.

Moreover, it is possible to define a system of deduction using, as rules, the usual rules of sequent calculus, together with the $\boldsymbol{\beta}$-rule (10.2.27).

(iii) The $\boldsymbol{\beta}$-rule, as described in (ii) has a rather uncommon feature: in usual rules, we say: assume that the sequents $\Gamma_i \vdash \Delta_i$ are *provable*, then... . (We do not need to know anything about the proofs of the sequents $\Gamma_i \vdash \Delta_i$.) Here we must look at the given proofs $\boldsymbol{\pi}_i$ of

$\Gamma_i \vdash \Delta_i$ $(= \Gamma \vdash A(i), \Delta)$ and verify an inner functoriality condition (10.2.7). This is a very rare situation, since the applicability of the rule depends not only on the provability of the premises, but also on their proofs.

We now investigate the more general case of F-proofs, where F is arbitrary; this case is of interest if we have in mind the cut-elimination theorem of Chapter 11, where F-proofs play an essential role.

10.2.16. Theorem.
We can define canoncial bijections between the following sets (F is a dilator):

(i) The set $F(0)$.

(ii) The set of dilators F' such that $F = F' + \underline{1} + F''$ for some F'', depending on F'.

(iii) The set of all $a \in F(0n)$ s.t. $a \in rg\, F(\mathbf{E}_{00n})$ (i.e. the set of all points $(z_0\,;\,;\,0n)_F$).

(iv) The set of all functions $0n$ from $0n$ to itself such that: $\forall x t(x) < F(x) \wedge \forall x \forall x' \forall f \in I(x, x')\ F(f)\big(t(x)\big) = t(x')$.

(v) The set $I^1(\underline{1}, F)$.

Proof. The theorem is rather trivial; for instance

(i) $\to$ (iii): To each $z_0 \in F(0)$ (hence $(z_0\,;\,0) \in \mathsf{Tr}(F)$), associate $(z_0\,;\,;\,0n)_F$.

(iii) $\to$ (iv): To $(z_0\,;\,;\,0n)_F$, associate the function $t(x) = (z_0\,;\,;\,x)_F$; then $F(f)\Big((z_0\,;\,;\,x)_F\Big) = (z_0\,;\,;\,x')_F$.

(iv) $\to$ (v): If t belongs to (iv), define $T \in I^1(\underline{1}, F)$, by $T(x)(0) = t(x)$; then the diagrams

$$\begin{tikzcd}
1 \arrow[r, "T(x)"] \arrow[d, "\mathbf{E}_1"'] & F(x) \arrow[d, "F(f)"] \\
1 \arrow[r, "T(y)"'] & F(y)
\end{tikzcd}$$

are commutative.

(v) $\to$ (ii): If $T \in I(\mathsf{Id}, F)$, write T as a sum $T = \sum_{i<f} T_i$; necessarily $f = \mathbf{E}_{1\,\mathbf{LH}(F)}$, and $T_0 = \mathbf{E}^1_{\underline{1}}$. Then

$$F = \sum_{i<f(0)} F_i + \underline{1} + \sum_{f(0)<i<\mathbf{LH}(F)} F_i \ ,$$

i.e. $F = F' + 1 + F''\ldots$.

The other bijections are obtained by combining these five basic bijections... .

□

10.2.17. <u>Definition</u>.
An F-**term** is a member of the set defined in 10.2.16 (iv).

10.2.18. <u>Remark</u>.
The reader who has still problems with dilators, can prefer to define F-terms by (iii), because this is in this variant that the operations on F-terms can be the most easily hand led by the beginner... .

10.2.19. <u>Proposition</u>.
Assume that for all $\alpha \in 0n$ $\Gamma_\alpha \vdash \Delta_\alpha$ is a sequent of $\boldsymbol{L}\,[F(\alpha)]$, and that ${}^{F(f)}(\Gamma_\beta \vdash \Delta_\beta) = \Gamma_\alpha \vdash \Delta_\alpha$ when $f \in I(\alpha, \beta)$; then all ordinal parameters of $\Gamma_\alpha \vdash \Delta_\alpha$ are of the form $t_1(\alpha), ..., t_k(\alpha)$, for some F-terms $t_1, ..., t_k$. (Conversely if $\Gamma_\alpha \vdash \Delta_\alpha$ is a sequent of the form

$$\Gamma\Big(\overline{t_1(\alpha)}, ..., \overline{t_k(\alpha)}\Big) \vdash \Delta\Big(\overline{t_1(\alpha)}, ..., \overline{t_k(\alpha)}\Big)$$

where $\Gamma(x_1^{\boldsymbol{O}}, ..., x_n^{\boldsymbol{O}}) \vdash \Delta(x_1^{\boldsymbol{O}}, ..., x_n^{\boldsymbol{O}})$ has no ordinal parameters, then ${}^{F(f)}(\Gamma_\beta \vdash \Delta_\beta) = \Gamma_\alpha \vdash \Delta_\alpha$ for all α, β and $f \in I(\alpha, \beta)$.)

<u>Proof</u>. Write $\Gamma_\alpha \vdash \Delta_\alpha$ as: $\Gamma(\bar{\xi}^1_\alpha, ..., \bar{\xi}^q_\alpha) \vdash \Delta(\bar{\xi}^{q+1}_\alpha, ..., \bar{\xi}^r_\alpha)$; then the hypothesis ${}^{F(f)}(\Gamma_\beta \vdash \Delta_\beta) = \Gamma_\alpha \vdash \Delta_\alpha$ shows that, for $i = 1, ..., r$ $F(f)(\xi^i_\alpha) = \xi^i_\beta$. Hence the ξ_i's are F-terms. □

10.2.20. <u>Definition</u>.
When $\boldsymbol{L}$ is a $\boldsymbol{\beta}$-language and F is a dilator, we define the language $\boldsymbol{L}\,[F]$ to consist of those families $(A_\alpha)_{\alpha \in 0n}$ such that $A_\alpha \in \boldsymbol{L}\,[F(\alpha)]$ for all α, and

${}^{F(f)}A_\beta = A_\alpha$ for all α, β and $f \in I(\alpha,\beta)$.

10.2.21. Remark.
Equivalently, a formula of $\boldsymbol{L}\,[F]$ is a formula whose ordinal parameters are F-terms (by 10.2.19). Using 10.2.16, it is possible to represent $\boldsymbol{L}\,[F]$ as an ordinary language, in several equivalent ways:

– by 10.2.16 (i) $\boldsymbol{L}\,[F]$ can be identified with the language consisting of all formulas with parameters in $F(0)$.

– by 10.2.16 (iii) these parameters can be chosen in $F(0n)$: they can be identified with the points $(z_0\,;;\,0n)_F$.

In particular, when $F = \underline{a} + \mathsf{Id}$, we see that $\boldsymbol{L}\,[F]$ can be identified with $L\,[\alpha]$.

10.2.22. Definition.
We identify, when F' is of the form $F \circ (\underline{a} + \mathsf{Id})$, the language $\boldsymbol{L}\,[F]$ with a sublanguage of $\boldsymbol{L}\,[F']$; the identification works as follows: to each F-term t, we associate an F'-term t' by $t'(x) = t(a+x)$. If A is a formula of $\boldsymbol{L}\,[F]$, we identify A with the formula of $\boldsymbol{L}\,[F']$ obtained by replacing the F-terms $t_1, ..., t_k$ occurring in A, by $t'_1, ..., t'_k$.

10.2.23. Remark.
The essential thing in 10.2.22 is the identification of F-terms with certain F'-terms; it may be of interest to see how this identification looks like when we consider F-terms through the equivalent viewpoints of 10.2.16:

(i) F'-terms can be identified with $F'(0) = F(a)$; then $(\ )'$ can be viewed as the function $F(\mathbf{E}_{0a})$ from $F(0)$ to $F'(0)$.

(ii) If $F = G + \underline{1} + G_1$, then $F' = G \circ (\underline{a} + \mathsf{Id}) + \underline{1} + G_1 \circ (\underline{a} + \mathsf{Id})$. If F-terms are identified with such G's, then $(\ )'$ appears as composition with $\underline{a} + \mathsf{Id}$.

(iii) If $(z\,;;\,0n)_F$ and $(z'\,;;\,0n)_{F'}$ are the points in $F(0n)$ and $F'(0n)$ corresponding to t and t' respectively, then observe that $0n = a + 0n$, hence

$$(z\,;\,;\,0n)_f \;=\; F(\mathbf{E}_{a0n})\Big((z\,;\,;\,a)_F\Big) = F'(\mathbf{E}_{00n})\Big((z\,;\,;\,a)_F\Big) \;=\; F'(\mathbf{E}_{00n})\Big((z'\,;\,;\,0)_{F'}\Big) = (z'\,;\,;\,0n)_{F'}\ .$$

Hence $(z\,;\,;\,0n)_F = (z'\,;\,;\,0n)_{F'}$ (the ordinal classes corresponding to these points are equal; moreover $z' = (z\,;\,;\,a)_F$).

(iv) $T \in I^1(\underline{1}, F)$ is changed into $T \circ \mathbf{E}^1_{\underline{a}+\mathsf{Id}} \in I^1(\underline{1}, F')$.

Observe that in case (iii) everything is slightly simpler, since the ordinal classes corresponding to t and t' are exactly the same... .

10.2.24. <u>Definition</u>.
Assume that $\boldsymbol{\pi}$ is an F-proof; then we define the last rule of $\boldsymbol{\pi}$, the conclusion of $\boldsymbol{\pi}$, the premises of $\boldsymbol{\pi}$, and their proofs as follows:

(i) The last rule of $\boldsymbol{\pi}$ is defined exactly as in 10.2.3 (i).

(ii) If $\Gamma_\alpha \vdash \Delta_\alpha$ is the conclusion of $\boldsymbol{\pi}_\alpha$, then clearly the family $(\Gamma_\alpha \vdash \Delta_\alpha)$ defines a sequent of $\boldsymbol{L}\,[F]$.

(iii) The premises of $\boldsymbol{\pi}$ and their proofs are defined exactly as in 10.2.3, when the arity of (R) is 0, 1, or 2. Let us look at the case:

4. (R) is $(r\forall\boldsymbol{o})$ or $(l\exists\boldsymbol{o})$; write

$$\boldsymbol{\pi}_\alpha \;=\; \left\{\begin{array}{c} \boldsymbol{\lambda}_{\alpha,\beta}\ \vdots \\ \dots\ \Lambda_{\alpha,\beta}\ \vdash\ \Pi_{\alpha\beta}\ \dots\ \mathsf{all}\ \beta < F(\alpha) \\ \hline \Gamma\ \vdash\ \Delta \end{array}\right. R\ .$$

If $\beta < F(0n)$, write $\beta = (z_0\,;\,x_0,\dots,x_{n-1}\,;\,0n)_F$, and let c_β be the smallest ordinal such that $x_0,\dots,x_{n-1} < c_\beta$. Let $F_\beta = F \circ (\underline{c}_\beta{+}\mathsf{Id})$; then, we can define a F_β-proof $\boldsymbol{\chi}^\beta$ by: $\boldsymbol{\chi}^\beta_\alpha = \boldsymbol{\lambda}_{c_\beta+\alpha,(z_0\,;\,x_0,\dots,x_{n-1}\,;\,c_\beta+\alpha)_F}$.

The fact that $\boldsymbol{\chi}^\beta$ defines an F_β-proof is left to the reader.

10.2.25. <u>Proposition</u>.
Assume that $f \in I(c_\beta, c_{\beta'})$ is such that $\beta' = F(f + \mathbf{E}_{0n})(\beta)$ (i.e. $\beta =$

$(z_0\,;\, x_0, ..., x_{n-1}\,;\, 0n)_F$, and $\beta' = (z_0\,;\, f(x_0), ..., f(x_{n-1})\,;\, 0n)_F)$; consider $F_f \in I(F_\beta, F_{\beta'})$: $F_f = \mathbf{E}_F \circ (\underline{f} + \mathsf{Id})$; then

$$^{F_f}\boldsymbol{\chi}^\beta = \boldsymbol{\chi}^\beta \; .$$

Proof. Left to the reader. □

10.2.26. Remarks.

(i) In the particular case where F is $a + \mathsf{Id}$, then β is an ordinal, c_β computed as in 10.2.24 equals the c_β of 10.2.3, $F_\beta = \underline{a} + \underline{c}_\beta + \mathsf{Id}$, i.e. $\underline{b}_\beta + \mathsf{Id}$, etc.... .

(ii) Hence the last rule of $\boldsymbol{\pi}$ (in 10.2.24 (iii) 4) can be written as:

$$\boldsymbol{\pi} \quad = \quad \left\{ \begin{array}{ll} \quad \boldsymbol{\chi}^\beta \;\vdots & \\ ...\ \Theta_\beta \;\vdash\; \Xi_\beta\ ...\ \beta \in F(0n) & \\ & R \\ \quad \Gamma \;\vdash\; \Delta & \end{array} \right. \; .$$

The sequents $\Theta_\beta \vdash \Xi_\beta$ are in the language $\boldsymbol{L}\,[F_\beta]$; $\Theta_\beta \vdash \Xi_\beta$ is the family

$$\left(\Lambda_{c_\beta+\alpha,(z_0\,;\,x_0,...,x_{n-1}\,;\,c_\beta+\alpha)_F} \vdash \Pi_{c_\beta+\alpha,(z_0\,;\,x_0,...,x_{n-1}\,;\,c_\beta+\alpha)_F}\right)$$

when $a \in 0n$.

(iii) Let us rewrite the premises in the two possible cases for (R), with the convention that F-terms are represented by points $< F(0n)$; one gets

$$\boldsymbol{\pi} \quad = \quad \left\{ \begin{array}{ll} \quad \boldsymbol{\chi}_\beta \;\vdots & \\ ...\ \Gamma \;\vdash\; A(\bar{\beta}), \Delta'\ ...\ \beta \in F(0n) & \\ & r\forall \boldsymbol{o} \\ \quad \Gamma \;\vdash\; \forall x^{\boldsymbol{o}}\; A(x^{\boldsymbol{o}}), \Delta' & \end{array} \right.$$

and

$$\boldsymbol{\pi} \quad = \quad \begin{cases} \qquad\qquad \boldsymbol{\chi}_\beta \ \vdots \\ \ldots\ \Gamma', A(\bar{\beta}) \ \vdash \ \Delta \ \ldots\ \beta \in F(0n) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad l\exists\boldsymbol{o} \\ \Gamma', \exists x^{\boldsymbol{o}}\ Ax^{\boldsymbol{o}} \vdash \ \Delta \end{cases}$$

and we see that the premises of (R), provided we choose representation 10.2.16 (iii) of F-terms, are very naturally written!

(iv) The analogue of 10.2.14 still holds here; essentially, this means that, if $(\boldsymbol{\chi}_\beta)_{\beta<F(0n)}$ is a family of F_β-proofs of $\Gamma \ \vdash \ A(\bar{\beta}), \Delta$, with the property 10.2.25, then it is possible to construct an F-proof of $\Gamma \vdash \forall x^{\boldsymbol{o}} \ A(x^{\boldsymbol{o}}), \Delta$, as follows:

$$\boldsymbol{\pi}_\alpha \quad = \quad \begin{cases} \boldsymbol{\lambda}_{\alpha,\beta} \ \vdots \\ \ldots\ \Lambda \ \vdash \ A(\bar{\beta}), \Delta \ \ldots \ \text{all } \beta < F(\alpha) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad r\forall\boldsymbol{o} \\ \quad \Gamma \vdash \ \forall x^{\boldsymbol{o}} \ A(x^{\boldsymbol{o}}), \Delta \end{cases} .$$

(By defining, when $\beta = (z_0\,;\, x_0, ..., x_{n-1}\,;\, \alpha)_F$, $\boldsymbol{\lambda}_{\alpha,\beta} = \boldsymbol{\chi}_{\alpha-c_\beta}^{(z_0\,;\,x_0,...,x_{n-1}\,;\,0n)_F}$.) The details are left to the reader.

10.2.27. Remark.
One question essentially remains; we have been able to replace locally functorial proofs by rules of a new kind; what happens if we iterate that process? Clearly we obtain a new sequent calculus, whose rules are exactly the usual ones, except for $(r\forall\boldsymbol{o})$ and $(l\forall\boldsymbol{o})$. The passage from a proof in this calculus to a functorial proof (and its converse) is easily done, and left to the reader. But one last remark: the rules $(r\forall\boldsymbol{o})$ and $(l\forall\boldsymbol{o})$ can only be applied when the proofs of the premises satisfy compatibility conditions (10.2.7 or 10.2.25), and it is necessary to define ${}^T\boldsymbol{\pi}$ (10.2.4) when $\boldsymbol{\pi}$ is a proof in the sequent calculus just considered; this can be easily defined, for instance:

$$\text{if } \boldsymbol{\pi} \text{ is} \quad \dfrac{\begin{array}{c}\boldsymbol{\chi}^{\beta} \ \vdots \\ \ldots\ \Theta_\beta \vdash \Xi_\beta \ \ldots \text{ all } \beta < F'(0n)\end{array}}{\Gamma \vdash \Delta}\ r\forall \boldsymbol{o}$$

then

$$^{T}\boldsymbol{\pi} \ = \ \left\{ \dfrac{\begin{array}{c}{}^{T_\beta}\boldsymbol{\chi}^{T(0n)(\beta)} \ \vdots \\ \ldots\ {}^{T}\Theta_\beta \vdash {}^{T}\Xi_\beta \ \ldots \text{ all } \beta < F(0n)\end{array}}{{}^{T}\Gamma \vdash {}^{T}\Delta}\ r\forall \boldsymbol{o} \right.$$

with $T_\beta \in I(F_\beta, F'_{T(0n)(\beta)})$ defined by $T_\beta = T \circ \mathbf{E}_{c_\beta + \mathsf{Id}} \ldots)$.

This basic step can be iterated to define $^{T}\boldsymbol{\pi}$; the details are left to the reader. Observe that we have succeeded in writing $\boldsymbol{\beta}$-logic, $\boldsymbol{\beta}$-proofs in tree-like form, and then there is a natural mutilation process for these trees, but *the mutilation is done w.r.t. natural transformations.*

I have only indicated the possibility of such an approach; I am not convinced, at this stage of the work, of the interestof this alternative presentation; it is important to have it in mind, but presumably this alternative approach is too complicated to be really useful; in the sequel, we shall prefer to work only "locally" in this calculus, i.e. extract the last rule, the last premises, and their proofs, but express them as functorial proofs.

10.2.28. Exercise.

(i) Adapt the main results of this sections to $\omega\boldsymbol{\beta}$-logic.

(ii) Is it possible to do the same thing with pre-$\boldsymbol{\beta}$-proofs, i.e. is it possible to find last rules, premises, conclusions, and their pre-$\boldsymbol{\beta}$-proofs?

Annex 10.A. The calculus $\boldsymbol{L}_{\beta\omega}$

I introduced this calculus in 1978, in analogy with $\boldsymbol{L}_{\omega_1\omega}$; the precise mathematical results which follow have been proved in the following years (1979–82) by several people (M.C. Ferbus, J.F. Husson, J. Vauzeilles).

10.A.1. Definition.
We start with a language $\boldsymbol{L}$ containing two types of objects $\boldsymbol{o}$ and $\boldsymbol{\tau}$; we assume that the only terms of $\boldsymbol{L}$ of type $\boldsymbol{o}$ are variables; then we construct, for any ordinal α, a language $\boldsymbol{L}_{\alpha\omega}$, as follows (the notation is not very satisfactory: in the context of Π^1_1-logic, $\boldsymbol{L}_{\omega_1\omega}$ means the use of conjunctions and disjunctions of length $< \omega_1$, whereas here $\boldsymbol{L}_{\alpha\omega}$ means the use of conjunctions and disjunctions of length α...):

(i) The *terms* of $\boldsymbol{L}_{\alpha\omega}$ are defined by

1. the variables $x_n^{\boldsymbol{\tau}}$ $(n \geq 0)$ are terms of type $\boldsymbol{\tau}$.
2. for all $\xi < \alpha$, $\bar{\xi}$ is a term of type $\boldsymbol{Oo}$.
3. if f is a (n, m)-ary function letter of $\boldsymbol{L}$ (hence taking as values objects of type $\boldsymbol{\tau}$), if $\bar{\xi}_1, ..., \bar{\xi}_n$ are terms of type $\boldsymbol{o}$, if $t_1, ..., t_m$ are terms of type $\boldsymbol{\tau}$, then $f(\bar{\xi}_1, ..., \bar{\xi}_n, t_1, ..., t_m)$ is a term of type $\boldsymbol{\tau}$.
4. all terms of $\boldsymbol{L}_{\alpha\omega}$ are given by 1–3.

(ii) The *formulas* of $\boldsymbol{L}_{\alpha\omega}$ are defined by:

1. if p is a (n, m)-ary predicate letter of $\boldsymbol{L}$, if $t_1, ..., t_m$ are terms of type $\boldsymbol{\tau}$, if $\xi_1, ..., \xi_n < \alpha$, then $p(\bar{\xi}_1, ..., \bar{\xi}_n, t_1, ..., t_m)$ is a(n atomic) formula of $\boldsymbol{L}_{\alpha\omega}$.
2. if A is a formula of $\boldsymbol{L}_{\alpha\omega}$, so is $\neg A$.
3. if A, B are formulas of $\boldsymbol{L}_{\alpha\omega}$, so are $A \wedge B$, $A \vee B$, $A \rightarrow B$.
4. if A is a formula of $\boldsymbol{L}_{\alpha\omega}$, if x is a variable of type $\boldsymbol{\tau}$, then $\forall x A$ and $\exists x A$ are formulas of $\boldsymbol{L}_{\alpha\omega}$.
5. if $(A_\xi)_{\xi<\alpha}$ is a family of formulas of $\boldsymbol{L}_{\alpha\omega}$, involving only finitely many free variables of type $\boldsymbol{\tau}$: $x_1, ..., x_k$, then $\underset{\xi<\alpha}{\boldsymbol{M}} A_\xi$ and $\underset{\xi<\alpha}{\boldsymbol{W}} A_\xi$ are formulas of $\boldsymbol{L}_{\alpha\omega}$.

6. the only formulas of $\boldsymbol{L}_{\alpha\omega}$ are those given by 1–5.

10.A.2. Definition.

(i) Assume that t is a term of $\boldsymbol{L}_{\beta\omega}$ and that $f \in I(\alpha, \beta)$; then we (eventually) define a term $f^{-1}(t) \in \boldsymbol{L}_{\alpha\omega}$, as follows:
$f^{-1}(t)$ is defined if and only if all ordinal parameters of t are of the form $\overline{f(\xi)}$ for some $\xi < \alpha$; if this condition is fulfilled, then $f^{-1}(t)$ is defined as the result of the replacement in t of all subterms $\overline{f(\xi)}$ by terms $\bar{\xi}$, for all $\xi < \alpha$.

(ii) Assume that A is a formula of $\boldsymbol{L}_{\beta\omega}$ and that $f \in I(\alpha, \beta)$; then we (eventually) define a formula $f^{-1}(A) \in \boldsymbol{L}_{\alpha\omega}$, as follows:

1. When A is atomic, $f^{-1}(A)$ is the result of the replacement of all ordinal parameters of A by their inverse image under f; hence $f^{-1}(A)$ is defined iff all ordinal parameters of A belong to the range of f.
2. $f^{-1}(\neg A)$ is defined iff $f^{-1}(A)$ is; in that case $f^{-1}(\neg A) = \neg f^{-1}(A)$.
3. $f^{-1}(A \wedge B)$, $f^{-1}(A \vee B)$, $f^{-1}(A \to B)$ are defined iff $f^{-1}(A)$ and $f^{-1}(B)$ are both defined; in that case $f^{-1}(A \wedge B) = f^{-1}(A) \wedge f^{-1}(B)$, $f^{-1}(A \vee B) = f^{-1}(A) \vee f^{-1}(B)$, $f^{-1}(A \to B) = f^{-1}(A) \to f^{-1}(B)$.
4. $f^{-1}(\forall x A)$, $f^{-1}(\exists x A)$ are defined iff $f^{-1}(A)$ is; in that case $f^{-1}(\forall x A) = \forall x f^{-1}(A)$, $f^{-1}(\exists x A) = \exists x f^{-1}(A)$.
5. $f^{-1}\Big(\underset{\xi<\beta}{\boldsymbol{M}} A_\xi\Big)$, $f^{-1}(\boldsymbol{W} A_\xi)$ are defined iff for all $\xi \in rg(f)$, $f^{-1}(A_\xi)$ is defined; in that case $f^{-1}\Big(\underset{\xi<\beta}{\boldsymbol{M}} A_\xi\Big) = \underset{\xi<\alpha}{\boldsymbol{M}} f^{-1}(A_{f(\xi)})$,
$f^{-1}\Big(\underset{\xi<\beta}{\boldsymbol{W}} A_\xi\Big) = \underset{\xi<\alpha}{\boldsymbol{W}} f^{-1}(A_{f(\xi)})$.

10.A.3. <u>Definition</u>.

(i) We define a category $\mathbf{FOR}_{\boldsymbol{L}}$ as follows:

- *objects*: pairs (α, A), where A is a formula of
- *morphisms*: the sets $I(\alpha, A\,;\,\beta, B)$ consisting of those $f \in I(\alpha, \beta)$ such that $f^{-1}(B)$ is defined and $f^{-1}(B) = A$.

(ii) The language $\boldsymbol{L}_{\boldsymbol{\beta}_\omega}$ consists of all functors F from $\mathbf{ON}$ to $\mathbf{FOR}_{\boldsymbol{L}}$ such that:

- $F(\alpha)$ is of the form (α, F_α) $(\alpha \in 0n)$.
- $F(f) = f$ (for all $f \in I(\alpha, \beta)$).

10.A.4. <u>Remarks</u>.

(i) The analogue of $\boldsymbol{L}_{\boldsymbol{\beta}_\omega}$ when $\boldsymbol{L}$ contains several types of objects $\neq \boldsymbol{o}$ can be easily defined. One can also consider the combination $\boldsymbol{L}_{\boldsymbol{\beta},\omega_1\omega}$ of $\boldsymbol{L}_{\boldsymbol{\beta}_\omega}$ and $\boldsymbol{L}_{\omega_1\omega}$, i.e. when denumerable conjunctions $\dot{\boldsymbol{M}}_{n<\omega} A_n$ are allowed (with $f^{-1}\Big(\dot{\boldsymbol{M}}_{n<\omega} A_n\Big) = \dot{\boldsymbol{M}}_{n<\omega} f^{-1}(A_n)$). All these things are minor and easy-going variants of our definition.

(ii) Another variant would consist in allowing partial conjunctions and disjunctions, when $\lambda \leq \mu \leq \alpha$:

$$\underset{\lambda<\xi<\mu}{\boldsymbol{M}} A_\xi\,, \quad \underset{\lambda\leq\xi<\mu}{\boldsymbol{M}} A_\xi\,, \quad \underset{\lambda<\xi\leq\mu}{\boldsymbol{M}} A_\xi\,, \quad \underset{\lambda\leq\xi\leq\mu}{\boldsymbol{M}} A_\xi$$

with

$$f^{-1}\Big(\underset{\lambda<\xi<\mu}{\boldsymbol{M}} A_\xi\Big) = \underset{f^{-1}(\lambda)<\xi<f^{-1}(\mu)}{\boldsymbol{M}} f^{-1}(A_{f(\xi)})$$

$$f^{-1}\Big(\underset{\lambda<\xi\leq\mu}{\boldsymbol{M}} A_\xi\Big) = \underset{f^{-1}(\lambda)<\xi\leq f^{-1}(\mu)}{\boldsymbol{M}} f^{-1}(A_{f(\xi)})$$

$$f^{-1}\Big(\underset{\lambda\leq\xi<\mu}{\boldsymbol{M}} A_\xi\Big) = \underset{f^{-1}(\lambda)\leq\xi<f^{-1}(\mu)}{\boldsymbol{M}} f^{-1}(A_{f(\xi)}) \qquad \text{etc.\dots}$$

However, if we have specific symbols $\boldsymbol{t}$ and $\boldsymbol{f}$ for true and false, we can replace such partial conjunctions by total ones, e.g. replace $\underset{\lambda\le\xi<\mu}{\boldsymbol{M}} A_\xi$ by $\underset{\xi<\alpha}{\boldsymbol{M}} A'_\xi$, with $A'_\xi = \boldsymbol{t}$ if $\xi < \lambda$ or $\mu \le \xi < \alpha$, $A'_\xi = A_\xi$ if $\lambda \le \xi < \mu$.

(iii) In contrast with $\boldsymbol{L}_{\omega_1\omega}$, the binary connectives $\wedge$ and $\vee$ cannot any longer be defined from $\boldsymbol{M}$ and $\boldsymbol{W}$ in $\boldsymbol{L}_{\boldsymbol{\beta}\omega}$. However, the formula

$$F_\alpha = \underset{\zeta<\alpha}{\boldsymbol{M}} \; \underset{\xi<\alpha}{\boldsymbol{M}} \; G_{\zeta\xi\alpha}$$

(with $G_{\zeta\xi\alpha} = A_\alpha$ if $\zeta \le \xi$, $G_{\zeta\xi\alpha} = B_\alpha$ if $\xi < \zeta$) defines, given two formulas $A(\alpha) = (\alpha, A_\alpha)$, $B(\alpha) = (\alpha, B_\alpha)$, another formula $F(\alpha) = (\alpha, F_\alpha)$ of $\boldsymbol{L}_{\boldsymbol{\beta}\omega}$, which has the property that $F_\alpha \leftrightarrow A_\alpha \wedge B_\alpha$ for all $\alpha \ge 2 \ldots$. This is the best that we can do in that direction.

(iv) Of course, formulas of $\boldsymbol{L}_{\boldsymbol{\beta}\omega}$ can be viewed as families $(F_\alpha)_{\alpha\in 0n}$ such that:

1. for all $\alpha \in 0n$, F_α is a formula of $\boldsymbol{L}_{\alpha\omega}$.
2. for all $\alpha, \beta \in 0n$ and $f \in I(\alpha,\beta)$, $f^{-1}(F_\beta)$ is defined and equals F_α.

(v) It would have been bad taste to consider the quantifiers $\forall x^{\boldsymbol{O}} \; A(x^{\boldsymbol{O}})$ and $\exists x^{\boldsymbol{O}} \; A(x^{\boldsymbol{O}})$, since we have the obvious translations:

$$\forall x^{\boldsymbol{O}} \; A(x^{\boldsymbol{O}}) \rightsquigarrow \underset{\xi<\alpha}{\boldsymbol{M}} \; A(\bar{\xi})$$

$$\exists x^{\boldsymbol{O}} \; A(x^{\boldsymbol{O}}) \rightsquigarrow \underset{\xi<\alpha}{\boldsymbol{W}} \; A(\bar{\xi}) \; .$$

This shows that, for instance, the usual $\boldsymbol{\beta}$-logic of Section 10.1 can be translated in $\boldsymbol{L}_{\boldsymbol{\beta}\omega} \ldots$.

10.A.5. Example.

Let us give one of the most typical examples of formula, deeply connected with the results of Chapter 11: we shall consider the case where $\boldsymbol{L}$ makes no use of the type $\boldsymbol{O}$, i.e. when no atomic formula of $\boldsymbol{L}_{\alpha\omega}$ contains any ordinal parameter $\bar{\xi}$, for all α and $\xi < \alpha$: let $\boldsymbol{\Phi} = \boldsymbol{\Phi}(X, x)$ be a *positive operator* in $\boldsymbol{L}$; then we can consider, for all ξ and $\alpha > \xi$ formulas $A_{\xi\alpha}(x)$: assume that $A_{\zeta\alpha}(x)$ has been defined for all $\zeta < \xi$; the formula obtained by replacing in $\boldsymbol{\Phi}$ all atoms $X(t)$ by formulas $A_{\zeta\alpha}(t)$ is denoted by $\boldsymbol{\Phi}(\lambda y A_{\zeta\alpha}(y), x)$; we define

$$A_{\xi\alpha} = \underset{\zeta<\xi}{\boldsymbol{W}} \ \boldsymbol{\Phi}(\lambda y A_{\zeta\alpha}(y), x)$$

(partial conjunctions have been introduced as abbreviations in 10.A.4 (iii)). One easily shows (induction on ξ) that, if $f \in I(\alpha, \beta)$, then $f^{-1}(A_{f(\xi)\beta})$ is defined and equals $A_{\xi\alpha}$.

The formulas $A_\alpha = \underset{\xi<\alpha}{\boldsymbol{W}} \ A_{\xi\alpha}$ are such that:

$$f^{-1}(A_\beta) = \underset{\xi<\alpha}{\boldsymbol{W}} \ f^{-1}(A_{f(\xi)\beta}) = \underset{\xi<\alpha}{\boldsymbol{W}} \ \xi\alpha = A_\alpha \ .$$

Hence $A(\alpha) = (\alpha, A_\alpha)$ is a formula of $\boldsymbol{L}_{\boldsymbol{\beta}\omega}$. (As usual it is deceiving to say that the construction is made by induction on ξ; there is a tree-like version of $\boldsymbol{L}_{\boldsymbol{\beta}\omega}$, and the construction just made can be handled in this framework, without using induction on ordinals ... see 10.A.15.)

10.A.6. Definition.

Let D be a dilator; then we define the language $\boldsymbol{L}_{D\omega}$ to consist of those functors F from **ON** to $\mathbf{FOR}_{\boldsymbol{L}}$ of the form: (also called D-formulas)

$$F(\alpha) = (D(\alpha), F_\alpha)$$
$$F(f) = D(f) \ .$$

In particular, $\boldsymbol{L}_{\boldsymbol{\beta}\omega}$ can be identified with $\boldsymbol{L}_{\mathsf{Id}\,\omega}$.

10.A.7. Definition.

If F is a formula of $\boldsymbol{L}_{D\omega}$, then we define the **immediate subformulas** of F: (compare with 10.2.24)

writing $F(\alpha) = (\alpha, F_\alpha)$, it is possible to find a "first symbol" of F, namely the first symbol of all F_α's; let us call it by S:

(i) If $S = \neg$, the formula E of $\boldsymbol{L}_{D\omega}$ defined by $F_\alpha = \neg E_\alpha$ is the only immediate subformula of F.

(ii) If $S = \wedge, \vee \rightarrow$, the formulas E' and E'' of $\boldsymbol{L}_{D\omega}$ defined by: $F_\alpha = E'_\alpha \wedge E''_\alpha$ (resp. $E'_\alpha \vee E''_\alpha$, $E'_\alpha \rightarrow E''_\alpha$) are the only immediate subformulas of F.

(iii) If $S = \forall, \exists$, consider the formulas E_α such that $F_\alpha = \forall x E_\alpha$ (resp. $\exists x E\alpha$); then the immediate subformulas of F are all formulas G of $\boldsymbol{L}_{D\omega}$ of the form $G_\alpha = E_\alpha(t)$ for some term t of type $\boldsymbol{s}$.

(iv) If $S = \boldsymbol{M}, \boldsymbol{W}$, consider the formulas $E_{\xi\alpha}$ ($\xi < \alpha$) defined by $F_\alpha = \underset{\xi < D(\alpha)}{\boldsymbol{M}} E_{\xi\alpha}$ (resp. $\underset{\xi < D(\alpha)}{\boldsymbol{W}} E_{\xi\alpha}$). If $\xi < D(0n)$, write $\xi = (z_0\,; x_0, ..., x_{n-1}\,; 0n)_D$, and let $c_\xi = \mathsf{inf}\,\{u\,; x_0, ..., x_{n-1} < u\}$; we define a formula E^ξ of $\boldsymbol{L}_{D\,\circ\,(\underline{c_\xi}+\mathsf{Id})\omega}$ by:

$$E^\xi(\alpha) = E_{(z_0\,; x_0, ..., x_{n-1}\,; c_\xi+\alpha)_D c_\xi+\alpha}\ .$$

Then the immediate subformulas of F are exactly the formulas E^ξ.

A particular case is when D is of the form $\underline{a} + \mathsf{Id}$; then it is immediate that all immediate subformulas of F are in some $\boldsymbol{L}_{\underline{b}+\mathsf{Id}}$ ($b \in 0n$).

The **subformula** relation is the order relation generated by the strict subformula relation.

10.A.8. <u>Remarks</u>.

(i) If F is a formula of $\boldsymbol{L}_{D\omega}$ and $T \in I^1(D', D)$, then one defines (eventually) a formula $T^{-1}(F)$ of $\boldsymbol{L}_{D'\omega}$ by $T^{-1}(F)_\alpha = T(f)^{-1}(F_\alpha)$. In 10.A.7 (iv), if one takes $f \in I(c_\xi, c_{\xi'})$ such that $\xi = (z_0\,; x_0, ..., x_{n-1}\,; 0n)_D$, $\xi' = (z_0\,; f(x_0), ..., f(x_{n-1})\,; 0n)_D$, then

$$f + \mathbf{E}^1_{\mathsf{Id}} \in I\Big(D \,\circ\, (c_\xi + \mathsf{Id}), D \,\circ\, (c_{\xi'} + \mathsf{Id})\Big)$$

and clearly $(f + \mathbf{E}^1_{\mathsf{Id}})^{-1}(E^{\xi'}) = E^\xi$.

(ii) The process of finding subformulas can be inverted into a process of building formulas:

1. If E is a formula of $\boldsymbol{L}_{D\omega}$, then it is possible to define $\neg E$ in $\boldsymbol{L}_{D\omega}$ as follows: $(\neg E)_\alpha = \neg E_\alpha$.
2. If E', E'' are formulas of $\boldsymbol{L}_{D\omega}$, one defines formulas $E' \wedge E''$, $E' \vee E''$, $E' \to E''$ of $\boldsymbol{L}_{D\omega}$ as follows:
$$(E' \wedge E'')_\alpha = E'_\alpha \wedge E''_\alpha \ ,$$
$$(E' \vee E'')_\alpha = E'_\alpha \vee E''_\alpha \ ,$$
$$(E' \to E'')_\alpha = E'_\alpha \to E''_\alpha \ .$$
3. If E is a formula of $\boldsymbol{L}_{D\omega}$ and x is a variable of type $\boldsymbol{\tau}$, we define formulas $\forall x E$ and $\exists x E$ of $\boldsymbol{L}_{D\omega}$ by $(\forall x E)_\alpha = \forall x E_\alpha$, $(\exists x E)_\alpha = \exists x E_\alpha$.
4. If E^ξ is a $D \circ (\underline{c}_\xi + \mathsf{Id})$-formula for all $\xi < D(0n)$, enjoying the condition of (i) above, then we define formulas $\underset{\xi<D(0n)}{\boldsymbol{M}} E^\xi$ and $\underset{\xi<D(0n)}{\boldsymbol{W}} E^\xi$ of $\boldsymbol{L}_{D\omega}$ as follows:
$$\Big(\underset{\xi<D(0n)}{\boldsymbol{M}} E^\xi\Big)_\alpha = \underset{\xi<D(\alpha)}{\boldsymbol{M}} E^{\xi'}_{\alpha'} \, , \quad \Big(\underset{\xi<D(0n)}{\boldsymbol{W}} E^{\xi'}\Big) = \underset{\xi<D(\alpha)}{\boldsymbol{W}} E^{\xi'}_{\alpha'}$$
(with, when $\xi = (z_0\,;\, x_0, ..., x_{n-1}\,;\, \alpha)_D$: $\xi' = (z_0\,;\, x_0, ..., x_{n-1}\,;\, 0n)_D$, and $\alpha = c_{\xi'} + \alpha'$).

It is possible to prove many results which are generalizations of results already obtained for $\boldsymbol{L}_{\omega_1\omega}$ and for usual $\boldsymbol{\beta}$-logic. We shall indicate some of them, but we shall present these things as exercises... .

10.A.9. Exercise (trace).
We define the functor Tr from $\mathbf{FOR}_{\boldsymbol{L}}$ to $\mathbf{ON}$ by $\mathsf{Tr}\big((\alpha, f)\big) = \alpha$, $\mathsf{Tr}(f) = f$; show that

(i) If $\big((\alpha_i, F_i), f_{ij}\big)$ is a direct system in $\mathbf{FOR}_{\boldsymbol{L}}$, and $\big((\alpha, F), f_i\big)$ enjoys 8.1.11 (i)–(iii) w.r.t. $\big((\alpha_i, F_i), f_{ij}\big)$; then the direct system has a direct limit $\big((\beta, G), g_i\big)$ in $\mathbf{FOR}_{\boldsymbol{L}}$, and $(\beta, g_i) = \varinjlim (\alpha_i, f_{ij})$.

(ii) Assume that $f_i \in I(\alpha_i, F_i\,;\, \beta, G)$ $(i = 1, 2)$; then the pull-back $f_1 \wedge f_2$ in $\mathbf{FOR}_{\boldsymbol{L}}$ exists and equals $f_1 \wedge f_2$ in $\mathbf{ON}$.

(iii) If FG is a formula in $\boldsymbol{L}_{D\omega}$, show that F preserves direct limits and pull-backs. Conclude that the family $(F_n)_{n<\omega}$ determines F unambiguously.

10.A.10. <u>Exercise</u>.
If $\alpha \in 0n$ and $X \subset \alpha$, we define the concept of an (α, X)-formula as follows:

- $p(\xi_1, ..., \xi_n, t_1, ..., t_m)$ is an (α, X)-formula iff all the ordinal parameters occurring belong to X.
- If A is an (α, X)-formula, so are $\neg A$, $\forall x A$, $\exists x A$.
- If A and B are (α, X)-formulas, so are $A \wedge B$, $A \vee B$, $A \to B$.
- If A_ξ is a $(\alpha, X \cup \{\xi\})$-formula for all $\xi < \alpha$, $\underset{\xi<\alpha}{\boldsymbol{M}} A_\xi$ and $\underset{\xi<\alpha}{\boldsymbol{W}} A_\xi$ are (α, X)-formulas.

(i) Assume that $f \in I(\alpha, \beta)$ and that $f(X) = Y$; if A is a (β, Y)-formula show that $f^{-1}(A)$ always exists and is an (α, X)-formula. What does happen when $f(X) \supset Y$? When $f(X) \subset Y$?

(ii) Show that, if F is an (α, X_i)-formula for all $i \in I$ $(I \neq \emptyset)$, then F is an $\left(a, \bigcap_i X_i\right)$-formula. Conclude that there exists a smallest subset $X_0 \subset \alpha$ such that $f^{-1}(F)$ exists for all f s.t. $rg(f) \supset X_0$. If F is a formula of $\boldsymbol{L}_{\boldsymbol{\beta}\omega}$, show that F is an $(\alpha, \emptyset)$-formula for all $\alpha \in 0n$.

(iii) If D is a dilator, define a concept of $D(\alpha, X)$-formula, with the following properties:

  - If G is a $D(\beta, Y)$-formula and $f(X) = Y$, then $D(f)^{-1}(G)$ exists and is a $D(\alpha, X)$-formula.
  - If G is a formula of $\boldsymbol{L}_{D\omega}$, then for all $\alpha \in 0n$ $G(\alpha)$ is a $D(\alpha, \emptyset)$-formula.

10.A.11. Exercise (completeness theorem; Husson).

(i) Define for all α a sequent calculus $\boldsymbol{L}K_{\alpha\omega}$ corresponding to the obvious intended meaning of the connectives $\boldsymbol{M}$ and $\boldsymbol{W}$. Define the mutilations as well as functorial proofs: the resulting system is named $\boldsymbol{L}K_{\boldsymbol{\beta}_\omega}$. A model for $\boldsymbol{L}K_{\boldsymbol{\beta}_\omega}$ is a structure $\boldsymbol{m}$ for $\boldsymbol{L}$ (but $\boldsymbol{m}(\boldsymbol{o}) = \emptyset$ is possible). Define a notion $\boldsymbol{m} \models A$, when A is a closed formula of $\boldsymbol{L}[\boldsymbol{m}]_{\boldsymbol{\beta}_\omega}$, in such a way that: if $\vdash A$ is provable in $\boldsymbol{L}K_{\boldsymbol{\beta}_\omega}$ then all closed instantiations of A in $\boldsymbol{m}$ are true in $\boldsymbol{m}$.

(ii) Prove the converse of (i), i.e. completeness.
(*Hint. More precisely, if $\boldsymbol{T}$ is a theory in the language $\boldsymbol{L}_{\boldsymbol{\beta}_\omega}$ with only denumerably many proper axioms, and if the closed formula A is true in all models of $\boldsymbol{T}$, then we construct a functorial proof of the sequent $\vdash A$ in $\boldsymbol{T} + \boldsymbol{L}K_{\boldsymbol{\beta}_\omega}$.*)

10.A.12. Exercise (cut-elimination; Husson).

(i) Define, for any two proofs $\boldsymbol{\pi}$, $\boldsymbol{\pi}'$ of $\boldsymbol{L}K_{\alpha\omega}$, and any two finite sets of integers $\{i_0, ..., i_{n-1}\} = I$ $\{j_0, ..., j_{m-1}\} = J$, a proof $N_{IJ}(\boldsymbol{\pi}, \boldsymbol{\pi}')$, with the following properties:

1. If $\Gamma \vdash \Delta$ and $\Gamma' \vdash \Delta'$ are the respective conclusions of $\boldsymbol{\pi}$ and $\boldsymbol{\pi}'$, and if $\Delta = (A_0, ..., A_k)$, $\Gamma' = (B_0, ..., B_l)$ and $A_{i_0} = ... = A_{i_{n-1}} = B_{j_0} = ... = B_{j_{m-1}}$, then $N_{IJ}(\boldsymbol{\pi}, \boldsymbol{\pi}')$ is a proof of $\Gamma, \Gamma'_1 \vdash \Delta_1, \Delta'$, where Γ'_1 (resp. Δ_1) has been obtained from Γ' (resp. Δ) by removing $A_0, ..., A_k$ (resp. $B_0, ..., B_l$).
2. If $f \in I(\alpha', \alpha)$ and $f^{-1}(\boldsymbol{\pi})$ and $f^{-1}(\boldsymbol{\pi}')$ are defined then $f^{-1}\Big(N_{IJ}(\boldsymbol{\pi}, \boldsymbol{\pi}')\Big) = N_{IJ}\Big(f^{-1}(\boldsymbol{\pi}), f^{-1}(\boldsymbol{\pi}')\Big)$.
3. The cut-degree of $N_{IJ}(\boldsymbol{\pi}, \boldsymbol{\pi}')$ is $\leq$ the supremum of the cut-degrees of $\boldsymbol{\pi}$, $\boldsymbol{\pi}'$ and the degree of A_0, B_0.

Why is it delicate to define when A is a formula of $\boldsymbol{L}_{\alpha\omega}$, a proof $N_A(\boldsymbol{\pi}, \boldsymbol{\pi}')$ of $\Gamma, \Gamma' \vdash \Delta - A, \Delta'$, in such a way that $f^{-1}\Big(N_A(\boldsymbol{\pi}, \boldsymbol{\pi})\Big) = N_{f^{-1}(A)}\Big(f^{-1}(\boldsymbol{\pi}), f^{-1}(\boldsymbol{\pi}')\Big)$?

(ii) Define, for any proof $\boldsymbol{\pi}$ in $\boldsymbol{L}K_{\alpha\omega}$, a cut-free proof $N(\boldsymbol{\pi})$ of the same sequent, enjoying the following property:
if $f \in I(\alpha', \alpha)$ and $f^{-1}(\boldsymbol{\pi})$ is defined, then $f^{-1}\Big(N(\boldsymbol{\pi})\Big)$ is defined and equals $N\Big(f^{-1}(\boldsymbol{\pi})\Big)$.

(iii) Show the existence, for any proof $\boldsymbol{\pi}$ in $\boldsymbol{L}K_{\boldsymbol{\beta}_\omega}$, of a cut-free proof $N(\boldsymbol{\pi})$ of the same sequent. Does the result hold for $\boldsymbol{L}_{D\omega}$?

10.A.13. Exercise (interpolation; Vauzeilles, [96]).

(i) Define, for any sequent $\Gamma \vdash \Delta$ of $\boldsymbol{L}_{\alpha\omega}$ (with $\Gamma = (A_0, ..., A_{n-1})$, $\Delta = (B_0, ..., B_{m-1})$), any $I \subset n$, $J \subset m$, any proof $\boldsymbol{\pi}$ of $\Gamma \vdash \Delta$ in $\boldsymbol{L}_{\alpha\omega}K$, proofs $\mathsf{Int}'_{I,J}(\boldsymbol{\pi})$, $\mathsf{Int}''_{I,J}(\boldsymbol{\pi})$ in $\boldsymbol{L}_{\alpha\omega}$, such that:

1. If $\Gamma' = (A_{i_0}, ..., A_{i_{p-1}})$, $\Gamma'' = (A_{i'_0, ..., i'_{p'-1}})$, if $\Delta' = (B_{j_0}, ..., B_{j_{q-1}})$, $\Delta'' = (B_{j'_0}, ..., B_{j'_{q'-1}})$ (with $(i_0, ..., i_{p-1})$, $(i'_0, ..., i'_{p'-1})$, $(j_0, ..., j_{q-1})$, $(j'_0, ..., j'_{q'-1})$ enumerations of I, $n - I$, J, $m - J$, in strictly increasing order), then $\mathsf{Int}'_{J,J}(\boldsymbol{\pi})$ is a proof of a sequent of the form $\Gamma' \vdash C, \Delta'$ whereas $\mathsf{Int}''_{I,J}(\boldsymbol{\pi})$ is a proof of a sequent of the form $\Gamma'', C \vdash \Delta''$ (the same C) and all predicate letters which occur positively (resp. negatively) in C occur positively (resp. negatively) in $\Gamma' \vdash \Delta'$ and negatively (resp. positively) in $\Gamma'' \vdash \Delta''$.
2. If $f \in I(\alpha', \alpha)$ and $f^{-1}(\boldsymbol{\pi})$ exists, then $f^{-1}\Big(\mathsf{Int}'_{I,J}(\boldsymbol{\pi})\Big)$ and $f^{-1}\Big(\mathsf{Int}''_{I,J}(\boldsymbol{\pi})\Big)$ exist and they are respectively equal to $\mathsf{Int}'_{I,J}\Big(f^{-1}(\boldsymbol{\pi})\Big)$ and $\mathsf{Int}''_{I,J}\Big(f^{-1}(\boldsymbol{\pi})\Big)$.

(ii) Prove the interpolation lemma for $\boldsymbol{L}_{\boldsymbol{\beta}_\omega}$: if $A \vdash B$ has a proof in $\boldsymbol{L}_{\boldsymbol{\beta}_\omega}$, one can find an interpolant C such that:

1. $A \vdash C$ and $C \vdash B$ are provable in $\boldsymbol{L}_{\boldsymbol{\beta}_\omega}$.
2. The predicates occurring positively (resp. negatively) in C occur positively (resp. negatively) in both of A and B.

(iii) We want to extend this result to a sharper version: we assume that the only function letters of $\boldsymbol{L}$ are constants $c_0, ..., c_n, ...$ of type $\boldsymbol{\tau}$. Why is it impossible to directly adapt (i) and (ii) above in such a

way that the interpolation holds for constants as well?
(*Hint. In the case where the last rule is* ($l\forall$), *one must distinguish several subcases that are not preserved by* $f^{-1}(\cdot)$.)
In order to prove interpolation in that case, we consider instead of formulas, pairs (F, X), where X is a subset of the set $\{c_0, ..., c_n, ...\}$ such that all constants of F are among the constant in X. Show that every proof in $\boldsymbol{L}_{\boldsymbol{\beta}_\omega}$ can be modified in such a way that formulas are replaced by pairs (F, X) as above, without losing functoriality. We prove the interpolation as follows: let

$$\Gamma' \vdash \Delta' = (A'_0, X'_0), ..., (A'_{p-1}, X'_{p-1}) \vdash (B'_0, Y'_0), ..., (B'_{q-1}, Y'_{q-1})$$
$$\Gamma'' \vdash \Delta'' \ = \ (A''_0, X''_0), ..., (A''_{p'-1}, X''_{p'-1}) \ \vdash$$
$$(B''_0, Y''_0), ..., (B''_{q'-1}, Y''_{q'-1}) \ .$$

Then all parameters of type $\boldsymbol{\tau}$ of the interpolant C belong to $(X'_0 \cup ... \cup X'_{p-1} \cup Y'_0 \cup ... \cup Y'_{q-1}) \cap (X''_0 \cup ... \cup X''_{p'-1} \cup Y''_0 \cup ... \cup Y''_{q-1})$.

10.A.14. <u>Exercise</u> (bounds for cut-elimination; Ferbus, [97]).

(i) Show that the majoration theorems obtained in Chapter 6 for $\boldsymbol{L}_{\omega_1\omega}$ are still true for the calculi $\boldsymbol{L}K_{\alpha\omega}$; moreover, show that these results (mainly 6.B.6) are compatible with mutilations: assume that $f \in I(\alpha_1, \alpha)$, that $\boldsymbol{\pi}$ is an α-proof, and that $f^{-1}(\boldsymbol{\pi})$ exists; assume that (ϑ, λ), (ϑ_1, λ_1) are majorations of $\boldsymbol{\pi}$ and $\boldsymbol{\pi}_1$ $(= f^{-1}(\boldsymbol{\pi}))$ respectively, and that $h \in I(\lambda_1, \lambda)$ is such that:

$$\begin{tikzcd} T_1 \arrow[r, "\vartheta_1"] \arrow[d, "\varphi"'] & \lambda_1 \arrow[d, "h"] \\ T \arrow[r, "\vartheta"'] & \lambda \end{tikzcd}$$

is a commutative diagram (T and T_1 are the trees associated with $\boldsymbol{\pi}$ and $\boldsymbol{\pi}_1$, φ is the function from T_1 to T corresponding to mutilation w.r.t. f; eventually see 10.A.15). Assume too that $\delta^s, \mu)$ and $\delta^s_1, \mu_1)$ are graduations for $\boldsymbol{\pi}$ and $\boldsymbol{\pi}_1$ respectively and that $k \in I(\lambda_1, \lambda)$ renders all diagrams

$$\begin{tikzcd} U_{1,s} \arrow[r, "\delta_1^s"] \arrow[d, "\psi_s"'] & \mu_1 \arrow[d, "k"] \\ U_{\varphi(s)} \arrow[r, "\delta^s"'] & \mu \end{tikzcd}$$

commutative (with: $s \in T_1$ s.t. the rule applied "at stage s" is a cut; $U_{1,s}$ the underlying tree of the cut-formula of this cut, $U_{\varphi(s)}$ its homologue under the mutilation function; if $A_{1,s}$ and $A_{\varphi(s)}$ are the corresponding formulas, then $f^{-1}(A_{\varphi(s)}) = A_{1,s}$, hence the mutilation function induces a function ψ_s from $U_{1,s}$ to $U_{\varphi(s)}$...).

Assume that all these conditions are fulfilled; let $\boldsymbol{\pi}' = (T', \varphi')$, $\boldsymbol{\pi}'_1 = (T'_1, \varphi'_1)$ be the associated cut-free proofs constructed in 6.B.6; show that $f^{-1}(\boldsymbol{\pi}')$ exists and equals $\boldsymbol{\pi}'_1$. Furthermore, if $\big(\vartheta, V(\mu, \lambda)\big)$ and $\big(\vartheta'_1, V(\mu_1, \lambda_1)\big)$ are the associated majoration, show that

$$\begin{tikzcd} T'_1 \arrow[r, "\vartheta'_1"] \arrow[d, "\chi"'] & V(\mu_1, \lambda_1) \arrow[d, "V(k,h)"] \\ T' \arrow[r, "\vartheta'"'] & V(\mu, \lambda) \end{tikzcd}$$

is commutative, χ being the function from T'_1 to T' induced by the mutilation of $\boldsymbol{\pi}'$ w.r.t. f.
(*Hint. The complete result may be rather long to prove, and it is perhaps sufficient to prove the analogue of 6.2.5–6.2.8–6.3.6 to have a good idea of what is going on....*)

(ii) Prove the majoration theorem for functorial proofs: assume that $\boldsymbol{\pi}$ is a proof in $\boldsymbol{L}K_{\boldsymbol{\beta}_\omega}$; a **majoration** for $\boldsymbol{\pi}$ is a pair (ϑ_α, D) where

- D is a dilator.
- $\big(\vartheta_\alpha, D(\alpha)\big)$ is a majoration of $\boldsymbol{\pi}_\alpha$ for all $\alpha \in 0n$.
- The diagrams

$$
\begin{tikzcd}
T_\alpha \arrow[r, "\vartheta_\alpha"] \arrow[d, "\varphi_f"'] & D(\alpha) \arrow[d, "D(f)"] \\
T_\beta \arrow[r, "\vartheta_\beta"'] & D(\beta)
\end{tikzcd}
$$

are commutative (when $f \in I(\alpha, \beta)$, and φ_f is the function from T_α to T_β induced by the mutilation of $\boldsymbol{\pi}_\beta$ w.r.t. f).

Show that there exists at least one majoration of $\boldsymbol{\pi}$, when $\boldsymbol{\pi}$ is an arbitrary proof in $\boldsymbol{L}K_{\boldsymbol{\beta}_\omega}$.

A **graduation** for $\boldsymbol{\pi}$ is a pair (δ^s_α, E) where

- E is a dilator.
- For all $\alpha \in 0n$, $\Big(\delta^s_\alpha, E(\alpha)\Big)$ is a majoration of $\boldsymbol{\pi}_\alpha$.
- The diagrams

$$
\begin{tikzcd}
U_{\alpha,s} \arrow[r, "\delta^s_\alpha"] \arrow[d, "\varphi^s_f"'] & E(\alpha) \arrow[d, "E(f)"] \\
U_{\beta'\varphi_f(s)} \arrow[r, "\delta_\beta^{\varphi_f(s)}"'] & E(\beta)
\end{tikzcd}
$$

are commutative (the precise meaning of the symbols will easily be found by the reader, using (i)). Show that, if $\boldsymbol{\pi}$ is a proof in $\boldsymbol{L}K_{\boldsymbol{\beta}_\omega}$, there is at least one graduation for $\boldsymbol{\pi}$.

Given $\boldsymbol{\pi}$, together with a majoration (ϑ_α, D) and a graduation (δ^s_α, E) for $\boldsymbol{\pi}$, construct a cut-free proof $\boldsymbol{\pi}'$ of the same sequent in $\boldsymbol{L}K_{\boldsymbol{\beta}_\omega}$, together with a majoration (ϑ'_α, F), where F is the dilator

$$
\begin{aligned}
F(\alpha) &= V\Big(E(\alpha), D(\alpha)\Big) \\
F(f) &= V\Big(E(f), D(f)\Big)\ .
\end{aligned}
$$

(iii) What happens for proofs in $\boldsymbol{L}K_{P\omega}$, when P is an arbitrary dilator? If $\boldsymbol{\pi}' = T^{-1}(\boldsymbol{\pi})$, state a result involving commutative diagrams of majorations and graduations, in the spirit of (i).

(iv) The cut-free proofs constructed in (i) and (ii) depend heavily on the choice of the graduation. Prove a variant of these results where the cut-free proofs obtained do not any longer depend on the graduations, but V is replaced by the original Veblen hierarchy:

$$W(\alpha, \beta) = V(\omega^\alpha, \beta)$$
$$W(f, g) = V(\omega^f, g) \ .$$

10.A.15. Exercise.

(i) Define the concepts of formula, of proof in $\boldsymbol{L}K_{\boldsymbol{\beta}_\omega}$ using the tree-like spirit of 6.A.3, 6.A.7; in particular the associated trees are *quasi-dendroids.* Define the concept of prim. rec. formulas and proofs in this context. Define similarly (using 10.A.14) prim. rec. majorations and graduations. Show that the constructions of 10.A.14 (ii), when applied to prim. rec. data, yield prim. rec. cut-free proofs and majorations.

(ii) Define the concept of preformula, preproof by dropping all well-foundedness assumptions. Can we still formulate and prove 10.A.13 and 10.A.14 in this new context?

Annex 10.B. Generalized $\boldsymbol{\beta}$-rules

The following generalization of the $\boldsymbol{\beta}$-rules was suggested by several persons (Feferman, Jervell, Ressayre): Consider a class C of models of a given finite or denumerable language $\boldsymbol{L}$, which is closed under submodels; from $\tilde{C}$, we can form a category $\tilde{C}$ as follows:

– objects: models $\boldsymbol{m}$ of $\boldsymbol{T}$ belonging to C.

– morphisms from $\boldsymbol{m}$ to $\boldsymbol{m}'$: the set $I_C(|\boldsymbol{m}|, |\boldsymbol{m}'|)$ consisting of all functions f from $|\boldsymbol{m}|$ to $|\boldsymbol{m}'|$ such that:

+ f is injective.
+ $\boldsymbol{m} \models A(\bar{a}_1, ..., \bar{a}_n) \leftrightarrow \boldsymbol{m}' \models A(\overline{fa}_1, ..., \overline{fa}_n)$ for all atomic formula $A(x_1, ..., x_n)$.

A C-**language** is nothing but a language with several types objects, whose restriction to type C is (isomorphic to) $\boldsymbol{L}$; a C-**theory** is a (denumerable) theory in a C-language; a C-**model** of a C-theory $\boldsymbol{T}$ is a model $\boldsymbol{m}$ whose restriction to $\boldsymbol{L}$ belongs to C.

If $\boldsymbol{m} \in C$, then an $\boldsymbol{m}$-proof is the following:

– It uses axioms $\vdash A(\bar{a}_1, ..., \bar{a}_n)$ (resp. $A(\bar{1}_1, ..., \bar{a}_n) \vdash$) when $\vdash A(a_1, ..., a_n)$ is a true (resp. false) atomic formula of $\boldsymbol{L}[\boldsymbol{m}]$.

– It uses the $\boldsymbol{m}$-rule, i.e. the following:

$$\frac{\Gamma, A(\bar{a}_0) \vdash \Delta}{\Gamma, \forall x^C A \vdash \Delta}\, l\forall C \qquad \frac{... \ \Gamma \vdash A(\bar{a}), \Delta \ ... \ \text{all } a \in |\boldsymbol{m}|}{\Gamma \vdash \forall x^C A, \Delta}\, r\forall C$$

with $a_0 \in |\boldsymbol{m}|$ in the $(l\forall C)$.

$$\frac{... \ \Gamma, A(\bar{a}) \vdash \Delta \ ... \ (\text{all } a \in |\boldsymbol{m}|}{\Gamma, \exists x^C A \vdash \Delta}\, l\exists C \qquad \frac{\Gamma \vdash A(\bar{a}_0) \vdash \Delta}{\Gamma \vdash \exists x^C A \vdash \Delta}\, r\exists C$$

with $a_0 \in |\boldsymbol{m}|$ in the $(r\exists C)$.

When $f \in I_C(\boldsymbol{m}, \boldsymbol{n})$ and $\boldsymbol{\pi}$ is a $\boldsymbol{n}$-proof, we (eventually) define an $\boldsymbol{m}$-proof $f^{-1}(\boldsymbol{\pi})$, in the now familiar way. A C-**proof** is a family $\boldsymbol{\pi}(\boldsymbol{m})$ s.t.:

(i) For all $\boldsymbol{m} \in C$ $\boldsymbol{\pi}(\boldsymbol{m})$ is an $\boldsymbol{m}$-proof.

(ii) If $f \in I_C(\boldsymbol{m}, \boldsymbol{n})$, then $f^{-1}\Big(\boldsymbol{\pi}(\boldsymbol{m})\Big) = \boldsymbol{\pi}(\boldsymbol{n})$.

Then the reader will prove the:

10.B.1. Exercise (C-completeness; Feferman, Jervell, Ressayre 1978).
$\Gamma \vdash \Delta$ is valid in all C-models of $\boldsymbol{T}$ iff there is a C-proof of this sequent in $\boldsymbol{T}$.

10.B.2. Remark.
Of course, 10.B.1 is of no interest, unless we can relativize it to the recursive context: $\boldsymbol{T}$ recursive, and recursive C-proofs. What is a recursive C-proof essentially depends of structural properties of C:

(i) For obvious (direct limit) reasons, a C-proof is completely determined by its values on denumerable objects; hence it is likely that one can extend the family $\boldsymbol{\pi}(\boldsymbol{m})$ to all $\boldsymbol{m}$ which are models of $\boldsymbol{L}$, encoded by a subset of $\mathbb{N}$. (Of course, if $\boldsymbol{m} \notin C$, then $\boldsymbol{\pi}(\boldsymbol{m})$ needs not to be well-founded... .) If 10.B.1 is proved by following our method of 10.1.23, then $\boldsymbol{\pi}(\boldsymbol{m})$ is ipso facto defined on arbitrary models of $\boldsymbol{L}$. Then one easily checks that $\boldsymbol{m} \rightsquigarrow \boldsymbol{\pi}(\boldsymbol{m})$ is a continuous type 2 functional, and so it makes sense to style it recursive, prim. rec... . This is the abstract answer to the question raised.

(ii) But in practice, we are more interested in specific categories, for instance when C is a category of ptykes, as in Chapter 12. Then the main property of these categories is the existence of a denumerable subset (finite dimensional objects) which is dense w.r.t. direct limits; moreover, finite dimensional objects (and their morphisms) can be enumerated in a prim. rec. way. Since a C-proof will be uniquely determined by its restriction to finite dimensional objects, it can be encoded by: $n \to \boldsymbol{\pi}(\boldsymbol{m}_n)$, where $(\boldsymbol{m}_n)$ is the prim. rec. enumeration of the finite dimensional objects... . From this one gets another notion of recursive, prim. rec. C-proof.

(iii) It is not difficult to extend the Theorem 10.B.1 in such a way that, if $\boldsymbol{T}$ is recursive, then the C-proof is prim. rec... .

10.B.3. Examples.

(i) The category **ON** is of the form $\tilde{C}$: the language $\boldsymbol{L}$ consists of $\leq$ and $\bar{0}$, and the models are of the form: an ordinal, ordered by $\boldsymbol{m}(\leq)$ and whose bottommost element is $\boldsymbol{m}(\bar{0})$... . Then $\boldsymbol{m}$ can be identified with $1+x$; the embeddings $f \in I(1+x, 1+y)$ can be identified with the functions $\mathbf{E}_1 + g$, where $g \in I(x, y)$.

(ii) The same thing holds with the category **DIL**: the language $\boldsymbol{L}$ consists of:

- a predicate $\leq$.
- for any n and linear order σ on n, a binary predicate p_σ.
- a constant $\bar{0}$.

And we require that:

1. $\boldsymbol{m}(\leq)$ is a linear order, with bottommost elements $\boldsymbol{m}(\bar{0})$.
2. If $a \leq b$ in $\boldsymbol{m}(\leq)$, then one and only one of the formulas $p_\sigma(\bar{a}, \bar{b})$ holds in $\boldsymbol{m}$; $p_\emptyset(\bar{0}, \bar{0})$, where $\emptyset$ is the void order on 0.
3. $\boldsymbol{m} \models \forall x, y, z \left(x \leq y \leq z \wedge p(x, y) \wedge p(y, z) \rightarrow p_{\sigma \wedge \tau}(x, z)\right)$ (Conditions 1–3 express that $\boldsymbol{m}$ encodes a predilator of the form $\underline{1} + F$, in the sense of 8.G.10.)
4. The predilator encoded by $\boldsymbol{m}$ is a dilator.

One easily checks that embeddings can be identified with natural transformations $\mathbf{E}_1 + T$... .

(iii) The same thing holds for more general categories of ptykes, but the relation is rather abstract, compared to the elegant expressions (i) and (ii). The details are left to the reader.

10.B.4. <u>Exercise</u> (Π^1_n-completeness of ptykes).

(i) Define the concept of **DIL**-proof (C-proof, when $\tilde{C}$ = **DIL**), and prove that the concept of prim. rec. **DIL**-proof is Π^1_3-complete.

(ii) If $\boldsymbol{\pi}$ is a **DIL**-proof, construct a ptyx of type $(\boldsymbol{O} \to \boldsymbol{O}) \to \boldsymbol{O}$, **LIN**($\boldsymbol{\pi}$), which "majorizes" $\boldsymbol{\pi}$.
(*Hint. We want to linearize* $\boldsymbol{\pi}(F)$ *for all* F*, in a functorial way; the essential idea is to give a functorial well-order of the set* $\mathsf{Tr}(F)$*, which occurs in the "*F*-rule"*

$$\frac{\ldots\ \Gamma \vdash A(a), \Delta\ \ldots\ \text{all } a \in \mathsf{Tr}(F)}{\Gamma \vdash \forall x^C A, \Delta}\ r\forall C$$

We can well order $\mathsf{Tr}(F)$ *by:*

$$a \leq b \leftrightarrow (z_0\,;\,0,\ldots,n-1\,;\,\omega)_F < (z_1\,;\,0,\ldots,m-1\,;\,\omega)_F$$

if $a = (z_0\,;\,n)$ *and* $b = (z_1\,;\,m)$*).*)
Conclude that the set of prim. rec. ptykes of type $2 = (\boldsymbol{O} \to \boldsymbol{O}) \to \boldsymbol{O}$, is Π^1_3-complete.

(iii) By induction on n ($= (n-1) \to \boldsymbol{O}$), show that the set of all prim. rec. ptykes of type n is Π^1_{n+1}-complete.

CHAPTER 11
INDUCTIVE DEFINITIONS

Had this book been written a few years ago, *inductive definitions* would have occupied the central chapter in the part concerning Π^1_1-logic. But the proof-theoretical analysis of inductive definitions by means of the concepts of Π^1_1-logic, although doable, is not very satisfactory; following Takeuti, [98], people proved cut-elimination theorems for inductive definitions, but these theorems are only *partial* results (the full-calculus does not enjoy cut-elimination); see annex 11.A.

In 1979 I initiated a new treatment of inductive definitions based upon Π^1_2-logic. This new method gives a full-cut-elimination theorem (and a subformula property) for theories of inductive definitions, and it is essentially this method and its applications that we shall consider in this chapter.

11.1. Inductive definitions

Inductive definitions are familiar from practice; for instance integers are defined by:

(1) $\bar{0}$ is an integer.

(2) If n is an integer, so is Sn.

(3) All integers given by (1)–(2).

The definition of O (5.A.3) is an inductive definition; but one can define wf-trees by an inductive definition too:

(1) If $\langle\ \rangle \in T$ and $\Big(\forall i(\langle i\rangle \in T \to T_{\langle i\rangle}$ is a wf-tree$)\Big)$, then is a wf-tree.

(2) All wf-trees are given by (1).

Let us give an example from current mathematical practice: we inductively define the concept of a *Borel set*:

(1) An interval $]r, r'[$ of $\mathbb{R}$ is a Borel set.

(2) If B is a Borel set, so is $\mathbb{R} - B$.

(3) If $(B_n)_{n\in\mathbb{N}}$ are Borel sets, so is $\bigcup\limits_n B_n$.

(4) All Borel sets are given by (1)–(4).

In the sequel we shall only be concerned with inductive definitions of sets of integers; this will enable us to study the integers, O, but also wf-trees, since one can express the *accessible part* of a tree T by:

(1) If $s * \langle n \rangle \in \mathsf{Acc}(T)$ for all n such that $s * \langle n \rangle \in T$, and $s \in T$, then $s \in \mathsf{Acc}(T)$.

(2) All elements of $\mathsf{Acc}(T)$ are given by (1).

And T is a wf-tree iff $\langle\ \rangle \in \mathsf{Acc}(T)$.

11.1.1. Definition.
Assume that $\boldsymbol{L}$ is a language; a **positive operator** in $\boldsymbol{L}$ is a formula $\boldsymbol{\Phi}(X, x)$ of the language $\boldsymbol{L}[X]$ obtained from $\boldsymbol{L}$ by adding a new unary predicate letter X, and x is a variable (in the case of several types, we require that Xx is a corrct formula of $\boldsymbol{L}[X]$) such that all occurrences of X in $\boldsymbol{\Phi}(X, x)$ are positive. (There may be free variables $\neq x$ in $\boldsymbol{\Phi}(X, x)$.)

11.1.2. Examples.

(i) $\boldsymbol{N}(X, x)$: $x = \bar{0} \vee \exists y < x\, (x = Sy \wedge Xy)$ is a positive operator in $\boldsymbol{L}_0$.

(ii) $O(X, x)$: $x = \bar{1} \vee \exists y < x\, (x = \bar{2} \cdot y \wedge Xy) \vee \exists y < x\, \exists z < x \Big(x = \langle y, x \rangle + \bar{1} \wedge \forall t\, \exists u \Big(T_1(z, t, u) \wedge XU(u) \Big) \wedge \forall t\, \forall u\, \forall u' \Big(T_1(z, t, u) \wedge T_1(z, St, u') \rightarrow U(u) <_0 U(u') \Big) \wedge Xy \Big)$ is a positive operator in $\boldsymbol{L}_{pr}$.

(iii) $T(X, x)$: $TR(f) \wedge f(x) = \bar{0} \wedge Seq(x) \wedge \forall n \Big(f(x * \langle n \rangle) = \bar{0} \rightarrow X(x * \langle n \rangle) \Big)$ is a positive operator in $\boldsymbol{L}^2_{pr}$.

11.1.3. Definition.
Assume that $\boldsymbol{m}$ is a model for the language $\boldsymbol{L}$ and $\boldsymbol{\Phi}$ $(= \boldsymbol{\Phi}(X, x))$ is a positive operator in $\boldsymbol{L}$. We assume that x is the only free variable in $\boldsymbol{\Phi}(X, x)$. We define a function $\boldsymbol{m}(\boldsymbol{\Phi})$ from $P(|\boldsymbol{m}|)$ to $P(|\boldsymbol{m}|)$ as follows: if $A \subset |\boldsymbol{m}|$, extend $\boldsymbol{m}$ to a model $\boldsymbol{m}(A)$ of $\boldsymbol{L}(X)$ as follows: $(\boldsymbol{m}(A) \models X\bar{a}) \leftrightarrow a \in A$; then let $\boldsymbol{m}(\boldsymbol{\Phi})(A) = \{a \in |\boldsymbol{m}|\,;\, \boldsymbol{m}(A) \models \boldsymbol{\Phi}(X, \bar{a})\}$.

11.1.4. Remarks.

(i) The function $\boldsymbol{m}(\boldsymbol{\Phi})$ is obviously increasing: $A \subset A' \rightarrow \boldsymbol{m}(\boldsymbol{\Phi})(A) \subset \boldsymbol{m}(\boldsymbol{\Phi})(A')$: this comes from the fact that $\boldsymbol{\Phi}$ is *positive*.

(ii) When $\boldsymbol{L}$ has several types of objects, then for instance x is of type $\boldsymbol{\tau}$ and X takes arguments of type $\boldsymbol{\tau}$. Then one must of course define $\boldsymbol{m}(\boldsymbol{\Phi})$ as a function from $P(|\boldsymbol{m}|_{\boldsymbol{\tau}})$ to itself. ($|\boldsymbol{m}|_{\boldsymbol{\tau}}$ is the set of objects of $\boldsymbol{m}$ of type $\boldsymbol{\tau}$.)

(iii) When $\boldsymbol{\Phi}(X,x)$ contains free variables $y_1, ..., y_m$ distinct from X, and if $c_1, ..., c_m$ are elements of $\boldsymbol{m}$ of the appropriate types, then one can define $\boldsymbol{m}\Big(\boldsymbol{\Phi}(c_1, ..., c_m)\Big)$, since the substitution of the $\bar{c}_i$'s for the y_i's define a positive operator $\boldsymbol{\Phi}(\bar{c}_1, ..., \bar{c}_m)(X,x)$ in $\boldsymbol{L}\,[\boldsymbol{m}]$ whose only free variable is x... .

11.1.5. <u>Definition</u>.

(i) Under the hypotheses of 11.1.3, we define subsets $\boldsymbol{m}(I\boldsymbol{\Phi}^{\alpha})$ of $P(|\boldsymbol{m}|)$, for all ordinals α:

$$\boldsymbol{m}(I\boldsymbol{\Phi}^{\alpha}) = \bigcup_{\alpha'<\alpha} \boldsymbol{m}(\boldsymbol{\Phi})\Big(\boldsymbol{m}(I\boldsymbol{\Phi}^{\alpha'})\Big) \ .$$

(ii) The ordinal α_0 defined by:

$$\alpha_0 = \mu\xi\Big(\boldsymbol{m}(I\boldsymbol{\Phi}^{\xi+1}) = \boldsymbol{m}(I\boldsymbol{\Phi}^{\xi})\Big)$$

is called the **closure ordinal** of $\boldsymbol{\Phi}$ (w.r.t. $\boldsymbol{m}$); in many contexts, $\boldsymbol{m}$ is clear, and we shall speak of the closure ordinal of $\boldsymbol{\Phi}$.

11.1.6. <u>Remarks</u>.

(i) The equality $\Big(\boldsymbol{m}(I\boldsymbol{\Phi}^{\xi}) = \boldsymbol{m}(I\boldsymbol{\Phi}^{\xi+1})\Big)$ implies $\boldsymbol{m}(I\boldsymbol{\Phi}^{\xi'}) = \boldsymbol{m}(I\boldsymbol{\Phi}^{\xi})$ for all $\xi' > \xi$, hence the iteration "stops" at the closure ordinal; nothing new happens after α_0.

(ii) One easily proves that $\boldsymbol{m}(I\boldsymbol{\Phi}^{\alpha}) \subset \boldsymbol{m}(\boldsymbol{\Phi})\Big(\boldsymbol{m}(I\boldsymbol{\Phi}^{\alpha})\Big)$ by induction on α.

(iii) Hence another description of the $\Big(\boldsymbol{m}(I\boldsymbol{\Phi}^{\alpha})\Big)$'s is

$$\begin{aligned}
&\boldsymbol{m}(I\boldsymbol{\Phi}^0)\\
&\boldsymbol{m}(I\boldsymbol{\Phi}^{\alpha+1}) = \boldsymbol{m}(\boldsymbol{\Phi})\big(\boldsymbol{m}(I\boldsymbol{\Phi}^{\alpha})\big)\\
&\boldsymbol{m}(I\boldsymbol{\Phi}^{\lambda}) = \bigcup_{\lambda'<\lambda} \boldsymbol{m}(I\boldsymbol{\Phi}^{\lambda'}) \qquad \text{for } \lambda \text{ limit .}
\end{aligned}$$

11.1.7. Examples.

(i) In 11.1.2 (i), one gets:

$$\boldsymbol{m}(I\boldsymbol{N}^{\alpha}) = \{c \mid c \in |\boldsymbol{m}| \wedge \exists n < \alpha \ \boldsymbol{m} \models \bar{c} = \bar{n}\} \ .$$

(ii) In 11.1.2, if $\boldsymbol{m}$ is the standard model of $\boldsymbol{L}_{pr}$, one gets:

$$\boldsymbol{m}(IO^{\alpha}) = \{e \mid e \in O \wedge \|e\| < \alpha\} \ .$$

(iii) In 11.1.2 (iii), if $\boldsymbol{m}$ is the standard model of $\boldsymbol{L}^2_{pr}$ and f is a function from $\mathbb{N}$ to $\mathbb{N}$ which is the characteristic function of a tree T, one gets

$$\boldsymbol{m}(T(f)^{\alpha}) = \{s \in T \, ; \, WTR(T_s) \wedge \|T_s\| < \alpha\} \ .$$

(iv) In the cases (i), (ii), (iii), the respective closure ordinals are ω, ω_1^{ck}, and $\alpha_0 = \mathsf{sup}\,(\|T_s\| \, ; \, s \in I \wedge T_s \ wf\text{-tree})$.

11.1.8. Definition.

(i) Under the hypothesis of 11.1.3, we define a new language $\boldsymbol{L}\,[\bar{\boldsymbol{\Phi}}]$ by adding to $\boldsymbol{L}$ a new unary predicate letter $\bar{\boldsymbol{\Phi}}$.

(ii) Under the hypothesis of 11.1.5, we define a model $\boldsymbol{m}\,[\bar{\boldsymbol{\Phi}}]$ for the language $\boldsymbol{L}(\bar{\boldsymbol{\Phi}})$ by saying that

$$\boldsymbol{m}\,[\bar{\boldsymbol{\Phi}}] \models \bar{\boldsymbol{\Phi}}\bar{c} \leftrightarrow c \in \boldsymbol{m}(I\boldsymbol{\Phi}^{\alpha_0})$$

where α_0 is the closure ordinal of $\boldsymbol{\Phi}$ w.r.t. $\boldsymbol{m}$.

11.1.9. Examples.
In 11.1.7 (i)–(iii), then the interpretation $\boldsymbol{m}(\bar{\boldsymbol{\Phi}})$ of $\bar{\boldsymbol{\Phi}}$ in $\boldsymbol{m}\,[\bar{\boldsymbol{\Phi}}]$ is:

(i) The set $\{c \in |\boldsymbol{m}|\,;\, \exists n\ \boldsymbol{m} \models \bar{c} = \bar{n}\}$.

(ii) O.

(iii) lThe set $\{s \in T\,;\, WTR(T_s)\}$.

11.1.10. Remark.
In general, when $\boldsymbol{\Phi}$ depnds on additional variables $y_1, ..., y_k$, then $\bar{\boldsymbol{\Phi}}$ is a $k+1$-ary predicate letter such that $\bar{\boldsymbol{\Phi}}y_1...y_kx$ is a correct formula of the language.

11.1.11. Theorem.
$\boldsymbol{m}(\bar{\boldsymbol{\Phi}})$ is the smallest fixed point of the function $\boldsymbol{m}(\boldsymbol{\Phi})$ (w.r.t. inclusion):

$$\boldsymbol{m}(\bar{\boldsymbol{\Phi}}) = \bigcap \{A\,;\, A \subset |\boldsymbol{m}| \wedge \boldsymbol{\Phi}(\boldsymbol{m})(A) = A\}\ .$$

Proof. If $\boldsymbol{a}^\alpha = \boldsymbol{m}(I\boldsymbol{\Phi}^\alpha)$, it is clear from 11.1.6 (iii) that, if α_0 is the closure ordinal of $\boldsymbol{\Phi}$, then $\boldsymbol{a}^{\alpha_0} = \boldsymbol{\Phi}(\boldsymbol{m})(\boldsymbol{a}^{\alpha_0})$. Hence $\boldsymbol{a}^{\alpha_0}$ $(= \boldsymbol{m}(\bar{\boldsymbol{\Phi}}))$ is a fixed point of $\boldsymbol{m}(\boldsymbol{\Phi})$. Now assume that $\boldsymbol{m}(\boldsymbol{\Phi})(A) \subset A$; we prove by induction on α that $a^\alpha \subset A$; the non trivial case is the successor case: if $\boldsymbol{a}^\alpha \subset A$, then $\boldsymbol{m}(\boldsymbol{\Phi})(\boldsymbol{a}^\alpha) \subset \boldsymbol{m}(\boldsymbol{\Phi})(A) \subset A\ldots$. By the way observe that we have proved slightly more: $\boldsymbol{m}(\bar{\boldsymbol{\Phi}})$ is the smallest $A \subset |\boldsymbol{m}|$ such that $\boldsymbol{m}(\boldsymbol{\Phi})(A) \subset A$. □

11.1.12. Corollary.
Assume that $\boldsymbol{m}$ is a denumerable model of $\boldsymbol{L}$; then the set $\boldsymbol{m}(\bar{\boldsymbol{\Phi}})$ is Π^1_1 in $\boldsymbol{m}$.

Proof. $z \in \boldsymbol{m}(\bar{\boldsymbol{\Phi}})$ can be written

$$\forall A \subset |\boldsymbol{m}|\ \forall c \in |\boldsymbol{m}|\ \Big(\big(\boldsymbol{m}(A) \models \boldsymbol{\Phi}(A, \bar{c})\big) \to c \in A\Big)$$

and this is clearly Π^1_1 in the data, i.e. $\boldsymbol{m}$. □

11.1.13. Remark.

11.1.12 expresses the general way of writing an inductive definition under a Π_1^1 form; one sees that "inductive definitions are Π_1^1".

11.2. Theories of inductive definitions

There are many theories of inductive definitions (in the familiar sense of Σ^0_1-logic); we shall essentially consider theories of the form $\mathbf{ID}_1$ (non-iterated inductive definitions), which are the interesting case. In practice, it is often useful to iterate inductive definitions, and this yields various systems that we shall also consider: these iterated systems areconnected with the Π^1_1-comprehension axiom, Π^1_1-CA. Classical results (equivalence of systems of inductive definitions with systems of Π^1_1-comprehension...) can be found in [99].

11.2.1. Definition.
Let $\boldsymbol{L}$ be a first order language containing the language $\boldsymbol{L}_0$ of arithmetic, let $\mathbf{\Phi}$ be a positive operator in $\boldsymbol{L}$; then we define a theory $\mathbf{ID}_1(\boldsymbol{T}, \mathbf{\Phi})$ in the language $\boldsymbol{L}\,[\bar{\mathbf{\Phi}}]$, by adding to $\boldsymbol{T}$ the following axioms:

(i) Usual induction axioms involving $\bar{\mathbf{\Phi}}$:

$$A(\bar{0}) \wedge \forall x \Big(A(x) \rightarrow A(Sx)\Big) \rightarrow \forall z\ A(\bar{z})\ .$$

(ii) The **closure** axiom

$$\mathbf{\Phi}(\bar{\mathbf{\Phi}}, x) \rightarrow \bar{\mathbf{\Phi}}(x)\ .$$

(iii) The **generalized $\mathbf{\Phi}$ induction** axioms:

$$\forall y \Big(\mathbf{\Phi}(\lambda x B(x), y) \rightarrow B(y)\Big) \rightarrow \forall z \Big(\bar{\mathbf{\Phi}}(z) \rightarrow B(z)\Big)\ .$$

(In (i) and (iii), the formulas A, B, are arbitrary in $\boldsymbol{L}\,[\mathbf{\Phi}]$.)

11.2.2. lemma.
The equality axiom $x = y \rightarrow \Big(\bar{\mathbf{\Phi}}(x) \rightarrow \bar{\mathbf{\Phi}}(y)\Big)$ is a theorem of $\mathbf{ID}_1(\boldsymbol{T}, \mathbf{\Phi})$ + equality axioms of $\boldsymbol{T}$.

Proof. Consider the formula $B(x)$: $\forall y \Big(y = x \rightarrow \bar{\mathbf{\Phi}}(y)\Big)$; if $\mathbf{\Phi}(\lambda x B(x), y)$, then since the only occurrence of $\bar{\mathbf{\Phi}}$ in this formula are inside occurrences of $B(t)$ for some terms t, it is immediate that $\mathbf{\Phi}(\lambda x B(x), y) \wedge y = y' \rightarrow$

$\mathbf{\Phi}(\lambda x B(x), y')$. Using the fact that $\mathbf{\Phi}$ is positive, and $\forall z \left(B(z) \rightarrow \mathbf{\Phi}(z)\right)$, we obtain: $\mathbf{\Phi}(\lambda x B, y) \rightarrow \forall y' \left(y = y' \rightarrow \mathbf{\Phi}(\bar{\mathbf{\Phi}}, y')\right)$, and so by the closure axiom (ii): $\mathbf{\Phi}(\lambda x B, y) \rightarrow B(y)$; hence we may apply generalized induction to B, and we get:

$$\forall z \left(\bar{\mathbf{\Phi}}(z) \rightarrow B(z)\right) ,$$

which can be read as:

$$\forall z \ \forall y \left(y = z \wedge \bar{\mathbf{\Phi}}(z) \rightarrow \bar{\mathbf{\Phi}}(y)\right) . \qquad \square$$

11.2.3. Example.

Assume that $\boldsymbol{T}$ is $\mathbf{PA}_{pr}$; then the theory $\mathsf{ID}_1(O, \boldsymbol{T})$ contains a closure axiom (ii) that can be rewritten as:

- $\bar{O}(\bar{1})$.
- $\bar{O}(x) \rightarrow \bar{O}(2x)$.
- $\Big[\bar{O}(y) \wedge \forall t \ \exists u \left(T_1(z,t,u) \wedge \bar{O}\big(U(u)\big)\right) \wedge \forall t \ \forall u \ \forall u' \left(T_1(z,t,u) \wedge T_1(z, St, u') \rightarrow U(u) \ <_0 \ U(u')\right)\Big] \rightarrow \bar{O}(\langle y, z\rangle + 1)$.

11.2.4. Remarks.

(i) The natural idea is to iterate this construction of $\mathbf{ID}_1(\boldsymbol{T}, \mathbf{\Phi})$, obtaining thus, from a positive operator $\mathbf{\Psi}$ in $\boldsymbol{L}\,[\bar{\mathbf{\Phi}}]$ a theory $\mathbf{ID}_2(\boldsymbol{T}, \mathbf{\Phi}\mathbf{\Psi})$, etc.... This process can even be iterated transfinitely many times, along, say, a recursive ordinal V; this leads to the iterated theory of inductive definitions $\mathbf{ID}_V$, which are studied in [3]; see 11.4.12.

(ii) However, the pattern of iteratins is slightly tricky: *it is not true that* $\mathbf{ID}_2(\boldsymbol{T}, \mathbf{\Phi}, \mathbf{\Psi}) = \mathbf{ID}_1(\mathbf{ID}_1(\boldsymbol{T}, \mathbf{\Phi}), \mathbf{\Psi})$. The reason is simple: if we do twice the $\mathbf{ID}_1$ construction, then *the only provable generalized* $\mathbf{\Phi}$ *inductions will be the ones where* B *does not contain* $\bar{\mathbf{\Psi}}$*, and this will be definitely too weak.* $\mathbf{ID}_2(\boldsymbol{T}, \mathbf{\Phi}, \mathbf{\Psi})$ is therefore the theory containing:

1. usual induction axioms on all formulas.
2. closure axioms for $\mathbf{\Phi}$ and $\mathbf{\Psi}$.

3. $\mathbf{\Phi}$ and $\mathbf{\Psi}$-inductions on all formulas.

(iii) Another way of overcoming the difficulty rised in (ii) would be to consider theories of inductive definitions as **schematical theories**, i.e. the principle of $\mathbf{\Phi}$-induction has to apply to any extension of the language... .

(iv) If we look more closely, it is plain that it would be also possible to define $\mathbf{ID}_0(\boldsymbol{T}, \mathbf{\Phi})$ simply as $\mathbf{ID}_1(\boldsymbol{T}, \mathbf{\Phi})$, but without the usual induction axioms (i). Then for instance $\mathbf{ID}_0(\mathbf{EA}, \boldsymbol{N})$ is obtained from $\mathbf{EA}$ by adding the extra principles:

- $\bar{\boldsymbol{N}}(\bar{0})$
- $\bar{\boldsymbol{N}}(x) \to \bar{\boldsymbol{N}}(Sx)$
- $B(\bar{0}) \wedge \forall z \Big(B(z) \to B(Sz)\Big) \to \forall y\, \bar{\boldsymbol{N}}(y) \to B(y)$

and this theory (if all quantifiers are restricted to $\bar{\boldsymbol{N}}(\cdot)$) is essentially Peano arithmetic.

Peano arithmetic is therefore the simplest case of theory of inductive definitions; the theories ID_1 are already theories of inductive definitions iterated twice, and this is the reason why (recall (ii) above) we must add to the closure and $\mathbf{\Phi}$-inductions, the usual induction axioms which are just $\boldsymbol{N}$-inductions... .

11.2.5. <u>Definition</u>.
In the language $\boldsymbol{L}_2^2$ of second-order arithmetic, we consider the following positive operator:

$$\mathbf{\Phi}_Y(X, t) : \ t \in Y \wedge \mathsf{Tr}(Y) \wedge \forall n\, (t * \langle n \rangle \in Y) \to t \in X \ .$$

($\mathsf{Tr}(Y)$ is the formula: $\langle\ \rangle \in Y \wedge \forall s\, \forall s'\, (s \in Y \wedge s <^* s' \to s' \in Y)$.)

We define a theory $\mathbf{IND}$ as follows: the language consists of $\boldsymbol{L}^2$, together with a new function letter mapping objects of type $\boldsymbol{s}$ into themsleves: if T is a term of type $\boldsymbol{s}$, so is Acc_T. The precise definition of the terms of type $\boldsymbol{s}$ is as follows:

- Variables $X, Y, Z, ...$ of type $\boldsymbol{s}$ are terms of type $\boldsymbol{s}$.

- If T is a term of type $\boldsymbol{s}$, so is Acc_T.
- If A is a formula involving no quantifiers of type $\boldsymbol{s}$, then $\lambda x T$ is a term of type $\boldsymbol{s}$.
- The only terms of type $\boldsymbol{s}$ are given by these conditions.

The atomic formulas are formulas $t \in T$, where t and T are terms of respective types $\boldsymbol{i}$ and $\boldsymbol{s}$. We do the usual identification between the formulas $t \in \lambda x A(x)$ and $A(t)$.

The axioms of **IND** are the following:

(i) The sequent calculus corresponding to the language; the rules for the quantifiers of type $\boldsymbol{s}$ being:

$$\frac{\Gamma \vdash A(X), \Delta}{\Gamma \vdash \forall X \; A(X), \Delta} \; r\forall\boldsymbol{s} \qquad \frac{\Gamma, A(T) \vdash \Delta}{\Gamma, \forall X \; A(X) \vdash \Delta} \; l\forall\boldsymbol{s}$$

$$\frac{\Gamma \vdash A(T), \Delta}{\Gamma \vdash \exists X \; A(X), \Delta} \; r\exists\boldsymbol{s} \qquad \frac{\Gamma, A(X) \vdash \Delta}{\Gamma, \exists X \; A(X) \vdash \Delta} \; l\exists\boldsymbol{s}$$

(in $(r\forall\boldsymbol{s})$ and $(l\exists\boldsymbol{s})$ X is not free in $\Gamma \vdash \Delta$; in $(l\forall\boldsymbol{s})$ and $(r\exists\boldsymbol{s})$ T is a term of the language, of type $\boldsymbol{s}$).

(ii) The full induction axiom:

$$A(\bar{0}), \forall z \Big(A(z) \to A(Sz) \Big) \vdash A(t) \; .$$

(An arbitrary formula of the language.)

(iii) The axioms of **PRA**.

(iv) Equality axioms $x \in X \wedge x = y \to y \in X$.

(v) Closure axioms:

$$\boldsymbol{\Phi}_T(\mathsf{Acc}_T, x) \to x \in \; \mathsf{Acc}_T \; .$$

(vi) $\mathbf{\Phi}$-induction axioms:

$$\forall y\left(\mathbf{\Phi}_T(\lambda x B, y) \to B(y)\right) \to \forall z\left(z \in \mathsf{Acc}_T \to B(z)\right)$$

where B is an arbitrary formula of the language; hence $\lambda x B$ need not to be a term!

11.2.6. <u>Interpretation</u>.
Let us look at the standard interpretation of **IND** in the structure mainly consisting of $I\!N$ and $P(I\!N)$. First the interpretation of $\mathbf{\Phi}$: if Y is not a tree, then $\mathbf{\Phi}_Y(X,n)$ is always false, hence we may assume that Y is a tree; then $\mathbf{\Phi}_Y(X,s)$ is satisfied iff $s \in Y$ is s.t. all its immediate predecessors $s * \langle n \rangle$ in Y are already in X. In other terms, let $X^\xi = \boldsymbol{m}(\mathbf{\Phi}^\xi_Y)$ ($\boldsymbol{m}$ is the standard model $\left(I\!N, P(I\!N)\right)$, $\boldsymbol{m}(\mathbf{\Phi}^\xi_Y)$ is defined as in 11.1.5); obviously:

$$s \in X^\xi \leftrightarrow s \in Y \land \forall n\,(s * \langle n \rangle \in Y \to \exists \xi' < \xi s \in X^\xi)\ .$$

The reader will have no difficulty in establishing that:

$$s \in X^\xi \leftrightarrow Y_s \text{ is a } wf\text{-tree and } \|Y_s\| < \xi\ .$$

The standard interpretation $\boldsymbol{m}(\bar{\mathbf{\Phi}}_Y)$ is therefore (11.1.8)

$$\boldsymbol{m}(\bar{\mathbf{\Phi}}_Y) = \emptyset \text{ if } Y \text{ is not a tree}$$

$$\boldsymbol{m}(\bar{\mathbf{\Phi}}_Y) = \{s\,;\, Y_s \text{ is a } wf\text{-tree}\} \text{ if } Y \text{ is a tree}\ .$$

But if we apply the $\mathbf{\Phi}_Y$-induction axiom to the set $\boldsymbol{m}(\bar{\mathbf{\Phi}}_Y) = Z$, we see that

$$\forall y\,(\mathbf{\Phi}_Y(Z,y) \to y \in Z)\ , \qquad \text{hence } \forall z\,(z \in \mathsf{Acc}_Y \to z \in Z)\ .$$

But Z is the smallest set such that $\mathbf{\Phi}_Y(Z) \subset Z$ (11.1.11), and $\mathbf{\Phi}_Y(\mathsf{Acc}_Y) \subset \mathsf{Acc}_Y$: hence $Z = \mathsf{Acc}_Y$.

We have thus established that, in the standard model $\left(I\!N, P(I\!N)\right)$ the interpretation of Acc_Y, when Y is a tree, is the set of all $s \in Y$ s.t. Y_s is a wf-tree.

11.2.7. <u>Theorem</u> (Feferman, [99]).

The Π^1_1*-comprehension axiom is a theorem of* **IND**.

Proof. *Let A be a Π^1_1 formula; we shall explicitly construct a term T in the language of* **IND** *such that:* $\forall z \left(z \in T \leftrightarrow A(z)\right)$ *is a theorem of* **IND***; then of course*

$$\frac{\vdash \forall z \left(z \in T \leftrightarrow A(z)\right)}{\vdash \exists X\, \forall z \left(z \in X \leftrightarrow A(z)\right)}\, r\exists \boldsymbol{s}$$

gives the desired proof.

We start with a formula A of the form $\forall X\ B(X)$, where B does not contain any second order quantifiers; then it is possible to build a term U such that the equivalence

$$\forall X\ B(X) \leftrightarrow WTR(U)$$

is a theorem of **IND**.

(Sketch of the proof. *By replacing the set quantifiers $\forall X$ by quantifiers over functions, we can place ourselves inside* $\mathbf{PRA}^2$*, where a similar result (5.2.4) has already been obtained; we use Σ^0_1-CA^* to reduce the formula to the form $\forall f\ \exists n\ t(f, n) = \bar{0}$. The next step is to translate back this result in* **IND***: the function quantifiers can be replaced by quantifiers $\forall X\, (\mathsf{Fnc}(X) \to -)$ where $\mathsf{Fnc}(X)$ stands for $\forall n\ \exists! m\ \langle n, m\rangle \in X$; the possibility of translating terms and proving Σ^0_1-CA^* comes from the obvious remark that arithmetical comprehension is provable in* **IND***, and in order to do these translations all we need is arithmetical comprehension. Finally, we obtain a term U s.t. $\mathsf{Tr}(U)$ is provable, and $\forall X\ B(X)$ is provably equivalent to the following formula (abbreviated as $WTR(U)$)*

$$\forall X \left(\mathsf{Fnc}(X) \to \exists s\, (\forall n < lh(s)\, \langle n, (s)_n\rangle \in X \wedge s \notin U)\right). \qquad \square)$$

We now establish in **IND** *the formal equivalence*

$$WTR(U) \leftrightarrow \langle\ \rangle \in \mathsf{Acc}\ .$$

(Proof. $\leftarrow$ *Consider the formula $B(x)$: $x \in U \wedge WTR(U_x)$. Then assume that $\forall n x * \langle n\rangle \in U \to B(x * \langle n\rangle)$; then if $\mathsf{Fnc}(X)$ let n_0 be the unique* *integer s.t.* $\langle 0, n_0\rangle \in X$ *and define* X' *by:* $\langle a.m\rangle \in X' \leftrightarrow \langle Sa, m\rangle \in X$ *(i.e.* $X' = \lambda x\, \langle S(x)_0, (x)_1\rangle \in X$*); the hypothesis* $B(x * \langle n_0\rangle) \vee x * \langle n_0\rangle \notin U$ *shows that:*

– *If* $x * \langle n_0\rangle \notin U$ *then* $s \notin U_x$*, with* $s = \langle n_0\rangle$*, and since* $\langle 0, n_0\rangle \in X$*, one concludes that* $B(x)$*.*

– *If* $B(x * \langle n_0\rangle)$*, apply the definition of* $WTR(U_{x*\langle n_0\rangle})$ *with* X' *defined as above; then there is a* s *s.t.* $s \notin U_{x*\langle n_0\rangle}$ *and* $\forall n < lh(s)\ \langle n, (s)_n\rangle \in X'$*. Let* $s' = \langle n_0\rangle * s$*; then* $s' \notin U_{x*\langle n_0\rangle} \wedge \forall n < lh(s')\ \langle n, (s')_n\rangle \in X$*. Summing up we obtain a proof of:*

$$\mathbf{\Phi}_U(\lambda x B(x), z) \to B(Z)$$

hence by the $\mathbf{\Phi}_U$*-induction axiom:*

$$x \in \mathsf{Acc}_U \to WTR(U_x)$$

hence

$$\langle\ \rangle \in \mathsf{Acc}_U \to WTR(U)\ .$$

$\to$ *Roughly speaking, the idea wholly lies in the Kleene Basis theorem (5.6.7): we shall construct a s.d.s. in* U*, which is recursive in* Acc_U *(hence encodable by a term of the language); assume that* $\langle\ \rangle \notin \mathsf{Acc}_U$*; then, by the closure axiom, we obtain* $\neg\,\mathbf{\Phi}_U(\mathsf{Acc}_U, \langle\ \rangle)$*, i.e.* $\exists n\,(\langle n\rangle \in U \wedge \langle n\rangle \notin \mathsf{Acc}_U)$*. By a trivial induction on* p *we obtain:*

$$\forall p\ \exists s\,(lh(s) = p \wedge s \in U \wedge s \notin \mathsf{Acc}_U)\ .$$

Therefore define a set X *by:*

$$s \in X \leftrightarrow \mathsf{Seq}(s) \wedge s \in U \wedge s \notin \mathsf{Acc}_U \wedge$$
$$\forall i < s\ \forall n < (s)_i \Big((s{\restriction} i) * \langle n\rangle \in \mathsf{Acc}_U \vee (s{\restriction} i) * \langle n\rangle \notin U\Big)\ .$$

X *can be described by an abstraction term involving* Acc_U*. And if one considers* X' *defined by*

$$X' = \lambda x\ \exists s\,(\mathsf{Seq}(s) \wedge x = \langle lh(s), s\rangle \wedge s \in X)$$

then it is immediate how to prove that:

$$\forall s\left((\forall n < lh(s)\ \langle n, (s)_n\rangle \in X') \rightarrow s \in U\right)$$

(this follows from $\forall s\,(s \in X \rightarrow s \in U \wedge s \notin \mathsf{Acc}_U)$, proved by a trivial induction on $lh(s)$...).

We have just established that

$$\langle\ \rangle \notin \mathsf{Acc}_U \rightarrow \neg WTR(U)\ . \qquad \square)$$

Summing up we have obtained in **IND** *the provability of the equivalence*

$$A \leftrightarrow \langle\ \rangle \in \mathsf{Acc}_U\ .$$

If A depends on a variable z of type $\boldsymbol{i}$, let $T = \lambda z\,(\langle\ \rangle \in \mathsf{Acc}_{U(z)})$; then we have proved that

$$A(z) \leftrightarrow z \in T\ . \qquad \square$$

11.2.8. <u>Remarks</u>.

(i) The original result of Feferman [99] makes use of $\mathbf{ID}_\omega$ (ω-times iterated inductive definitions) which is a system roughly equivalent to the one we are presenting.* We have chosen our presentation because it is more flexible to be in the same language as second order arithmetic ... the formalism $\mathbf{ID}_\omega$ has the dubious advantage of eliminating second order variables... . Of course, what must absolutely be eliminated is the use of second order quantifiers inside comprehension axioms: our system appears as something formally predicative ... this point will be clarified when we shall use $\boldsymbol{\beta}$-logic... .

(ii) Something the reader must absolutely know, even if he never works with the formalism $\mathbf{ID}_\omega$, is that this theory *is not the union of the*

[1*] In fact, ID_ω is equivalent to (Π^1_1-CA) + (BI), where (BI) (Bar-Induction) stands for the scheme

$$\forall X\left(WTR(X) \rightarrow TI(X, A)\right)\ .$$

A here is arbitrary. In the framework of Π^1_2-logic, principles s.t. (BI) usually have a purely logical proof... .

$\mathbf{ID}_n$*'s*! The union of the $\mathbf{ID}_n$'s os something is denoted by $\mathbf{ID}_{<\omega}$. $\mathbf{ID}_\omega$ is a system where one has a predicate $P(n,m)$ together with axioms expressing that $P(n,\cdot)$ is the n^{th} iterated inductively defined predicate. It is clear on general ground, that $\mathbf{ID}_\omega$ is likely to be strictly stronger than $\mathbf{ID}_{<\omega}$.

(iii) In fact one would easily show that **IND** is nothing but the standard system for Π^1_1 comprehension. Using 11.1.12, it is easy to replace the sets Acc_Y by Π^1_1 comprehensions. Since

$$t \in \mathsf{Acc}_y \leftrightarrow \forall X\,(\forall x\,(\mathbf{\Phi}_Y(X,x) \to x \in X) \to t \in X$$

is provable by axioms (v) and (vi), we see that $t \in \mathsf{Acc}_Y$ is formally equivalent to a Π^1_1 formula $B(x,Y)$, and the use of Acc_Y can be eliminated if one assumes that $\exists X\ \forall x\,\Big(x \in X \leftrightarrow B(x,Y)\Big)$, i.e. the Π^1_1-comprehension axiom ... details are left to the reader.

(iv) It is legitimate to ask: what is the status of $\mathbf{ID}_1$ in this context; the answer is that $\mathbf{ID}_1$ obviously corresponds to a specific instance of Π^1_1-comprehension, $\exists X\ \forall x\,\Big(x \in X \leftrightarrow B(x)\Big)$, namely when B contains no free variable of type $\boldsymbol{s}$; this restricted form of comprehension is sometimnes styled as "Π^1_1-comprehension without parameters". In practice the study of $\mathbf{ID}_1$ (i.e. Π^1_1-comprehension without parameters) is more rewarding (and simpler) than the study of the iterated **IND** (or $\mathbf{ID}_\omega$). **IND** is especially useful if one is interested in the ordinal analysis of Π^1_1-CA ... but this a merely ideological question, which follows by a straightforward iteration of the basic pattern of $\mathbf{ID}_1$. The results on $\mathbf{ID}_1$ alone have a lot of applications outside proof-theory, that we shall also consider. After all these considerations, the reader will not be too surprised to discover that the emphasis in the next sections is placed on the theory $\mathbf{ID}_1$... .

11.3. Inductive logic

What we call inductive logic is just the use of $\boldsymbol{\beta}$-logic to yield a specific formalism for theories of inductive definitions; as explained above, we shall essentially work with the theories $\mathbf{ID}_1(\boldsymbol{T}, \boldsymbol{\Phi})$... .

11.3.1. Definition.

(i) Under the conditions of Definition 11.2.1, assume that α and β are two ordinals and that $\alpha < \beta$; then we introduce a language $\boldsymbol{L}^{\alpha\beta}$ (we must also denote $\boldsymbol{\Phi}$, for instnace $\boldsymbol{L}^{\alpha\beta}_{\boldsymbol{\Phi}}$, but we shan't): $\boldsymbol{L}^{\alpha\beta}$ consists of $\boldsymbol{L}$, together with the unary predicate $\bar{\boldsymbol{\Phi}}$ and $I\boldsymbol{\Phi}^{\lambda}$, for all $\lambda < \beta$.

(ii) Under the conditions of (i), we define a theory $\boldsymbol{T}^{\alpha\beta}$, whose axioms and rules are

1. the axioms of $\boldsymbol{T}$.
2. the rules of ω-logic for quantifiers of type $\boldsymbol{l}$

$$\frac{...\ \Gamma \vdash A(\bar{n}), \Delta\ ...\ \text{all } n < \omega}{\Gamma \vdash \forall x\ A(x), \Delta}\ r\forall\boldsymbol{l} \qquad \frac{\Gamma, A(\bar{n}_0) \vdash \Delta}{\Gamma, \forall x\ A(x) \vdash \Delta}\ l\forall\boldsymbol{l}$$

$$\frac{\Gamma \vdash A(\bar{n}_0), \Delta}{\Gamma \vdash \exists x\ A(x), \Delta}\ r\exists\boldsymbol{l} \qquad \frac{...\ \Gamma, A(\bar{n}) \vdash \Delta\ ...}{\Gamma, \exists x\ A(x) \vdash \Delta}\ l\forall\boldsymbol{l}$$

3. the rules expressing that $I\boldsymbol{\Phi}^{\lambda}$ is $\boldsymbol{\Phi}$ iterated λ times

$$\frac{\Gamma \vdash \boldsymbol{\Phi}(I\boldsymbol{\Phi}^{\mu}, t), \Delta}{\Gamma \vdash I\boldsymbol{\Phi}^{\lambda}(t), \Delta}\ rI\lambda \qquad \frac{...\ \Gamma, \boldsymbol{\Phi}(I\boldsymbol{\Phi}^{\mu}, t) \vdash \Delta\ ...}{\Gamma, I\boldsymbol{\Phi}^{\lambda}(t) \vdash \Delta}\ lI\lambda$$

In $(rI\lambda)$, μ is an arbitrary ordinal $< \lambda$, whereas in $(lI\lambda)$, μ varies through the set of all ordinals $< \lambda$.

4. the rules for $\bar{\boldsymbol{\Phi}}$:

$$\frac{\Gamma \vdash \mathbf{\Phi}(I\mathbf{\Phi}^{\mu}, t), \Delta}{\Gamma \vdash \bar{\mathbf{\Phi}}(t), \Delta}\ \bar{r} \qquad \frac{\ldots\ \Gamma, \mathbf{\Phi}(I\mathbf{\Phi}^{\mu}, t) \vdash \Delta\ \ldots}{\Gamma, \bar{\mathbf{\Phi}}(t) \vdash \Delta}\ \bar{l}$$

In $(\bar{r})$, μ is an arbitrary ordinal $< \beta$, whereas in $(\bar{l})$, μ varies through the set of all ordinals $< \alpha$.

11.3.2. <u>Comments</u>.

(i) The rules $(rI\lambda)$ and $(lI\lambda)$ exactly express what is expected, namely that:

$$I\mathbf{\Phi}^{\lambda} = \bigcup_{\mu<\lambda} \mathbf{\Phi}(I\mathbf{\Phi}^{\mu}, \cdot)\ .$$

(ii) The interpretation of the rules for $\bar{\mathbf{\Phi}}$ is more problematic, because these rules are asymmetric:

– if we allow the notation $I\mathbf{\Phi}^{\beta}$, then clearly $(\bar{r})$ says that

$$I\mathbf{\Phi}^{\beta} \subset \bar{\mathbf{\Phi}}\ .$$

– similarly, the meaning of $(\bar{l})$ is that

$$\bar{\mathbf{\Phi}} \subset I\mathbf{\Phi}^{\alpha}\ .$$

Since we are in systems with cuts, it is possible to derive from that, by transitivity of inclusion (this is the use of cut):

$$I\mathbf{\Phi}^{\beta} \subset \bar{\mathbf{\Phi}} \subset I\mathbf{\Phi}^{\alpha}$$

and since $\alpha < \beta$, it follows that $I\mathbf{\Phi} = I\mathbf{\Phi}^{\alpha+1}$, i.e. these rules express that α is the *closure ordinal* of $\mathbf{\Phi}$.

(iii) In practice, if one considers, say, $\boldsymbol{T} = \mathbf{PA}_{pr}$, $\mathbf{\Phi} = O$, then the theories $\boldsymbol{T}^{\alpha\beta}$ will be consistent iff $\alpha \geq \omega_1^{CK}$; in particular, for most of values α, $\boldsymbol{T}^{\alpha\beta}$ cannot enjoy any reasonable cut-elimination theorem (which would force $\boldsymbol{T}^{\alpha\beta}$ to be consistent).

11.3.3. Theorem.
If $\Gamma \vdash \Delta$ is provable in $\mathbf{ID}_1(\boldsymbol{T}, \mathbf{\Phi})$, then it is provable in $\boldsymbol{T}^{\alpha\beta}$ for all α and $\beta > \alpha$.

Proof. The usual induction axioms can be proved by the ω-rule, as usual; in order to prove the closure axioms, let us consider:

$$\begin{array}{c}
\mathbf{\Phi}(I\mathbf{\Phi}^{\lambda}, \bar{x}) \vdash \mathbf{\Phi}(I\mathbf{\Phi}^{\lambda}, \bar{x}) \\
\hspace{8em} rI\alpha \\
\ldots \quad \mathbf{\Phi}(I\mathbf{\Phi}^{\lambda}, \bar{x}) \vdash I\mathbf{\Phi}^{\alpha}(\bar{x}) \quad \ldots \quad \text{all } \lambda < \alpha \\
\hspace{16em} \bar{l} \\
\bar{\mathbf{\Phi}}(\bar{x}) \vdash I\mathbf{\Phi}^{\alpha}(\bar{x})
\end{array}$$

Now, using the positivity of $\mathbf{\Phi}$, it is not difficult to build a (cut-free) proof of

$$\mathbf{\Phi}(\bar{\mathbf{\Phi}}, \bar{y}) \vdash \mathbf{\Phi}(I\mathbf{\Phi}^{\alpha}, \bar{y})$$

and we conclude as follows:

$$\begin{array}{c}
\mathbf{\Phi}(\bar{\mathbf{\Phi}}, \bar{y}) \vdash \mathbf{\Phi}(I\mathbf{\Phi}^{\alpha}, \bar{y}) \\
\hspace{12em} \bar{r} \\
\mathbf{\Phi}(\bar{\mathbf{\Phi}}, \bar{y}) \vdash \bar{\mathbf{\Phi}}(\bar{y}) \\
\hspace{10em} r \rightarrow \\
\ldots \ \vdash \ \mathbf{\Phi}(\bar{\mathbf{\Phi}}, \bar{y}) \rightarrow \bar{\mathbf{\Phi}}(\bar{y}) \ \ldots \\
\hspace{10em} r\forall \boldsymbol{l} \\
\vdash \ \forall y \left(\mathbf{\Phi}(\bar{\mathbf{\Phi}}, y) \rightarrow \bar{\mathbf{\Phi}}(y)\right)
\end{array}$$

and we have therefore proved the closure axiom; in this proof the asymmetry between $(\bar{r})$ and $(\bar{l})$ was crucial. In order to prove $\mathbf{\Phi}$-induction, let $B(x)$ be a formula, and let B_0 be $\forall z \ \mathbf{\Phi}(\lambda x B, z) \rightarrow B(z)$; then, for each $\lambda < \alpha$, we produce a proof $\boldsymbol{\pi}^{z}_{\lambda}$ of the segments:

$$B_0, I\mathbf{\Phi}^{\lambda}(\bar{z}) \vdash B(\bar{z}) \ .$$

$$
\boldsymbol{\pi}^{z}_{\lambda} \quad \left\{
\begin{array}{cl}
\boldsymbol{\pi}^{z'}_{\mu} & \\
B_0, I\boldsymbol{\Phi}^{\mu}(\bar{z}') \vdash B(\bar{z}') & \\
\vdots & \\
B_0, \boldsymbol{\Phi}(I\boldsymbol{\Phi}^{\mu}, \bar{z}) \vdash \boldsymbol{\Phi}(\lambda x B, \bar{z}) \;\; B(\bar{z}) \vdash B(\bar{z}) & \\
 & l \rightarrow \\
B_0, \boldsymbol{\Phi}(I\boldsymbol{\Phi}^{\mu}, \bar{z}), \boldsymbol{\Phi}(\lambda x B, \bar{z}) \rightarrow B(\bar{z}) \vdash B(\bar{z}) & \\
 & l\forall \boldsymbol{l} \\
\ldots\; B_0, \boldsymbol{\Phi}(I\boldsymbol{\Phi}^{\mu}, \bar{z}) \vdash B(\bar{z}) \;\ldots\; \textsf{all } \mu < \lambda & \\
 & l I \lambda \\
B_0, I\boldsymbol{\Phi}^{\lambda}(\bar{z}) \vdash B(\bar{z}) &
\end{array}
\right.
$$

(As usual: either you say that the proof is constructed by induction on λ (this is the traditional way of expressing it) or you say that you start with the conclusion, then write some portion of the proof above, then do the same with the premises, etc...).

A similar construction would give:

$$
\begin{array}{cl}
\vdots & \\
\ldots\; B_0, \boldsymbol{\Phi}(I\boldsymbol{\Phi}^{\lambda}, \bar{z}) \vdash B(\bar{z}) \;\ldots\; \textsf{all } \lambda < \alpha & \\
 & \bar{l} \\
B_0, \bar{\boldsymbol{\Phi}}(\bar{z}) \vdash B(\bar{z}) & \\
 & r \rightarrow \\
\ldots\; B_0 \vdash \bar{\boldsymbol{\Phi}}(\bar{z}) \rightarrow B(\bar{z}) \;\ldots & \\
 & r\forall \boldsymbol{l} \\
B_0 \vdash \forall z \left(\bar{\boldsymbol{\Phi}}(z) \rightarrow B(z)\right) & \\
 & r \rightarrow \\
\vdash\; B_0 \rightarrow \forall z \left(\bar{\boldsymbol{\Phi}}(z) \rightarrow B(z)\right) &
\end{array}
$$

and we have therefore proved the $\boldsymbol{\Phi}$-induction axiom. □

11.3.4. Definition.

(i) Under the hypotheses of 11.2.1, let F be a dilator of the form $\underline{a} + \textsf{Id} + \underline{1} + F'$ (to simplify the understanding, the reader can imagine that $F = \textsf{Id} + \underline{1}$); then we define a language $\boldsymbol{L}^F$ as follows:
$\boldsymbol{L}^F$ consists of $\boldsymbol{L}$, togethjer with the unary predicates $\bar{\boldsymbol{\Phi}}$ and $\boldsymbol{\Phi}^t$, for

all F-terms t (10.2.17).

(ii) A proof in $\boldsymbol{T}^F$ of a sequent $\Gamma \vdash \Delta$ of $\boldsymbol{L}^F$ consists in a family $(\boldsymbol{\pi}_\alpha)_{\alpha \in 0n}$ such that:

1. For all α, $\boldsymbol{\pi}_\alpha$ proves $\Gamma(\alpha) \vdash \Delta(\alpha)$ (which is the result of replacing in Γ and Δ all F-terms t by their value $t(\alpha)$ at α) in the system $\boldsymbol{T}^{a+\alpha\, F(\alpha)}$.
2. For all $\alpha, \beta \in 0n$ and $f \in I(\alpha, \beta)$, $F(f^{-1})(\boldsymbol{\pi}_\beta)$ is defined and equals $\boldsymbol{\pi}_\alpha$. (The definition of $F(f)^{-1}$ is straightforward and therefore is left to the reader... .) These proofs are also called F-proofs.

11.3.4. Examples.

(i) The proof (in $\boldsymbol{T}^{\alpha,\alpha+1}$) of $\vdash \forall y \left(\boldsymbol{\Phi}(\bar{\boldsymbol{\Phi}}, y) \rightarrow \bar{\boldsymbol{\Phi}}(y)\right)$ given in 11.3.3 (let's call it $\boldsymbol{\lambda}_\alpha$) is such that:

$$(\mathsf{Id} + \underline{1})^{-1}(\boldsymbol{\lambda}_\beta) = \boldsymbol{\lambda}_\alpha \qquad \text{when } f \in I(\alpha, \beta)\ ,$$

hence $(\boldsymbol{\lambda}_\alpha)$ defines a $\mathsf{Id} + \underline{1}$-proof of the closure axiom.

(ii) In a similar way, the proofs (in $\boldsymbol{T}^{\alpha,\alpha+1}$) of the $\boldsymbol{\Phi}$-induction axiom on B define a proof in $\boldsymbol{T}^{\mathsf{Id}+\underline{1}}$ of this axiom.

11.3.5. Remark.

We have already remarked that the theories $\boldsymbol{T}^{\alpha\beta}$ are most often inconsistent, and in particular cannot enjoy cut-elimination; for the theories $\boldsymbol{T}^F$ the situation is different: they are all consistent (provided $\boldsymbol{T}$ is itself consistent), hence there is no a priori reason why the proofs in $\boldsymbol{T}^F$, considered as whole entities, should not enjoy reasonable cut-elimination theorems. We shall see in the next section that from a given proof in $\boldsymbol{T}^F$ it is possible to build a cut-free proof of the same thing, provided the dilator F is changed into some F', and the parameters are subsequently modified... .

11.3.6. Theorem.

Let $\boldsymbol{T}$ be a prim. rec. theory, $\boldsymbol{\Phi}$ a positive operator, and let $\Gamma \vdash \Delta$ be a closed sequent of $\boldsymbol{L}[\bar{\boldsymbol{\Phi}}]$; the following are equivalent:

(i) $\Gamma \vdash \Delta$ is true in any ω-model of $\boldsymbol{L}\,[\bar{\boldsymbol{\Phi}}]$ of the form $\boldsymbol{m}\,[\bar{\boldsymbol{\Phi}}]$ (see 11.1.8).

(ii) $\Gamma \vdash \Delta$ is provable in $\boldsymbol{T}^F$ for some dilator F of the form $\underline{a}+\mathsf{Id}+\underline{1}+F'$. (Moreover the proof can be chosen prim. rec. and one can assume that $F = \mathsf{Id} + \underline{1}$.)

Proof. (ii) $\rightarrow$ (i); choose an ω-model of $\boldsymbol{T}$, say $\boldsymbol{m}$; in $\boldsymbol{m}$, $\boldsymbol{\Phi}$ has a closure ordinal, say α_0; we show that $\boldsymbol{m}$ can be extended into a model of $\boldsymbol{T}^{a+\alpha_0,F(\alpha_0)}$: we simply define $\boldsymbol{m}(I\boldsymbol{\Phi}^\lambda)$ as in 11.1.5; of course we must verify that the rules $(rI\lambda)$, $(lI\lambda)$, $(\bar{r})$, $(\bar{l})$ are valid for this interpretation:

- The validity of $(rI\lambda)$ amounts to verifying that

$$\boldsymbol{m}(\boldsymbol{\Phi})\Big(\boldsymbol{m}(I\boldsymbol{\Phi}^\mu)\Big) \subset \boldsymbol{m}(I\boldsymbol{\Phi}^\lambda) \qquad \text{when } \mu < \lambda \ .$$

- The validity of $(lI\lambda)$ amounts to verifying that

$$\boldsymbol{m}(I\boldsymbol{\Phi}^\lambda) \subset \bigcup_{\mu<\lambda} \boldsymbol{m}(\boldsymbol{\Phi})\Big(\boldsymbol{m}(I\boldsymbol{\Phi}^\mu)\Big)$$

  and these two formulas are simply trivial by construction.

- The validity of $(\bar{r})$ amounts to verifying that

$$\boldsymbol{m}(I\boldsymbol{\Phi}^{F(\alpha_0)}) \subset \boldsymbol{m}(\bar{\boldsymbol{\Phi}}) \ .$$

- The validity of $(\bar{l})$ amounts to verifying that

$$\boldsymbol{m}(\bar{\boldsymbol{\Phi}}) \subset \boldsymbol{m}(I\boldsymbol{\Phi}^{a+\alpha_0})$$

  but since α_0 is the closure ordinal of $\boldsymbol{\Phi}$ in $\boldsymbol{m}$, it is plain that

$$\boldsymbol{m}(\bar{\boldsymbol{\Phi}}) = \boldsymbol{m}(I\boldsymbol{\Phi}^{a+\alpha_0}) = \boldsymbol{m}(I\boldsymbol{\Phi}^{F(\alpha_0)})$$

  (we use the fact that $a + \alpha_0 < F(\alpha_0)$).

Now, if $(\boldsymbol{\pi}_\alpha)$is any proof in $\boldsymbol{T}^F$ of $\Gamma \vdash \Delta$, then $\boldsymbol{\pi}_{\alpha_0}$ will be a proof in $\boldsymbol{T}^{a+\alpha_0\,F(\alpha_0)}$ of the sequent $\Gamma(\alpha_0) \vdash \Delta(\alpha_0)$, which is equal to $\Gamma \vdash \Delta$, and since $\boldsymbol{m}$ can be extended to a model of $\boldsymbol{T}^{a+\alpha_0\,F(\alpha_0)}$, it follows that $\Gamma \vdash \Delta$ is true in $\boldsymbol{m}\,[\boldsymbol{\Phi}]$.

(i) $\rightarrow$ (ii): The first thing is to obtain a completeness theorem w.r.t. the following theories $\boldsymbol{T}^\alpha$ consisting of:

1. $\boldsymbol{T}$.

2. The rules of ω-logic.

3. The rules $(rI\lambda)$, $(lI\lambda)$ for all $\lambda < \alpha$.

4. The rules $(\bar{r})$, $(\bar{l})$; in $(\bar{r})$ μ is $< \alpha$.

5. The additional axiom: $\vdash \forall y \Big(\boldsymbol{\Phi}(\bar{\boldsymbol{\Phi}}, y) \rightarrow \bar{\boldsymbol{\Phi}}(y)\Big)$.

Then a straightforward adaptation of the β-completeness theorem shows that, if $\Gamma \vdash \Delta$ is true in all ω-models of the form $\boldsymbol{m}\,[\bar{\boldsymbol{\Phi}}]$, then it has a primitive recursive $\boldsymbol{\beta}$-proof of the form $(\boldsymbol{\pi}_\alpha)$:

- $\boldsymbol{\pi}_\alpha$ is a proof of $\Gamma \vdash \Delta$ in $\boldsymbol{T}^\alpha$.
- If $f \in I(\alpha, \beta)$ then $f^{-1}(\boldsymbol{\pi}_\beta) = \boldsymbol{\pi}_\alpha$.

(Sketch of the proof. Take for instance a system of $\boldsymbol{\beta}\omega$-logic where $\boldsymbol{I\Phi}$ appears as a two variables predicate $\boldsymbol{I\Phi}(\bar{\lambda}, \bar{n})$l the $\boldsymbol{\beta}\omega$-models of such a theory are exactly those models whose restriction to $\boldsymbol{L}\,[\bar{\boldsymbol{\Phi}}]$ is an ω-model of the form $\boldsymbol{m}\,[\boldsymbol{\Phi}]$; of course the rules $(rI\lambda)$, $(lI\lambda)$, $(\bar{r})$, $(\bar{l})$ must be rewritten as axioms, say:

$$\text{(a)} \qquad \forall\lambda\ \forall n \Big(\boldsymbol{I\Phi}(\lambda, n) \leftrightarrow \exists\mu < \lambda\ \boldsymbol{\Phi}(\boldsymbol{I\Phi}(\mu, \cdot), n)\Big)$$

$$\text{(b)} \qquad \forall n \Big(\bar{\boldsymbol{\Phi}}(n) \leftrightarrow \exists\lambda\ \boldsymbol{\Phi}(\boldsymbol{I\Phi}(\lambda, \cdot), n)\Big)\ .$$

The $\boldsymbol{\beta}$-completeness theorem enables us to find a prim. rec. $\boldsymbol{\beta}\omega$-proof of $\Gamma \vdash \Delta$, in this modified formalism. Then, for instance, we can replace the axioms (a) and (b) by their obvious proofs by means of $(rI\lambda)$, $(lI\lambda)$, $(\bar{r})$, $(\bar{l})$. A straightforward cut-elimination procedure enables us to eliminate all ordinal quantifiers ... then replacing all the $\boldsymbol{I\Phi}(\bar{\lambda}, n)$ by $\boldsymbol{I\Phi}^\lambda(n)$ we obtained the desired proof. Of course, it is also possible to use the completeness for $\boldsymbol{L}_{\boldsymbol{\beta}_\omega}$, using translations:

$$\boldsymbol{I\Phi}^\lambda(\bar{n}) \ :\ \bigvee_{\mu<\lambda} \boldsymbol{\Phi}(\boldsymbol{I\Phi}^\mu, \bar{n})$$

$$\bar{\boldsymbol{\Phi}}(\bar{n}) \ :\ \bigvee_{\lambda<\alpha} \boldsymbol{\Phi}(\boldsymbol{I\Phi}^\lambda, \bar{n})\ .$$

But of course a direct adaptation of the $\boldsymbol{\beta}$-completeness argument is the simplest solution, if not the shortest! $\square$)

Now, we modify our proof $(\boldsymbol{\pi}_\alpha)$ simply by replacing all uses of the closure axiom 5 by its proof in $\boldsymbol{T}^{\alpha\alpha+1}$, as described in 11.3.3; the resulting proof $(\boldsymbol{\pi}'_\alpha)$ is a prim. rec. $\mathsf{Id} + \underline{1}$-proof of $\Gamma \vdash \Delta$. $\square$

11.3.7. Remark.

The simultaneous use of ω and $\boldsymbol{\beta}$-logics is slightly inelegant; we shall see later on how to define variants of these constructions (by considering $\mathbf{ID}_1$ as a twice-iterated inductive definition) making use of $\boldsymbol{\beta}$-logic only... .

11.4. The cut-elimination theorem

This section is devoted to a syntactial proof of the following:

11.4.1. Theorem (Girard, 1979, [100]).
Assume that $\Gamma \vdash \Delta$ is a sequent of $\boldsymbol{L}\,[\bar{\mathbf{\Phi}}]$, $\boldsymbol{\pi} = (\boldsymbol{\pi}_\alpha)$ is a proof of $\Gamma \vdash \Delta$ in $\boldsymbol{T}^F$, and F and $\boldsymbol{\pi}$ are prim. rec. (typically: $F = \mathsf{Id} + \underline{1}$); then it is possible to construct explicitly a recursive dilator F' together with a recursive proof $\boldsymbol{\pi}'$ of $\Gamma \vdash \Delta$ in $\boldsymbol{T}^{F'}$, such that all cut-formulas of the proofs $\boldsymbol{\pi}'_\alpha$ belong to $\boldsymbol{L}$.

Proof. The syntactial proof of this fact is, as in the case of most cut-elimination results, long and tedious; the reader is advised of the existence of a purely model-theoretic argument, discovered by Buchholz (1982, [101]); see 11.4.9 (iii). We first indicate the main ideas of the cut-elimination theorem; let us assume that $F = \mathsf{Id} + \underline{1} + F'$.

The reader already knows how to eliminate all cuts whose cut-formula is not atomic; the new cases arise with cut-formulas of the form $I\mathbf{\Phi}^\lambda(\bar{n})$ and $\bar{\mathbf{\Phi}}(\bar{n})$; the elimination of a $I\mathbf{\Phi}^\lambda$-cut is straightforward

$$\boldsymbol{\pi}_\alpha \quad \left\{ \dfrac{\dfrac{\begin{matrix}\vdots\ \boldsymbol{\pi}'_\alpha \\ \Gamma \vdash \mathbf{\Phi}(I\mathbf{\Phi}^\mu, \bar{n}), \Delta\end{matrix}}{\Gamma \vdash I\mathbf{\Phi}^\lambda(\bar{n}), \Delta}\, rI\lambda \qquad \dfrac{\begin{matrix}\vdots\ \boldsymbol{\pi}''_{\nu,\alpha} \\ \ldots\ \Gamma', \mathbf{\Phi}(I\mathbf{\Phi}^\nu, \bar{n}) \vdash \Delta'\ \ldots\ \mathsf{all}\ \nu < \lambda\end{matrix}}{\Gamma', I\mathbf{\Phi}^\lambda(\bar{n}) \vdash \Delta'}\, lI\lambda}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\, \mathsf{CUT} \right.$$

This typical example will obviously be replaced by:

$$\dfrac{\begin{matrix}\vdots\ \boldsymbol{\pi}'_\alpha \\ \Gamma \vdash \mathbf{\Phi}(I\mathbf{\Phi}^\mu, \bar{n}), \Delta\end{matrix} \qquad \begin{matrix}\vdots\ \boldsymbol{\pi}''_{\mu,\alpha} \\ \Gamma', \mathbf{\Phi}(I\mathbf{\Phi}^\mu, \bar{n}) \vdash \Delta'\end{matrix}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\, \mathsf{CUT}$$

The case of a cut whose cut-formula is $\bar{\mathbf{\Phi}}(\bar{n})$ is completely different:

$$\boldsymbol{\pi}_\alpha \quad \left\{ \dfrac{\dfrac{\begin{array}{c}\vdots\ \boldsymbol{\pi}'_\alpha \\ \Gamma \vdash \boldsymbol{\Phi}(I\boldsymbol{\Phi}^\mu, \bar{n}), \Delta\end{array}}{\Gamma \vdash \bar{\boldsymbol{\Phi}}(\bar{n}), \Delta}\,\bar{r} \qquad \dfrac{\begin{array}{c}\vdots\ \boldsymbol{\pi}''_{\nu,\alpha} \\ \ldots\ \Gamma', \boldsymbol{\Phi}(I\boldsymbol{\Phi}^\nu, \bar{n}) \vdash \Delta'\ \ldots\ \text{all } \nu < \alpha\end{array}}{\Gamma', \bar{\boldsymbol{\Phi}}(\bar{n}) \vdash \Delta'}\,\bar{l}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathsf{CUT} \right.$$

What makes the situation hopeless is that μ is any ordinal $< F(\alpha)$ (typically: $\mu = \alpha$), whereas the premises of the $(\bar{l})$ rule vary only over ordinals $< \alpha$. This technical remark is just another way of remarking that the theories $\boldsymbol{T}^{\alpha\alpha+1}$ cannot enjoy any reasonable cut-elimination (11.3.2 (iii)). But we have also remarked that no such a priori limitation applies to proofs-as-functors (11.3.5). So, instead of trying to eliminate the cuts in $\boldsymbol{\pi} = \boldsymbol{\pi}_\alpha$ for each separate value of the ordinal α, we make a cut-elimination procedure which is not pointwise in the sense that *it applies to the family* $(\boldsymbol{\pi}_\alpha)$ *as a whole!*

The idea is essentially to form $\boldsymbol{\pi}''_{\mu,F(\alpha)}$ and then to form the cut:

$$\dfrac{\begin{array}{c}\vdots\ \boldsymbol{\pi}'_\alpha \\ \Gamma \vdash \boldsymbol{\Phi}(I\boldsymbol{\Phi}^\mu, \bar{n}), \Delta\end{array} \qquad \begin{array}{c}\vdots\ \boldsymbol{\pi}''_{\nu,F(\alpha)} \\ \Gamma', \boldsymbol{\Phi}(I\boldsymbol{\Phi}^\mu, \bar{n}) \vdash \Delta'\end{array}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathsf{CUT}$$

If we look more closely, we see that:

1. the coefficients in $\Gamma' \vdash \Delta'$ have been modified.

2. the proof obtained is now a $F \circ F$-proof.

3. in fact in the rules $(\bar{l})$ above the right premise of the cut, we have $F(\alpha)$ premises, that is obviously too much ... but too many premises are no handicap ... simply chop all premises of index $\geq \alpha$.

4. in the more general case $F = \underline{a} + \mathsf{Id} + 1 + F'$, instead of forming $F \circ F$, we form $F \circ (\mathsf{Id} + 1 + F')\ldots$.

Let us now enter into the heart of the matter... .

11.4.2. Definition.
Assume that $\boldsymbol{\pi} = (\boldsymbol{\pi}_\alpha)$ is a F-proof whose *last rule* (10.2.24) is a cut

$$\dfrac{\Gamma(\alpha) \vdash A(\alpha), \Delta(\alpha) \quad \Gamma'(\alpha), A(\alpha) \vdash \Delta'(\alpha)}{\Gamma(\alpha), \Gamma'(\alpha) \qquad \vdash \Delta(\alpha), \Delta'(\alpha)}\ \mathsf{CUT}$$

This cut is said to be of:

kind I : when $A(\alpha)$ is a formula of $\boldsymbol{L}$.
kind II : when $A(\alpha)$ is neither a formula of $\boldsymbol{L}$, nor a formula $\bar{\boldsymbol{\Phi}}(\bar{n})$.
kind III : when $A(\alpha)$ is a formula of the form $\bar{\boldsymbol{\Phi}}(\bar{n})$.

11.4.3. Theorem.
Assume that $\boldsymbol{\pi} = (\boldsymbol{\pi}_\alpha)$ is a F-proof of a sequent $\Gamma \vdash \Delta$ of $\boldsymbol{L}^F$; then one can construct (recursively in the data) a F-proof $\boldsymbol{\pi}' = (\boldsymbol{\pi}'_\alpha)$ of the same sequent $\Gamma \vdash \Delta$ such that:

(i) all cuts in $\boldsymbol{\pi}'$ are of kind I or III.

(ii) if no cut-formula of $\boldsymbol{\pi}$ contains $\bar{\boldsymbol{\Phi}}$, then all cuts in $\boldsymbol{\pi}'$ are of kind I.

Proof. This is a rather straightforward theorem, not essentially different from cut-elimination for the sequent calculus $\boldsymbol{L}_{\boldsymbol{\beta}_\omega}$... . The idea is to replace any cut of kind II by other cuts of smaller complexity, until we obtain only cuts of kinds I and III... . We can for instance do this for all proofs $\boldsymbol{\pi}_\alpha$ separately; if the result of this process is denoted $N(\boldsymbol{\pi}_\alpha)$, then we simply observe that, if $f \in I(\alpha, \beta)$ and $F(f)^{-1}(\boldsymbol{\pi}_\beta) = \boldsymbol{\pi}_\alpha$, then $F(f)^{-1}\Big(N(\boldsymbol{\pi}_\beta)\Big) = N(\boldsymbol{\pi}_\alpha)$. The very details are boring, very close to proofs we have already produced several times in this book, and I don't think seriously that a reader who has succeeded in getting through the book up to this chapter can have the slightest hesitation on such a theorem! However, let us compute explicitly a concept of **degree**, which computes the number of steps from A to cut-formulas of kinds I or III:

$$d^0(A) = -1 \quad \text{when } A \in \boldsymbol{L} \quad \text{or } A = \bar{\boldsymbol{\Phi}}(\bar{n})$$

$$d^0\Big(\boldsymbol{I\Phi}^\lambda(\bar{n})\Big) = \omega \cdot \lambda$$

$$d^0(A \to B) = d^0(A \vee B) = d^0(A \vee B) = \mathsf{sup}\Big(d^0(A), d^0(B)\Big) + 1$$
when A and B are not both in *bol*

$$d^0(\forall x\ A) = d^0(\exists x\ A) = d^0(\neg A) = d^0(A) + 1 \quad \text{when } A \notin \boldsymbol{L}\ .$$

Then cuts of kind II correspond exactly to $d^0(A) \neq -1$; the degree is functorial, i.e.

$$\underline{\omega} \cdot F(f)^{-1}\Big(d^0(A)\Big) = d^0\Big(F(f)^{-1}(A)\Big) \quad (A \in \boldsymbol{L}^{a+\beta, F(\beta)}\ldots)\ ;$$

moreover, $d^0\Big(\boldsymbol{\Phi}(I\boldsymbol{\Phi}^{\mu}, \bar{n})\Big) < d^0\Big(I\boldsymbol{\Phi}^{\lambda}(\bar{n}\Big)$ when $\lambda < \mu \ldots$. And 11.4.3 is simply the fact that the global cut-degree can be lowered to 0, i.e. the only cuts are of degree $-1\ldots$. But it's enough for this! $\square$

11.4.4. Theorem.

(i) To each 3-uple $(\boldsymbol{\pi}, F, \Gamma \vdash \Delta)$ such that

1. F is a dilator of the form $\underline{a} + \mathsf{Id} + \underline{1} + F'$
2. $\Gamma \vdash \Delta$ is a sequent of $\boldsymbol{L}^F$
3. $\boldsymbol{\pi}$ is a proof of $\Gamma \vdash \Delta$ in $\boldsymbol{T}^F$ such that $\boldsymbol{\pi}$ has no cuts of kind II

we associate a 3-uple $\boldsymbol{\vartheta}(\boldsymbol{\pi}, F, \Gamma \vdash \Delta)$, improperly denoted by $(\tilde{\boldsymbol{\pi}}, \tilde{F}, \tilde{\Gamma} \vdash \tilde{\Delta})$ (the definition of $\tilde{\boldsymbol{\pi}}$ depends also on F; the definition of $\tilde{F}$ depends also on $\boldsymbol{\pi}$) such that:

1. $\tilde{F}$ is a dilator of the form $F + F''$.
2. $\tilde{\Gamma} \vdash \tilde{\Delta}$ is the result of replacing in $\Gamma \vdash \Delta$ all atomic formulas $I\boldsymbol{\Phi}^t(n)$ where t is a non- constant F-term (i.e. $t(\alpha) \geq a + \alpha$ for all α) by $\bar{\boldsymbol{\Phi}}(n)$.
3. $\tilde{\boldsymbol{\pi}}$ is a $\tilde{F}$-proof of $\tilde{\Gamma} \vdash \tilde{\Delta}$; the only cuts of $\tilde{\boldsymbol{\pi}}$ are of kind I.

(ii) The construction is functorial in the following sense: let $G = \underline{b}' + \mathsf{Id} + \underline{1} + G'$, and assume that $T \in I^1(G, F)$ is of the form $\underline{f} + \mathbf{E}^1_{\mathsf{Id}} + \mathbf{E}^1_{\underline{1}} + T'$, and assume that $T^{-1}(\Gamma \vdash \Delta) = \Gamma' \vdash \Delta'$ exists, as well as $T^{-1}(\boldsymbol{\pi}) = \boldsymbol{\pi}'$; then if we consider $(\tilde{\boldsymbol{\pi}}', \tilde{G}, \tilde{\Gamma}' \vdash \tilde{\Delta}')$, we can define $\tilde{T} \in I^1(\tilde{F}, \tilde{G})$ of the form $T + T''$ s.t. $\tilde{T}^{-1}(\tilde{\boldsymbol{\pi}}) = \tilde{\boldsymbol{\pi}}'$ ($\tilde{T}^{-1}(\tilde{\Gamma} \vdash \tilde{\Delta}) = \tilde{\Gamma}' \vdash \tilde{\Delta}'$ is trivial).

(iii) The construction $\tilde{}$ has the following extra properties:

a. $\widetilde{\mathbf{E}'_F} = \mathbf{E}^1_{\tilde{F}}$.

b. If $(\boldsymbol{\pi}, F, \Gamma \vdash \Delta) \xrightarrow{T} (\boldsymbol{\pi}', F', \Gamma' \vdash \Delta')$
and $(\boldsymbol{\pi}', F', \Gamma' \vdash \Delta') \xrightarrow{T'} (\boldsymbol{\pi}'', F'', \Gamma'' \vdash \Delta'')$
(standard notations for the situation of (ii)) then since one has $(\boldsymbol{\pi}', F, \Gamma \vdash \Delta) \xrightarrow{T'T} (\boldsymbol{\pi}'', F'', \Gamma'' \vdash \Delta'')$ one can define $\tilde{T}$, $\tilde{T}'$, $\widetilde{T'T}$; then $\widetilde{T'T} = \tilde{T}'\tilde{T}$.

c. If $(\boldsymbol{\pi}_i, F_i, \Gamma_i \vdash \Delta_i) \xrightarrow{T_i} (\boldsymbol{\pi}', F', \Gamma' \vdash \Delta')$ ($i \in I$, I directed) are such that:

$$i < j \to rg\big(\mathsf{Tr}(T_i)\big) \subset rg\big(\mathsf{Tr}(T_j)\big)$$

if $(\boldsymbol{\pi}, F, \Gamma \vdash \Delta) \xrightarrow{T} (\boldsymbol{\pi}', F', \Gamma' \vdash \Delta')$ and $rg\big(\mathsf{Tr}(T)\big) = \bigcup_{i \in I} rg\big(\mathsf{Tr}(T_i)\big)$ then $rg\big(\mathsf{Tr}(\tilde{T})\big) = \bigcup_{i \in I} rg\big(\mathsf{Tr}(\tilde{T}_i)\big)$.

d. If $(\boldsymbol{\pi}_i, F_i, \Gamma_i \vdash \Delta_i) \xrightarrow{T_i} (\boldsymbol{\pi}, F, \Gamma \vdash \Delta)$ ($i = 1, 2, 3$) and $rg\big(\mathsf{Tr}(T_1)\big) = rg\big(\mathsf{Tr}(T_2)\big) \cap rg\big(\mathsf{Tr}(T_3)\big)$ (i.e. $T_1 = T_2 \wedge T_3$) then $\tilde{T}_1 \wedge \tilde{T}_2 = \tilde{T}_3$.

(Part (iii) of the theorem, which is the longest to state, is also the obvious part; the proof will be omitted... .)

Proof. First a remark on notatins: we shall often need to compute $\tilde{F}$ in several "contexts", for instance in $(\boldsymbol{\pi}, F, \Gamma \vdash \Delta)$ and $(\boldsymbol{\pi}', F, \Gamma' \vdash \Delta')$, and of course we cannot use $\tilde{F}$ as a notation for both cases; so what we do is the following; we say "consider $(\boldsymbol{\pi}', F', \Gamma' \vdash \Delta')$, with $F' = F''$ ... then we use $\tilde{F}'$ to speak of $\tilde{F}$ in the context of $(\boldsymbol{\pi}', F, \Gamma' \vdash \Delta')$. In order to carry out the proof, it will be necessary to analyze proofs as we did in Sec. 10.2... . One must therefore give a last rule, and premises for all F-proofs... . Assuming this has been done, then the proof of the theorem requires looking through a certain number of cases: we work by *induction on* $\boldsymbol{\pi}$ (10.2.9):

1. If $\boldsymbol{\pi}$ is an axiom of $\boldsymbol{T}$: let $\tilde{\boldsymbol{\pi}} = \boldsymbol{\pi}$, $\tilde{F} = F$.
2. If the last rule (R) of $\boldsymbol{\pi}$ is unary, there are three subcases

subcase a (R) is $(\bar{r})$: let us write

$$\boldsymbol{\pi} \quad \left\{ \begin{array}{c} \boldsymbol{\pi}' \ \vdots \\ \Gamma \vdash \boldsymbol{\Phi}(I\boldsymbol{\Phi}^\lambda, \bar{n}), \Delta \\ \hline \Gamma \vdash \bar{\boldsymbol{\Phi}}(\bar{n}), \Delta \end{array} \bar{r} \right.$$

The induction hypothesis, applied to $(\boldsymbol{\pi}', F', \Gamma \vdash \boldsymbol{\Phi}(I\boldsymbol{\Phi}^\lambda, \bar{n}), \Delta)$ with $F' = F$, yields a 3-uple $(\tilde{\boldsymbol{\pi}}', \tilde{F}', \tilde{\Gamma} \vdash \boldsymbol{\Phi}(\widetilde{I\boldsymbol{\Phi}^\lambda}, \bar{n}), \tilde{\Delta})$ enjoying the properties of the theorem; if λ is a constant F-term, then $\boldsymbol{\Phi}(\widetilde{I\boldsymbol{\Phi}^\lambda}, \bar{n}) = \boldsymbol{\Phi}(I\boldsymbol{\Phi}^\lambda, \bar{n})$, hence one can define:

$$\tilde{\boldsymbol{\pi}} \quad = \quad \left\{ \begin{array}{c} \vdots \ \tilde{\boldsymbol{\pi}}' \\ \tilde{\Gamma} \vdash \boldsymbol{\Phi}(I\boldsymbol{\Phi}^\lambda, \bar{n}), \tilde{\Delta} \\ \hline \tilde{\Gamma} \vdash \bar{\boldsymbol{\Phi}}(\bar{n}), \tilde{\Delta} \end{array} \bar{r} \right.$$

and $\tilde{F} = \tilde{F}'$. But, if λ is not a constant F-term, then $\boldsymbol{\Phi}(\widetilde{I\boldsymbol{\Phi}^\lambda}, \bar{n}) = \boldsymbol{\Phi}(\bar{\boldsymbol{\Phi}}, \bar{n})$. Here we use a lemma:

11.4.5. <u>Lemma</u>.

Given a F-proof of $\Gamma \vdash \Delta$, say $\boldsymbol{\pi}$, it is possible to construct a $F + \underline{1}$-proof of $\Gamma_1 \vdash \Delta_1$, where $\Gamma_1 \vdash \Delta_1$ is obtained from $\Gamma \vdash \Delta$ by replacing some positive occurrences of $\bar{\boldsymbol{\Phi}}$ in $\Gamma \vdash \Delta$ by corresponding occurrences of $I\boldsymbol{\Phi}^t$, where t is the $F + \underline{1}$-term $t(x) = F(x)$.

Furthermore, the construction is functorial in the following sense: if $T = \underline{f} + \mathbf{E}^1_{\mathsf{Id}} + \mathbf{E}^1_{\underline{1}} + T' \in I^1(G, F)$ and $\Gamma' \vdash \Delta' = T^{-1}(\Gamma \vdash \Delta)$, then $\Gamma'_1 \vdash \Delta'_1 = T_1^{-1}(\Gamma_1 \vdash \Delta_1)$ if $T^{-1}(\boldsymbol{\pi}) = \boldsymbol{\pi}'$, then $T_1^{-1}(\boldsymbol{\pi}_1) = \boldsymbol{\pi}'_1$, with $T_1 = T + \mathbf{E}^1_{\underline{1}}$.

<u>Proof</u>. The formulation of the lemma is more complicated than its proof: simply replace all occurrences of $\bar{\boldsymbol{\Phi}}$ which are “ancestors” of those occurrences of $\bar{\boldsymbol{\Phi}}$ in $\Gamma \vdash \Delta$ one wants to modify, by corresponding occurrences of $I\boldsymbol{\Phi}^t$. (The precise formulation of this would be terribly pedantic.) Of course, this forces us to rename some rules: some $(\bar{r})$ appear now as rules (rIt). The very details are left to the reader. The functorial property is immediate. $\square$

We apply the lemma to $\tilde{\boldsymbol{\pi}}'$, in order to get a proof $\tilde{\boldsymbol{\pi}}'_1$ of $\tilde{\Gamma} \vdash \boldsymbol{\Phi}(I\boldsymbol{\Phi}^t, \bar{n}), \tilde{\Delta}$, with $t(\alpha) = F(\alpha)$ for all α. This is a $\tilde{F}' + \underline{1}$-proof, and one defines

$$\tilde{\boldsymbol{\pi}} \quad = \quad \left\{ \begin{array}{c} \tilde{\boldsymbol{\pi}}' \;\vdots \\ \tilde{\Gamma} \vdash \boldsymbol{\Phi}(I\boldsymbol{\Phi}^t, \bar{n}), \tilde{\Delta} \\ \hline \tilde{\Gamma} \vdash \bar{\boldsymbol{\Phi}}(\bar{n}), \tilde{\Delta} \end{array} \bar{r} \right.$$

and one defines $\tilde{F} = \tilde{F}' + \underline{1}$.

The functorial part of the theorem is left to the reader. (Let $\tilde{T} = T + \mathbf{E}^1_{\underline{1}}$.)

<u>subcase b</u> (R) is $(rI\lambda)$ with λ a non constant F-term. Then

$$\boldsymbol{\pi} \quad \left\{ \begin{array}{c} \boldsymbol{\pi}' \;\vdots \\ \Gamma \vdash \boldsymbol{\Phi}(I\boldsymbol{\Phi}^\mu, \bar{n}), \Delta \\ \hline \Gamma \vdash I\boldsymbol{\Phi}^\lambda(\bar{n}), \Delta \end{array} rI\lambda \right.$$

The situation is very close to subcase a:

- If μ is a constant F-term, let

$$\tilde{\boldsymbol{\pi}} \quad = \quad \left\{ \begin{array}{c} \vdots\; \tilde{\boldsymbol{\pi}}' \\ \tilde{\Gamma} \vdash \boldsymbol{\Phi}(I\boldsymbol{\Phi}^\mu, \bar{n}), \tilde{\Delta} \\ \hline \tilde{\Gamma} \vdash \bar{\boldsymbol{\Phi}}(\bar{n}), \tilde{\Delta} \end{array} \bar{r} \right.$$

  and $\tilde{F} = \tilde{F}'$.

- $$\tilde{\boldsymbol{\pi}} \quad \begin{array}{c} \tilde{\boldsymbol{\pi}}'_1 \;\vdots \\ \tilde{\Gamma} \vdash \boldsymbol{\Phi}(I\boldsymbol{\Phi}^t, \bar{n}), \tilde{\Delta} \\ \hline \tilde{\Gamma} \vdash \bar{\boldsymbol{\Phi}}(\bar{n}), \tilde{\Delta} \end{array} \bar{r}$$

  with $\tilde{F} = \tilde{F}' + \underline{1}$.

<u>subcase c</u> in all other cases, one can write

$$\boldsymbol{\pi} \quad \left\{ \begin{array}{c} \vdots\ \boldsymbol{\pi}' \\ \Gamma' \vdash \Delta' \\ \hline \Gamma \vdash \Delta \end{array} R \right.$$

and it will be possible to define $\tilde{F} = \tilde{F}'$,

$$\tilde{\boldsymbol{\pi}} \quad \left\{ \begin{array}{c} \vdots\ \boldsymbol{\pi}' \\ \tilde{\Gamma}' \vdash \tilde{\Delta}' \\ \hline \tilde{\Gamma} \vdash \tilde{\Delta} \end{array} R \right.$$

3. The last rule (R) is binary; one must consider two subcases:

<u>subcase a</u> (R) is not a cut of kind III; assume that

$$\boldsymbol{\pi} \quad \left\{ \begin{array}{c} \begin{array}{cc} \vdots\ \boldsymbol{\pi}' & \vdots\ \boldsymbol{\pi}'' \\ \Gamma' \vdash \Delta' & \Gamma'' \vdash \Delta'' \end{array} \\ \hline \Gamma \vdash \Delta \end{array} R \right.$$

Let us apply the induction hypothesis to the 3-uples $(\boldsymbol{\pi}', F', \Gamma' \vdash \Delta')$, $(\boldsymbol{\pi}'', F''$,
$\Gamma'' \vdash \Delta'')$, with $F' = F'' = F$. We obtain $(\tilde{\boldsymbol{\pi}}', \tilde{F}', \tilde{\Gamma}' \vdash \tilde{\Delta}')$, $(\tilde{\boldsymbol{\pi}}'', .\tilde{F}'', \tilde{\Gamma}'' \vdash \tilde{\Delta}'')$; the main problem comes from the fact that, in general, $\tilde{F}' \neq \tilde{F}''$. Here again, we need a lemma:

11.4.6. <u>Lemma</u>.
Given a F-proof $\boldsymbol{\pi}$ of $\Gamma \vdash \Delta$ such that:

1. $\boldsymbol{\pi}$ contains no cuts of kind II or III.

2. if $I\boldsymbol{\Phi}^{\lambda}$ occurs in $\Gamma \vdash \Delta$, then λ is a constant F-term

and given G and $T \in I^1(F, G)$, $T = \mathbf{E}^1_{\underline{a}} + \mathbf{E}^1_{\mathsf{Id}} + \mathbf{E}^1_{\underline{1}} + T'$, then it is possible to define a G-proof of $\Gamma \vdash \Delta$, say $T(\boldsymbol{\pi})$, in such a way that $T^{-1}\big(T(\boldsymbol{\pi})\big) = \boldsymbol{\pi}$.

The construction is functorial in the following sense: assume that the diagram

$$
\begin{tikzcd}
F \arrow[r, "T"] \arrow[d, "U"'] & G \arrow[d, "V"] \\
F_1 \arrow[r, "T_1"'] & G_1
\end{tikzcd}
$$

is commutative ($U = \underline{f} + \mathbf{E}^1_{\mathsf{Id}} + \mathbf{E}^1_1 + U'$, $V = \underline{f} + \mathbf{E}^1_{\mathsf{Id}} + \mathbf{E}^1_1 + V'$...) and $V \wedge T_1 = VT = T_1 U$, then

$$V^{-1}\Big(T_1(\boldsymbol{\pi}_1)\Big) = T\Big(U^{-1}(\boldsymbol{\pi}_1)\Big) \ .$$

<u>Proof</u>. One easily checks that all negative occurrences of some $I\boldsymbol{\Phi}^\lambda$ in $\boldsymbol{\pi}_\alpha$ correspond to $\lambda < a + \alpha$. (Easy subformula argument: the premises of the rules $(\bar{l})$ and $(lI\mu)$ for $\mu < a+\alpha$ introduce negative $I\boldsymbol{\Phi}^\lambda$'s with $\lambda < a+\alpha$!!!) We define $\boldsymbol{\pi}' = T(\boldsymbol{\pi})$ as follows: in$\boldsymbol{\pi}_\alpha$, replace everywhere all $I\boldsymbol{\Phi}^\lambda$'s by $I\boldsymbol{\Phi}^{\lambda'}$, with $\lambda' = T(\alpha)(\lambda)$; we rename the rules accordingly: $(rI\lambda)$ becomes $\Big(rIT(\alpha)(\lambda)\Big)$, whereas $(lI\lambda)$ is unchanged, since, as remarked, $\lambda < a + \alpha$, hence $T(\alpha)(\lambda) = \lambda$. The fact that $\boldsymbol{\pi}'_\alpha$ is a $a + \alpha$, $G(\alpha)$-proof is trivial; but it is essential to remark that it works because the negative $I\boldsymbol{\Phi}^\lambda$'s are unchanged! The functorial property is immediate: assume that $U^{-1}(\boldsymbol{\pi}_1)$ is defined, then $T\Big(U^{-1}(\boldsymbol{\pi}_1)\Big)$ is so to speak "the image of $\boldsymbol{\pi}_1$ under the partial morphism TU^{-1}"; the hypothesis $V \wedge T_1 = VT = T_1 U$ is another way of expressing that $TU^{-1} = V^{-1}T$.... $\square$

We use Lemma 11.4.6 as follows: write $\tilde{F}' = F + F_1$, $\tilde{F}'' = F + F_2$; then we define $\tilde{F} = F + F_1 + F_2$ and we consider the natural transformations $T' \in I^1(\tilde{F}', \tilde{F})$, $T'' \in I^1(\tilde{F}'', \tilde{F})$ defined by: $T' = \mathbf{E}^1_{\tilde{F}'} + \mathbf{E}^1_{\underline{0}F_2}$, $T'' = \mathbf{E}^1_F + \mathbf{E}^1_{\underline{0}F_1} + \mathbf{E}^1_{F_2}$. Then we can obviously consider:

$$
\tilde{\boldsymbol{\pi}} \quad \left\{ \begin{array}{ccc}
\vdots \; T(\tilde{\boldsymbol{\pi}}') & \vdots \; T''(\tilde{\boldsymbol{\pi}}'') & \\
\tilde{\Gamma}' \vdash \tilde{\Delta}' & \tilde{\Gamma}'' \vdash \tilde{\Delta}'' & \\
 & & R \\
\multicolumn{2}{c}{\tilde{\Gamma} \vdash \tilde{\Delta}} &
\end{array} \right.
$$

The functorial property is left to the reader. (If

$$(\boldsymbol{\pi}', F', \Gamma' \vdash \Delta') \ \xrightarrow{T'_2} \ (\boldsymbol{\pi}'_2, G, \Gamma'_2 \vdash \Delta'_2)$$

$$(\boldsymbol{\pi}'', F, \Gamma'' \vdash \Delta'') \ \xrightarrow{T''_2} \ (\boldsymbol{\pi}''_2, G, \Gamma''_2 \vdash \Delta''_2)$$

$$(\boldsymbol{\pi}, F, \Gamma \vdash \Delta) \xrightarrow{T} (\boldsymbol{\pi}_2, G, \Gamma_2 \vdash \Delta_2)$$

with $T_2' = T_2'' = T$, and if $\tilde{T}_2'' = T + T_2''$, let $\tilde{T} = \tilde{T}_2' + T_2''$.)

<u>subcase b</u> (R) is a cut of kind III (*CRUCIAL CASE*)

$$\boldsymbol{\pi} \quad \left\{ \begin{array}{ccc} \vdots\ \boldsymbol{\pi}' & \vdots\ \boldsymbol{\pi}'' & \\ \Gamma' \vdash \bar{\boldsymbol{\Phi}}(\bar{n}), \Delta' & \Gamma'' \vdash \bar{\boldsymbol{\Phi}}(\bar{n}) \vdash \Delta'' & \\ & & \mathsf{CUT} \\ \multicolumn{2}{c}{\Gamma, \Gamma' \vdash \Delta, \Delta'} & \end{array} \right.$$

The induction hypothesis applied to $(\boldsymbol{\pi}', F', \Gamma' \vdash \bar{\boldsymbol{\Phi}}(\bar{n}), \Delta')$ and $(\boldsymbol{\pi}'', F'', \Gamma'', \bar{\boldsymbol{\Phi}}(\bar{n}) \vdash \Delta'')$ yields $(\tilde{\boldsymbol{\pi}}', \tilde{F}', \tilde{\Gamma}' \vdash \bar{\boldsymbol{\Phi}}(\bar{n}), \tilde{\Delta}')$ and $(\tilde{\boldsymbol{\pi}}'', \tilde{F}'', \tilde{\Gamma}'', \bar{\boldsymbol{\Phi}}(\bar{n}) \vdash \tilde{\Delta}'')$. Write $\tilde{F}' = \underline{a} + F_1'$, and let $\tilde{F} = \tilde{F}'' \circ F_1'$. Observe that, since $\tilde{F}''$ is of the form $\underline{a} + \mathsf{Id} + \underline{1} + ...$, $\tilde{F}$ is of the form $\underline{a} + F_1' + \underline{1} + ...$, i.e. $\tilde{F}' + \underline{1} + \mathsf{something}$, say $\tilde{F} = \tilde{F}' + \underline{1} + F_2'$.

• We first apply Lemma 11.4.5 to $\tilde{\boldsymbol{\pi}}'$: we obtain a $\tilde{F}' + \underline{1}$-proof $\tilde{\boldsymbol{\pi}}_1'$ of $\tilde{\Gamma}' \vdash I\boldsymbol{\Phi}^t(\bar{n}), \tilde{\Delta}'$, with $t(\alpha) = \tilde{F}'(\alpha)$ for all α. Next we apply Lemma 11.4.6 to $\tilde{\boldsymbol{\pi}}_1'$, with $T = \mathbf{E}^1_{\tilde{F}'+\underline{1}} + \mathbf{E}^1_{0F_2'}$: we obtain therefore a $\tilde{F}$-proof $\boldsymbol{\lambda}'$ of $\tilde{\Gamma}' \vdash I\boldsymbol{\Phi}^t(\bar{n}), \tilde{\Delta}'$.

• Consider the proof $\tilde{\boldsymbol{\pi}}''$. We use here an analogue of Lemma 11.4.5 for negative occurrences of $\bar{\boldsymbol{\Phi}}$.

11.4.7. <u>Lemma</u>.
Given a F-proof $\boldsymbol{\pi}$ of $\Gamma \vdash \Delta$, it is possible to construct a F-proof $\boldsymbol{\pi}_2$ of $\Gamma_2 \vdash \Delta_2$, where $\Gamma_2 \vdash \Delta_2$ is obtained by replacing some negative occurrences of $\bar{\boldsymbol{\Phi}}$ in $\Gamma \vdash \Delta$ by corresponding occurrences of $I\boldsymbol{\Phi}^u$, where u is the F-term $u(\alpha) = a + \alpha$.

Furthermore the construction is functorial in the following sense: if $T = f + \mathbf{E}^1_{\mathsf{Id}} + \mathbf{E}^1_{\underline{1}} + T' \in I^1(G, F)$ and $\Gamma' \vdash \Delta' = T^{-1}(\Gamma \vdash \Delta)$, $\Gamma_2' \vdash \Delta_2' = T^{-1}(\Gamma_2 \vdash \Delta_2)$, if $T^{-1}(\boldsymbol{\pi}) = \boldsymbol{\pi}'$, then $T_1^{-1}(\boldsymbol{\pi}_1) = \boldsymbol{\pi}_1'$.

<u>Proof</u>. Straightforward; we only use the fact that $(\bar{l})$ and $(lIa + \alpha)$ have the same premises... . □

Applying 11.4.7 to $\tilde{\boldsymbol{\pi}}''$, we obtain a $\tilde{F}''$-proof $\tilde{\boldsymbol{\pi}}_2''$ of $\tilde{\Gamma}''$, $I\boldsymbol{\Phi}^u(\bar{n}) \vdash \tilde{\Delta}''$. Our problem is to render u equal to t! Consider the $\tilde{F}'(\alpha)$, $\tilde{F}(\alpha)$-proofs

$(\tilde{\boldsymbol{\pi}}''_2)_{F'_1(\alpha)}$. These proofs can easily be changed into $a+\alpha$, $\tilde{F}(\alpha)$-proofs $\boldsymbol{\lambda}''_\alpha$: simply in each use of $(\bar{l})$ in $(\tilde{\boldsymbol{\pi}}_2)_{F'_1(\alpha)}$, chop all premises of index $\geq a+\alpha$! One easily checks that $(\boldsymbol{\lambda}''_\alpha)$ defines a $\tilde{F}$-proof of $\tilde{\Gamma}''$, $I\boldsymbol{\Phi}^t(\bar{n}) \vdash \tilde{\Delta}''$ (because $u\Big(F'_1(\alpha)\Big) = a + F'_1(\alpha) = \tilde{F}(\alpha) = t(\alpha)$).

• Then we define

$$\boldsymbol{\pi}_1 \quad \left\{ \begin{array}{ccc} \vdots\ \boldsymbol{\lambda}' & \vdots\ \boldsymbol{\lambda}'' & \\ \tilde{\Gamma}' \vdash I\boldsymbol{\Phi}^t(\bar{n}), \tilde{\Delta}' & \tilde{\Gamma}'', I\boldsymbol{\Phi}^t(n) \vdash \tilde{\Delta}'' & \\ & & \mathsf{CUT} \\ \multicolumn{2}{c}{\tilde{\Gamma}', \tilde{\Gamma}'' \vdash \tilde{\Delta}', \tilde{\Delta}''} & \end{array} \right.$$

The cut used here is of kind II and its cut-formula does not involve $\bar{\boldsymbol{\Phi}}$... . Finally $\boldsymbol{\pi}$ is obtained from $\boldsymbol{\pi}_2$ by means of 11.4.3.

The functoriality is trivial, as usual. (If $T = \underline{f} + \mathbf{E}^2_{\mathsf{Id}} + ...$ and $\tilde{T}' = \underline{f} + T'_2$, $\tilde{T}''$ denotes $\tilde{T}$ computed in the "contexts" $\boldsymbol{\pi}'$, $\boldsymbol{\pi}''$, then $\tilde{T}$ computed w.r.t. $\boldsymbol{\pi}$ is $\tilde{T}'' \circ T'_2$.)

4. The last rule (R) of $\boldsymbol{\pi}$ is b-ary for some $b \in 0n$. (This case covers the following rules:

– the ω-rules $(r\forall \boldsymbol{L})$ and $(l\exists \boldsymbol{L})$: $b = \omega$.

– the rules (lIt) when t is a constant F-term (hence $t = b < a$).)

The treatment is completely similar to the subcase 3a assume that

$$\boldsymbol{\pi} \quad \left\{ \begin{array}{cc} \vdots\ \boldsymbol{\pi}' & \\ ...\ \Gamma_i \vdash \Delta_i\ ...\ i < b & \\ & R \\ \Gamma \vdash \Delta & \end{array} \right.$$

then we apply the induction hypothesis to the 3-uples $(\boldsymbol{\pi}^i, F_i, \Gamma_i \vdash \Delta_i)$ with $F_i = F$: we obtain 3-uples $(\tilde{\boldsymbol{\pi}}^i, \tilde{F}_i, \tilde{\Gamma}_i \vdash \tilde{\Delta}_i)$, and let us write $\tilde{F}_i = F + F'_i$; we define $\tilde{F} = F + \sum_{j<b} F'_j$, and we consider the natural transformations:

$$U_i = \mathbf{E}^1_F + \mathbf{E}^1_{\underline{0}\ \sum_{j<i} F'_j} + \mathbf{E}^1_{\underline{0}\ \sum_{i<j<b} F'_j}$$

and we define:

$$\boldsymbol{\pi} \quad \left\{ \begin{array}{c} U_i(\boldsymbol{\pi}^i) \ \vdots \\ \dots\ \tilde{\Gamma}_i \vdash \tilde{\Delta}_i\ \dots\ i < b \\ \hline \tilde{\Gamma} \vdash \tilde{\Delta} \end{array} \right. R$$

The functoriality property is proved as follows:

• If (R) is $(r\forall \boldsymbol{L})$ or $(l\exists \boldsymbol{L})$, consider $G_i = G$, $T_i = T$ $(i < \omega)$; write $\tilde{T}_i = T + T'_i$; then

$$\tilde{T} = T + \sum_{i<\omega} T'_i\ .$$

• If (R) is (lIt), then t is the constant F term which equals b; if $T = \underline{f} + \dots$ then the fact that $T^{-1}(\boldsymbol{\pi})$ is defined implies that $b \in rg(f)$, say $b = f(b')$. Define $g \in I(b', b)$ to be the restriction of f, and G_i $(i < b')$ to be G, $T_i \in I^1(G_i, F_{g(i)})$ to be T. Write $\tilde{T}_i = T + T'_i$; then $\tilde{T} = T + \sum_{i<g} T'_i$.

5. The last rule (R) of $\boldsymbol{\pi}$ is $0n$-ary or more. (This case coverse the following rules:

– $(\bar{l})$ $(0n$-ary).

– (lIt) when t is a non-constant F-term.)

Let us treat these two subcases

<u>subcase a</u> $(R) = (\bar{l})$

$$\boldsymbol{\pi} \quad \left\{ \begin{array}{c} \boldsymbol{\pi}^\lambda \ \vdots \\ \dots\ \Gamma, \boldsymbol{\Phi}(I\boldsymbol{\Phi}^{\underline{\lambda}}, \bar{n}) \vdash \Delta\ \dots \text{ all } \lambda \in 0n \\ \hline \Gamma, \bar{\boldsymbol{\Phi}}(\bar{n}) \vdash \Delta \end{array} \right. \bar{l}$$

(Here we denote by $\underline{\lambda}$ the constant term equal to λ.) $\boldsymbol{\pi}'_\lambda$ is a F_λ-proof: if $a + \mu = \mathsf{sup}\,(a, \lambda + 1)$, then $F_\lambda = F \circ (\underline{\mu} + \mathsf{Id})$ (with the notations of 10.2.3 $\mu = c_\lambda$). We apply the induction hypothesis to the 3-uples $(\boldsymbol{\pi}^\lambda, F_\lambda, \Gamma.\boldsymbol{\Phi}(I\boldsymbol{\Phi}^\lambda, \bar{n}) \vdash \Delta)$, and we obtain $(\tilde{\boldsymbol{\pi}}^\lambda, \tilde{F}_\lambda, \tilde{\Gamma}, \boldsymbol{\Phi}(I\boldsymbol{\Phi}^\lambda, \bar{n}) \vdash \tilde{\Delta})$. Write $\tilde{F}_\lambda = F_\lambda + F'_\lambda$. We shall define G^y as follows:

$$G^y = F + \sum_{i<a} F'_i + \sum_{i<y} F'_{a+i}$$

and if $f \in I(y, y')$, we shall define $G^f \in I^1(G^y, G^{y'})$ by: $G^f = \mathbf{E}^1_F + \sum_{i<a} \mathbf{E}^1_{F'_i} + \sum_{i<f} T'_{a+i}$ where the transformations $T'_{a+i} \in I^1(F'_{a+i}, F'_{a+f(i)})$ are defined by: define natural transformations $T_\lambda \in I^1(F_\lambda, F_{(a+f)(\lambda)})$ as follows: $T_\lambda = \mathbf{E}_{F_\lambda}$ when $\lambda < a$, $T_{a+\mu} = \mathbf{E}^1_F \circ (\underline{\varphi} - \mu + \mathbf{E}^1_{\mathsf{Id}})$, where $\varphi_\mu \in I(\mu+1, f(\mu)+1)$ is defined by: $\varphi_\mu(x) = f(x)$ for all $x < \mu$.

Then, from the general results of 10.2.7, we obtain: $T^{-1}_{a+\mu}(\boldsymbol{\pi}^{a+f(\mu)}) = \boldsymbol{\pi}^{a+\mu}$ for all $\mu < y$. Write $\tilde{T}_{a+\mu} = T + T'_{a+\mu}$ ... this defines the T'_{a+i}'s. Now observe that G is

- a functor from **ON** to **DIL** (this results essentially from the induction hypothesis (iii) (a,b) applied to the $\tilde{T}_\lambda$'s).
- a functor preserving direct limits and pull-backs: (this results essentially from the induction hypothesis (iii) (c,d)...).
- a functor preserving $\mathbf{E}$: $G^{\mathbf{E}_{yy'}} = \mathbf{E}_{G^y G^{y'}}$.

Hence the functor G, viewed as a functor from $\mathbf{ON}^2$ to $\mathbf{ON}$ is something like a bilator; more precisely, either G does not at all depend on y, or G is a bilator. In both cases one can define $\tilde{F} = \mathbf{UN}(G)$ (if G does not depend on y, $\tilde{F} = G$).

The construction of G using sums makes it possible to give an explicit definition of $\mathbf{UN}(G)$:

$$\Big(\mathbf{UN}(G)\Big)(x) = F(x) + \sum_{i<a} F'_i(x) + \sum_{i<x} F'_{a+i}\Big(x - (i+1)\Big)$$

(similar formula for functions); as a corollary

$$\begin{aligned}
&\mathbf{UN}(G) \circ (\underline{b+1} + \mathsf{Id}) \ = \\
&F \circ (\underline{b+1} + \mathsf{Id}) + \sum_{i<a} F'_i \circ (\underline{b+1} + \mathsf{Id}) + \\
&\sum_{i<b+1} F'_{a+i} \circ \Big(\underline{(b+1)(i+1)} + \mathsf{Id}\Big) + ...
\end{aligned}$$

in other terms:

$$\mathbf{UN}(G) \circ (\underline{b+1} + \mathsf{Id}) = F + ... + F'_{a+b} + ...$$

and from this one can easily construct a natural transformation U from $\tilde{F}_{a+b}$ to $\mathbf{UN}(G) \circ (\underline{b+1} + \mathsf{Id})$, since $\tilde{F}_{a+b} = F^{a+b} + F'_{a+b}$.

In a similar way, since $\mathbf{UN}(G) = F + ... + F'_\lambda + ...$, when $\lambda < a$, it is possible to define $U_\lambda \in I^1\Big(\tilde{F}_\lambda, \mathbf{UN}(G)\Big)$. Hence we have obtained natural transformations:

$$U_\lambda \in I^1\Big(\tilde{F}_\lambda, \tilde{F} \circ (\underline{c}_\lambda + \mathsf{Id})\Big)$$

(recall that $\lambda + c_\lambda = \mathsf{sup}(a, \lambda + 1)$.

We apply now Lemma 11.4.6, and we obtain proofs $\boldsymbol{\lambda}^\lambda = U_\lambda(\tilde{\boldsymbol{\pi}}^\lambda)$; $\boldsymbol{\lambda}^\lambda$ is a $\tilde{F} \circ (\underline{c}_\lambda + \mathsf{Id})$-proof, and, when $f \in I(y, y')$ we have

$$(\underline{f} + \mathbf{E}_1 + \mathbf{E}^1_{\mathsf{Id}})^{-1}(\boldsymbol{\lambda}^{a+y'}) = \boldsymbol{\lambda}^{a+y} \ ;$$

this proves that

$$\left\{\begin{array}{cc} \begin{array}{c} \vdots\ \boldsymbol{\lambda}^\lambda \\ ...\ \tilde{\Gamma}, \mathbf{\Phi}(I\mathbf{\Phi}^\lambda, \bar{n}) \vdash \tilde{\Delta}\ ...\ \mathsf{all}\ \lambda \in 0n \\ \hline \tilde{\Gamma}, \bar{\mathbf{\Phi}}(\bar{n}) \vdash \tilde{\Delta} \end{array} & \bar{l} \end{array}\right.$$

defines a $\tilde{F}$-proof, which is by definition $\tilde{\boldsymbol{\pi}}$.

If $V \in I^1(F_1, F)$, let us look at the definition of $\tilde{V}$; write $V = \underline{f} + \mathbf{E}^1_{\mathsf{Id}+\underline{1}} + V'$, $f \in I(a_1, a)$; then we consider $\tilde{V}_\lambda$, computed from

$$(\boldsymbol{\pi}^\lambda_1, F_{1,\lambda}, \Gamma_1, \mathbf{\Phi}(I\mathbf{\Phi}^\lambda, \bar{n}) \vdash \Delta_1) \ \xrightarrow{V_\lambda}\ (\boldsymbol{\pi}_{\lambda'}, F_{\lambda'}, \Gamma, \mathbf{\Phi}(I\mathbf{\Phi}^{\lambda'}, \bar{n}) \vdash \Delta)$$

with $\lambda' = f(\lambda)$ if $\lambda < a_1$, $\lambda' = a + \mu$ if $\lambda = a_1 + \mu$, $\boldsymbol{\pi}^\lambda_1 = V^{-1}_\lambda(\boldsymbol{\pi}^\lambda)$, $V_\lambda = V = \underline{f} + \mathbf{E}^1_{\mathsf{Id}+\underline{1}} + V'$, $F_{1,\lambda} = F_1$, $\Gamma_1 \vdash \Delta_1 = V^{-1}(\Gamma \vdash \Delta)$. We can write $\tilde{V}_\lambda = V + V'_\lambda$, and it is therefore possible to define $W^y \in I^1(G^y_1, G^y)$ by:

$$W^y = V + \sum_{i < f + \mathbf{E}_y} V'_i \ .$$

Then W defines a natural transformation from G_1 to G, and it suffices to define $\tilde{V} = \mathbf{UN}(W)$.

<u>subcase b</u> $(R) = (lIt)$ with t non constant F-term; since $t(0n) \geq 0n$, it will be possible to extract from $\boldsymbol{\pi}$ proofs $\boldsymbol{\pi}_\lambda$ ($\lambda \in 0n$), in such a way that:

$$\boldsymbol{\pi}' \quad \left\{ \begin{array}{c} \vdots\ \boldsymbol{\pi}_\lambda \\ \dots\ \Gamma, \boldsymbol{\Phi}(I\boldsymbol{\Phi}^\lambda, \bar{n}) \vdash \Delta\ \dots\ \lambda < 0n \\ \hline \Gamma, \bar{\boldsymbol{\Phi}}(\bar{n}) \vdash \Delta \end{array} \bar{l} \right.$$

is a F-proof. (Simply we omit all premises of (lIt) whose indices are $\geq 0n$...). Then we apply subcase a to $(\boldsymbol{\pi}', F', \Gamma, \bar{\boldsymbol{\Phi}}(\bar{n}) \vdash \Delta)$, and we obtain a 3-uple $(\tilde{\boldsymbol{\pi}}', \tilde{F}', \tilde{\Gamma}, \bar{\boldsymbol{\Phi}}(\bar{n}) \vdash \tilde{\Delta})$; but since $I\widetilde{\boldsymbol{\Phi}^t}(\bar{n}) = \bar{\boldsymbol{\Phi}}(\bar{n})$, one can define:

$$\tilde{\boldsymbol{\pi}} = \tilde{\boldsymbol{\pi}}' \ , \qquad \tilde{F} = \tilde{F}' \ . \qquad \Box$$

End of the proof of 11.4.1. Apply 11.4.3, then 11.4.4. $\Box$

11.4.8. Remark.

A traditional technique of proof-theory is that of *ordinal assignment*, which goes back to the work of Gentzen; roughly speaking, we are given a proof in ordinary logic (more usually: ω-logic), and we want to prove a syntactic result, typically a cut-elimination theorem. Then we "assign" ordinals to each node of the original proof-tree, i.e. we define a function: $T \to 0n$ (usually increasing); the ordinal assignment is used to "measure" the sizes of some significant data in the proof and to construct new proofs, together with new (in general bigger) ordinal assignments.

(i) The advantage of the method is that we usually get simple proofs of syntactical results by transfinite induction on the ordinals involved in such assignments, e.g. ε_0, η_0.... This method is well-adapted for standard (abstract) metamathematical aims such as finding "the" ordinal of a theory, etc....

(ii) The obvious limitation of the method is its technicity; more precisely, besides the standard applications found in (i), it is hard to say what these "assignments" mean; strictly speaking, they hardly mean something.... In fact, they often reflect something close to the height of the trees involved, which is not so bad, but which is perhaps too much linked with the syntax of proofs, and for this reason, not a very flexible notion.

Our treatment of cut-elimination for inductive definitions implicitly contains the use of ordinal assignments: the "ordinal" assignment (it would be more correct to speak of a dilator assignment!) corresponds tot he values t occurring in the positive occurrences of the $I\boldsymbol{\Phi}^t$'s... . Let us make it more precise: it is possible to consider a variant of our systems of inductive logic where the rule $(\bar{r})$ is replaced by:

$$\frac{\Gamma \vdash \boldsymbol{\Phi}(\bar{\boldsymbol{\Phi}}, \bar{n}), \Delta}{\Gamma \vdash \bar{\boldsymbol{\Phi}}(\bar{n}), \Delta}\ \bar{r}'$$

Then one easily checks (see 11.4.9 (i)) that it is no longer necessary to use $I\boldsymbol{\Phi}$ positively! But the cut-elimination procedure cannot any longer be carried out, unless we *assign* ordinals t (or F-terms t for some F) to some occurrences of $\bar{\boldsymbol{\Phi}}$ in the proof, which (implicitly) become occurrences of $I\boldsymbol{\Phi}^t$ (see 11.4.9 (ii)), and the cut-elimination can therefore be performed ont he model of what we did. The advantage of the use of $I\boldsymbol{\Phi}^t$ (compared to: occurrences of $\bar{\boldsymbol{\Phi}}$ to which t is assigned) is that we have a clear semantic, syntactic world in which the $I\boldsymbol{\Phi}^t$'s can live and make sense (and have applications), whereas the roughly equivalent $\bar{\boldsymbol{\Phi}}$, assigned with t, are purely technical constructions. (Here again, the assignments are essentially heights of trees... .)

11.4.9. <u>Exercise</u>.
We consider a variant of inductive logic where:

– only negative occurrences of $I\boldsymbol{\Phi}$ are permitted.

– the rule $(\bar{r})$ is replaced by $(\bar{r}')$ (11.4.8).

(i) Show how to replace any F-proof of $\Gamma \vdash \Delta$ by a proof in this variant of $\hat{\Gamma} \vdash \hat{\Delta}$, where $\hat{\Gamma} \vdash \hat{\Delta}$ is obtained by replacing all positive occurrences of some $I\boldsymbol{\Phi}^t$ by corresponding occurrences of $\bar{\boldsymbol{\Phi}}$.

(ii) Conversely, given a reasonably cut-free proof of $\Gamma \vdash \Delta$ in this variant, construct an F, together with a F-proof of $\Gamma \vdash \Delta$ in inductive logic. (*Hint. Assume that the given proof is* $\boldsymbol{\pi} = (\boldsymbol{\pi}_\alpha)$, *and let* $F = \mathsf{Id} +$

$\underline{1}+\mathbf{LIN}(\boldsymbol{\pi})$*; in* $\boldsymbol{\pi}_\alpha$*, if* $\bar{\mathbf{\Phi}}$ *occurs positively in* $\Gamma' \vdash \Delta'$*, then follow the "descendants" of this occurrence, until such a descendant is "used" in the premise of some* $(\bar{r}')$*; let* $\Gamma'' \vdash \Delta''$ *be this premise, and* t *be the corresponding point in* $\mathsf{Lin}(\boldsymbol{\pi}_\alpha)$*: the given occurrence of* $\bar{\mathbf{\Phi}}$ *is replaced by* $I\mathbf{\Phi}^{\alpha+1+t}$*; if no descendant of the given occurrence is "used" in that way, then do nothing....*)

(iii) (Buchholz, 1982, [101].) Give a direct proof of cut-elimination by semantic methods.
(*Hint. Show that the variant considered here is complete w.r.t. truth in models* $\boldsymbol{m}\,[\mathbf{\Phi}]$*; the cuts must be of trivial nature; then restore a* F*-proof by means of (ii) above.*)

We now investigate into some of the variants of inductive logic we have already introduced, together with a few other ones; all the results are presented as exercises... .

11.4.10. <u>Exercise</u> (inductive logic without ω-rule).
(We consider $\mathbf{ID}_1$ as a twice iterated inductive definition, the first step being the inductive definition of the integers... .) We introduce $I\boldsymbol{N}^\lambda$ ($\lambda < F(\alpha)$), and $\bar{\boldsymbol{N}}$ as we did for $\mathbf{\Phi}$ in general; then given a positive operator $\mathbf{\Phi}$, we replace it by $\mathbf{\Phi}'$: simply all number theoretic quantifiers $\forall x$, $\exists x$ are restricted to $\bar{\boldsymbol{N}}$; then we introduce $I\mathbf{\Phi}'^\lambda$, $\bar{\mathbf{\Phi}}'$ as usual. The rules for the calculus are exactly $(rI\lambda)$, $(lI\lambda)$, $(\bar{r})$, $(\bar{l})$, written for $\mathbf{\Phi}$ and $\boldsymbol{N}$.

(i) Prove a completeness theorem for this calculus.

(ii) Prove a cut-elimination theorem; show that, if we start with weakly finite F, then the resulting $\tilde{F}$ is weakly finite.

11.4.11. <u>Exercise</u> (cut-elimination for **IND**).

(i) Express a concept of F-proof corresponding to the system **IND**.

(ii) Prove a cut-elimination theorem for the system introduced in (i). (*Hint. First establish the analogue of 11.4.3 for this system, which is not problematic. In order to find the analogue of 11.4.4, we must associate a degree to the formulas* $\bar{n}\,\varepsilon\,\mathsf{Acc}_T$ *(*$= \bar{\boldsymbol{\Phi}}_T(\bar{n})$*); this degree is defined to be:* $\mathsf{sup}\,\{p+1\,;\,p$ *varying over the degrees of subformulas* $t\,\varepsilon\,\mathsf{Acc}_U$ *of the term* T... *. Then we show that, given a proof* $\boldsymbol{\pi}$ *without cuts of kind II and of degree* $d \leq \omega$*, we can find another proof* $\boldsymbol{\pi}'$ *without cuts of kind III or II. This is essentially the iteration of 11.4.4 and 11.4.3....*)

(iii) Adapt (i) and (ii) to the context of systems without ω-logic, as in 11.4.10 (ii).

11.4.12. <u>Exercise</u> (cut-elimination for $\mathbf{ID}_\nu$) (Girard, [102]).
R is a fixed prim. rec. well-ordering of a subset $|R|$ of $I\!N$. We use μ, ν for elements of $|R|$.

A **fair operator** (this formalism is essentially due to Feferman [99]) is a fomula $Fr(X, Y, \mu, x)$ depending on the only variables X, Y, μ, x, X and Y being additional predicate variables, such that:

– X occurs only positively, and X is monadic.

– Y is a binary predicate.

The formalism of $\mathbf{ID}_R$ ($= \mathbf{ID}_R(Fr)$) is:

(1) $\forall\nu\ \forall x\,\Big(Fr(\lambda x\, P_\nu(x), \lambda\mu x(P_\mu(x) \wedge \mu \prec \nu), \nu, x) \to P_\nu(x)\Big)$.

(2) $\Big(\forall\nu\ \forall x\,(Fr\,\lambda x\, B_\nu(x), \lambda\mu x(P_\mu(x) \wedge \mu \prec \nu), \nu, x) \to B_\nu(x)\Big) \ \to$
$\forall\nu\ \forall x\,\Big(P_\nu(x) \to B_\nu(x)\Big)$.

(Explanations. $P_\nu(x)$ is a new atomic formula ($=\ P(\nu, x)$); the quantifiers $\forall\nu$ are short for: $\forall\nu \in |R|$, $B_\nu(x) = B(\nu, x)$ is an arbitrary formula, $\lambda x\, B_\nu(x)$ is the set $\{x\,;\, B_\nu(x)\}$, whereas $\lambda\mu x(...)$ is the set of all (μ, x) s.t.

(...).)

(Remark. This is exactly the system corresponding to the idea of iterating an inductive definition a transfinite number of times... . The standard way of speaking of these systems is to call them $\mathbf{ID}_\nu$; $\mathbf{ID}_\nu$ is a system equivalent to $\Pi^1_1 - CA$.)

Assume that $\alpha < \beta < \gamma$ are ordinals; we introduce a system $\mathbf{ID}_R(\alpha, \beta, \gamma)$, as follows:

- Let $\mathsf{Acc}(X, x)$ be the following positive operator:

$$\mathsf{Acc}(X, x) \leftrightarrow x \in |R| \wedge \forall y \left(y \prec x \rightarrow X(y) \right)$$

  then we introduce, for all $\lambda \leq \alpha$, the unary predicates $I\,\mathsf{Acc}^\lambda$ (but no $\overline{\mathsf{Acc}}\,!$), together with the rules $(rI\lambda)$, $(lI\lambda)$ corresponding to these predicates.

- If $\lambda \leq \alpha$, we introduce a new binary predicate letter $Q^\lambda(\nu, x)$ $(= Q^\lambda_\nu(x))$, and we introduce the positive operators Fr^λ_ν by:

$$Fr^\lambda_\nu(X, x) \leftrightarrow \mathsf{Acc}(I\,\mathsf{Acc}^\lambda, \nu) \wedge Fr(X, \lambda\mu y(Q^\lambda_\mu(y) \wedge \mu \prec \nu), \nu, x)$$

  together with the following rules:

$$\frac{\Gamma \vdash Fr^\lambda_\nu(I(Fr^\lambda_\nu)^\vartheta, \bar{n}), \Delta}{\Gamma \vdash I(Fr^\lambda_\nu)^{\vartheta'}(\bar{n}), \Delta}\; rI_\lambda\vartheta'$$

  when: $\lambda < \alpha$, $\vartheta < \vartheta' < \gamma$.

$$\frac{\ldots\ \Gamma, Fr^\lambda_\nu(I(Fr^\lambda_\nu)^\vartheta, \bar{n}) \vdash \Delta\ \ldots}{\Gamma, I(Fr^\lambda_\nu)^{\vartheta'}(\bar{n}) \vdash \Delta}\; lI_\lambda\vartheta'$$

  with: $\lambda < \alpha$, ϑ varying over all points $< \vartheta' < \gamma$.

$$\frac{\Gamma \vdash Fr^{\lambda'}_\nu(I(Fr^{\lambda'}_\nu)^\vartheta, \bar{n}), \Delta}{\Gamma \vdash Q^\lambda_\nu(\bar{n}), \Delta}\; \bar{r}_\lambda$$

with: $\lambda' < \lambda \leq \alpha$, $\vartheta < \gamma$.

$$\frac{\ldots\ \Gamma, Fr_\nu^{\lambda'}(I(Fr_\nu^{\lambda'})^\vartheta, \bar{n}) \vdash \Delta\ \ldots}{\Gamma, Q_\nu^\lambda(\bar{n}) \vdash \Delta}\ \bar{l}_\lambda$$

with: λ' varying over all points $< \lambda \leq \alpha$, ϑ varying over all points $< \beta$.

When F is a dilator of the form $\underline{a} + \mathsf{Id} + \underline{1} + F'$ and $\alpha < a$, then we define the theory $\mathbf{ID}_R(\alpha, F)$; the proofs in $\mathbf{ID}_R(\alpha, F)$ are families $(\boldsymbol{\pi}_x)_{x \in 0n}$, where $\boldsymbol{\pi}_x$ is a proof in $\mathsf{ID}_R\big(\alpha, a + x, F(x)\big)$ for all x... .

(i) Prove the following completeness theorem: if $\Gamma \vdash \Delta$ is true (in the standard interpretation), then $\Gamma, \mathsf{Acc}(R) \vdash \Delta$ is $\underline{\alpha + 1} + \mathsf{Id} + \underline{1}$-provable for all $\alpha \in 0n$ (in particular, with $\alpha = \|R\|$; observe that for this value $\mathsf{Acc}(R)$ is provable...), with: $\mathsf{Acc}(R)$: $\forall \nu \in |R|\, I\, \mathsf{Acc}^\alpha(\nu)$; furthermore, if $\boldsymbol{\pi}^\alpha$ is the proof constructed, and $f \in I(\alpha, \beta)$, we have ${}^T(\boldsymbol{\pi}^\beta) = \alpha$, with $T = \underline{f} + \mathbf{E}_1 + \mathbf{E}^1_{\mathsf{Id}} + \mathbf{E}^1_{\underline{1}}$.

(ii) Prove a cut-elimination theorem for the $\mathsf{ID}_R(\alpha, F)$'s (F changes, of course).

(iii) Adapt the formalism as to eliminate the ω-rule.

11.4.13. <u>Exercise</u> (Girard-Masseron [103]).
Our goal is to study *monotonic* inductive definitions; we start with an operator $\mathbf{\Phi}(X, x)$ with the property that:

$$\begin{aligned}&\boldsymbol{T} + \text{Induction on formulas involving}\\ &X, Y \vdash X \subset Y \to \mathbf{\Phi}(X) \subset \mathbf{\Phi}(Y)\ .\end{aligned}$$

We replace the rules $(rI\lambda)$ and $(lI\lambda)$ by:

$$\frac{\Gamma, I\mathbf{\Phi}^{\lambda'} \subset C \vdash \mathbf{\Phi}(X, \bar{n}), \Delta}{\Gamma \vdash I\mathbf{\Phi}^\lambda(\bar{n}), \Delta}\ rI_m\lambda$$

(X second order variable not occurring in $\Gamma \vdash \Delta$) and

$$\frac{\ldots\ \Gamma, X \subset I\mathbf{\Phi}^{\lambda'}, \mathbf{\Phi}(X,\bar{n}) \vdash \Delta\ \ldots\ \text{all } \lambda' < \lambda}{\Gamma, I\mathbf{\Phi}^{\lambda}(\bar{n}) \vdash \Delta}\ lI_m\lambda$$

(X not occurring in $\Gamma \vdash \Delta$).

The reader will explicitly write the corresponding asymmetric rules ($\bar{r}_m$) and ($\bar{l}_m$).

(i) Prove the axioms $I\mathbf{\Phi}^{\lambda}(\bar{n}) \vdash I\mathbf{\Phi}^{\lambda}(\bar{n})$ from the other axioms and rules. (Hint. Use proofs of $X \subset Y$, $\mathbf{\Phi}(X,\bar{m}) \vdash \mathbf{\Phi}(Y,\bar{m})$.)

(ii) Prove a completeness theorem corresponding to this calculus.

(iii) Prove the analogue of 11.4.3 for this calculus.

(iv) Prove the analogue of 11.4.4 for this calculus.

11.5. Ordinal bounds

Our goal here is to compute "the" ordinals of theories of inductive definitions, and of the theory $\Pi^1_1 - CA$.[(*)]

The first of these values was obtained by Howard ([87]), and in general, the equality $|\mathbf{ID}_n| = \eta_{n-1}$ was established by Pohlers [104]. The transfinite $\mathbf{ID}_\nu$'s were also analyzed by Pohlers [105], yielding ordinals η_ν (denoted in the system of Buchholz by $\bar{\Theta}(\varepsilon_{\Omega_\nu+1}, 0)$). In particular, the ordinal of $\Pi^1_1 - CA$[(*)], which is the same as the ordinal of $\mathbf{ID}_\omega$ is η_ω. All these results are exposed in detail in [3].

It is of course perfectly obvious that the cut-elimination theorem of Section 11.4 will directly lead to an ordinal analysis of the same theories. The only problem with the new methods coming from Π^1_1-logic is that they are too new: in particular, a certain number of very obivous things have not yet been done. Among these obvious things, I must first mention the problem of ordinal bounds for these cut-elimination results. There is not the slightest doubt as to the answer, but there is here a technical work to do, which requires a good familiarity with Π^1_2-logic... . The work of marie-Christine Ferbus [97], sketched in 10.A.14 partly fulfills this goal, by giving explicit bounds that can be used without essential modifications in 11.4.3; but the closely connected 11.4.4 has not yet been majorized.

However, since, as I said, the answer is not problematic, I shall try to describe it in the main lines:

11.5.1. Majorization of 11.4.4.

The important thing one must majorize in $(\tilde{\boldsymbol{\pi}}, \tilde{F}, \tilde{\Gamma} \vdash \tilde{\Delta})$ is of course $\tilde{F}$; now, if we look at the definition of $\tilde{F}$, we see that $\tilde{F}$ is defined by iterating the operation of *composition* (i.e. the case corresponding to a cut of kind III), and all other operations on $\tilde{F}$ are just here to make it work, i.e. to preserve increasivity, functoriality... . The iteration is done by induction on the structure of the (functorial) predecessors of $\boldsymbol{\pi}$.

It is now clear, on general grounds, that, if we replace $\boldsymbol{\pi}$ by a majorizing

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[1(*)] More precisely $(\Pi^1_1 - CA) + (BI)$, see 11.2.8 (i).

dilator D, then $\tilde{F}$ will still be expressible as the iteration of composition along the predecessors of D, i.e. by something close to $\mathbf{\Lambda}D$. The equation can therefore be written

$$\tilde{F} = \mathbf{\Lambda}_c D$$

where $\mathbf{\Lambda}_c$ is a variant (presumably very minor) of $\mathbf{\Lambda}$. The precise definition will be given by the person which will treat the question, but it will not be essentially different from $\mathbf{\Lambda}$... .

11.5.2. Majorization in the iterated case.
If one looks closely, it will not only be possible to majorize $\tilde{F}$ by $\mathbf{\Lambda}_c D$, but also $\tilde{\boldsymbol{\pi}}$ and why not the result of applying 11.4.3 to $\tilde{\boldsymbol{\pi}}$: we have only perhaps to be a little more liberal in our choice of $\mathbf{\Lambda}_c$... . (For instance 11.4.3 involves, by Ferbus's theorem, a Veblen hierarchy, which is nothing compared to a $\mathbf{\Lambda}$!)

The n-times iterated $\mathbf{ID}_n$ will therefore enjoy a majoration by means of $\mathbf{\Lambda}_c$ iterated n times, $\mathbf{\Lambda}_c^n$. Now, the precise definition of the cut-elimination procedure for $\mathbf{ID}_R$ will suggest an obvious way of "iterating $\mathbf{\Lambda}_c$ ν times" (similar to the product of ν copies of a nice flower), and we shall therefore obtain majorations by $\mathbf{\Lambda}_c^\nu$ for $\mathbf{ID}_\nu$.

11.5.3. Ordinals of theories.

(i) Assume that we work in $\mathbf{ID}_1(\mathbf{PA}, O)$; we want to investigate which $\underline{e} \in O$ are such that

$$\mathbf{ID}_1(\mathbf{PA}, O) \vdash \bar{O}(\bar{e}) \ .$$

We convert our given finite proof of $\bar{O}(\bar{e})$ into a $\mathsf{Id} + \underline{1}$-proof by means of 11.3.3; it is easily checked that this $\mathsf{Id} + \underline{1}$-proof, after elimination of all cuts of kind II, can be majorized by means of a dilator built up from Id, $\underline{1}$, $+$, $-$, exp, hence can be majorized by the dilator $\boldsymbol{\varepsilon}_0 = \mathsf{Id} + (\underline{1} + \mathsf{Id})^{\mathsf{Id}} + (\underline{1} + \mathsf{Id})^{(\underline{1}+\mathsf{Id})^{\mathsf{Id}}} + \ldots$. Now the results of 11.5.1 show that $\tilde{F}$ can be replaced by $\mathbf{\Lambda}_c \boldsymbol{\varepsilon}_0$; in particular we have a cut-free proof of

$$\vdash\ IO^{\mathbf{\Lambda}_c\boldsymbol{\varepsilon}_0(x)}(\bar{e})$$

for all x, so, with $x = 0$: $e \in IO^{\mathbf{\Lambda}_c\boldsymbol{\varepsilon}_0(0)}$, equivalently $\|e\| < \mathbf{\Lambda}_c\boldsymbol{\varepsilon}_0(0)$.

Conversely every ordinal $<\ \mathbf{\Lambda}_c\boldsymbol{\varepsilon}_0(0)$ can be shown to be a well-founded (for instance to be representable by a point in $\bar{O}$). See 12.6 for more details.

(ii) In (i) we "answered" the question of the ordinal of $\mathbf{ID}_1$, and we found $\mathbf{\Lambda}_c\boldsymbol{\varepsilon}_0(0)$, which is obviously the Howard ordinal η_0. The equation

$$\mathbf{\Lambda}_c\boldsymbol{\varepsilon}_0(0) = \eta_0$$

comes from our way of introducing η_0 in 9.A.30

$$|\Lambda L_n(\omega)| = [\eta_0]\,\eta$$

and all reasonable variants of $\mathbf{\Lambda}$ must give the same values at reasonable arguments, i.e.

$$\mathsf{sup}_n\ |\Lambda L_n(\omega)| = \mathsf{sup}_n\ [\eta_0]\,n = \eta_0 = \mathbf{\Lambda}_c\boldsymbol{\varepsilon}_0(0)\ .$$

(iii) In general, the same method will assign the ordinal $\eta_\nu = \mathbf{\Lambda}_c^\nu\boldsymbol{\varepsilon}_0(0)$ to the theories $\mathbf{ID}_\nu$; in particular, η_ω to $\Pi^1_1 - CA$. (Here we implicitly assume that we know how to iterate a transfinite number of times $\mathbf{\Lambda}$ and its variants. This offers no theoretical difficulty, but may be painful in practice... .)

(iv) These ordinals are $\boldsymbol{\lambda}$-ordinals; this means that they can be naturally equipped with a structure of Bachmann collections of type ω, i.e. a structure of elements of Kleene's O, in such a way that the provably total recursive functions of $\mathbf{ID}_\nu$ can be expressed as those recursive functins that are majorized by some $\boldsymbol{\lambda}_e$, for some $e < \eta_\nu$. Now, what happens with the $\boldsymbol{\gamma}$-ordinals? Two ways of answering:

- The way $\mathbf{\Lambda}$ is iterated yields something like

$$\mathbf{\Lambda}_c^{1+\nu} = \mathbf{\Lambda}_c \circ \mathbf{\Lambda}_c^\nu$$

and from the similarity of Λ and $\mathbf{\Lambda}_c$, it will not be hard to prove that:

$$\boldsymbol{\gamma}_{\eta_{1+\nu}} = \boldsymbol{\lambda}_{\eta_\nu} \ .$$

- Another way of doing the same thing is to replace $\mathbf{ID}_\nu$ by a system that does not use the ω-rule; this change essentially amounts to replacing ν by $1+\nu$; now if a formula $\forall x\ \exists y\ R(x,y)$ is provable in $\mathbf{ID}_\nu$, with R quantifier-free, then it is immediate that

  $$\forall x \in \bar{\boldsymbol{N}}\ \exists y \in \bar{\boldsymbol{N}}\ R(x,y)$$

  will be provable in the modified version of $\mathbf{ID}_\nu$. If we take a cut-free proof of this, and apply 11.4.5 and 11.4.7, then

  $$\forall x \in I\boldsymbol{N}^\alpha\ \exists y \in I\boldsymbol{N}^{\tilde{F}(\alpha)}\ R(x,y)$$

  in particular the probably total function

  $$f(x) = \mu z\ R(x,z)$$

  is bounded by $\tilde{F}(x+1)$. In other terms, we obtain

  $$f(n) \leq (\mathbf{\Lambda}_c^{1+\nu} F_p)(n) \qquad \text{for some } p\ ,$$

  with $F_0 = \mathsf{Id}$, $F_{k+1} = F_k + (\underline{1} + \mathsf{Id})^{F_k}$.

  But the values $(\mathbf{\Lambda}_c^{1+\nu} F)(n)$ and $\boldsymbol{\gamma}_{\Lambda^{1+\nu}L}(n)$ are closely related when F is the dilator induced by L... .

11.5.4. Final comment.

The situation of this section is a typical example of what I called a lack of modularity: for instance we have been led to introduce three variants of $\mathbf{\Lambda}$:

- $\mathbf{\Lambda}$ itself.

- Λ on ladders.

- $\mathbf{\Lambda}_c$ for the majoration business.

These variants are close one to another, but not exactly the same.... Hence the question of establishing precise relationships between these variants of the same idea is very painful and not very exciting.... It can be hoped that, in the future, a better framework will enable us to speak about **Λ**, without these initiating details of variants that change according to the context in which we are using it.

(It must, however, be noted that this is perhaps due to the fact that the typical proof-theoretic questions (provably total functions, provable ordinals) naturally lead to these questions of variants (because in these questions, all the provable objects of some kind must be organized along some ratherarbitrary linear principle), whereas the application of these methods outside proof-theory does not suffer from similar limitations...)

11.6. Equivalents of $\Pi^1_1 - CA$

This section is an analogue of 5.4, 6.4, where we listed a certain number of formal equivalents to the axiom $\Sigma^0_1 - CA^*$; one of these equivalents was: $\forall X$ (X well-order $\to 2^X$ well-order), in other terms $\Sigma^0_1; CA^* \leftrightarrow 2^{\mathsf{Id}}$ is a dilator. Here the main equivalence will be: $\Pi^1_1 - CA \leftrightarrow \mathbf{\Lambda}$ maps **DIL** into **BIL**. (This can also be rewritten, replacing **BIl** by **DIL**, as "the preptyx $\mathbf{\Lambda}$ of type $(\mathbf{O} \to \mathbf{O}) \to (\mathbf{O} \to \mathbf{O})$ is a ptyx".) In some sense, $\mathbf{\Lambda}$ is the Π^1_2 analogue of the exponential; this analogy is enhanced by the fact that the precise definition of blam (the iteration of composition) is of the kind "exponential". $\mathbf{\Lambda}$ is therefore the "Π^1_2 exponential".

11.6.1. Theorem.

There is a specific Π^1_1 formula $A(f, g, x, y)$ with the property that: (x, f, g are the only parameters of A)

$$\mathbf{PRA}_2 + \Sigma^0_1 - CA^* \vdash \{(x,y)\,;\, A(f,g,x,y)\} \;=$$

$$h \wedge \mathbf{pil}(f) \wedge \mathbf{WO}(g) \wedge \mathbf{WO}(f\,[h]) \to \mathbf{WO}([\boldsymbol{\lambda} f)\,[g]) \;.$$

($\{(x,y)\,;\, A(f,g,x,y)\} = h$ is the comprehension axiom: $\forall x\ \forall y\ \Big(h(x,y) = 0 \leftrightarrow A(f,g,x,y)\Big)$; $\mathbf{pil}(f)$ is a formula saying that f is (the code of) a predilator (there are many ways of encoding predilators; choose the one that you prefer; my favourite one is the one of Exercise 8.G.10...); when f is a predilator, g a linear order, $f\,[g]$ is (the code of) f applied to g; similarly, $\boldsymbol{\lambda} f$ is (the code of $\mathbf{\Lambda} p$, if f encodes P. (We consider $\mathbf{\Lambda}$ as a functor from **DIL** to **DIL**... . If we want to consider as a functor from **DIL** to **BIL**, then replace A by $A'(f,g,h,x,y)$... .))

Proof. *(sketched) The first thing is to express $\boldsymbol{\lambda}$: this is done by computing $\mathbf{\Lambda}$ on the category $\mathbf{DIL}_{fd}$, and then extending it by direct limits: no doubt that one gets the expression $\mathbf{\Lambda} P$, encoded by a prim. rec. term $\boldsymbol{\lambda}(f)$, where f is a code of P... . Hence if f is a predilator f is a well-defined predilator, and we want to make sure that $\boldsymbol{\lambda} f\,[g]$ is a well-order. Consider now all Σ^0_1 linear orders with f, g as parameters; they can be enumerated by a formula $B(f,g,x,y,z)$: $x \preceq y$ modulo the order of index z; let us abbreviate this into $x \;\preceq^z\; y$. The formula $C(f,g,z)$ is: the order $\preceq^z$ is a well-order.*

Hence the set $\{z\,;\,C(f,g,z)\}$ is nothing but the set of all wll-orders which are Σ^0_1 in g, g; of course Σ^0_1 is a Π^1_1 formula. Here the following formula $A(f,g,x,y)$ defines a Π^1_1 well-order:

$$x \le y \;\leftrightarrow\; C(f,g,(x)_0) \wedge C(f,g(y)_0) \wedge$$
$$\Big((x)_0 \le (y)_0 \vee \Big((x)_0 = (y)_0 \wedge (x)_1 \;\preceq^{(x)_0}\; (y)_1\Big)\Big)\ .$$

This typial Π^1_1-well-order has the order type of the first ordinal not recursive in g, g, say α; we shall use α rather than its encoding in the sequel… . But our hypothesis is that $\{(x,y)\,;\,x\le y\}$ exists, i.e. we can use this order as a two-places function of $\mathbf{PRA}_2$… . Now, the next hypothesis is $\mathbf{WO}(f\,[hg])$; if f encodes the predilator P, this means that $P(\alpha)$ is a well-order. Then we prove that $(\mathbf{\Lambda}P)(\beta,\gamma)$ is a well-order, for all $\beta,\gamma<\alpha$, i.e. for all well-orders of the form $\preceq^z, \preceq^{z'}$ … . The proof works by induction on the ordinal (well-order) $P(\alpha)$ when P varies through predilators recursive in f, g; later on, we shall justify the use of such an induction… .

(i) *If P is of kind $\mathbf{0}$, then $(\mathbf{\Lambda}P)(\beta,\gamma)=\gamma$ trivial.*

(ii) *If P is of kind $\mathbf{1}$, then write $P = Q+\underline{1}$; then $Q(\alpha) < P(\alpha)$; moreover $\beta+\gamma<\alpha$ (trivial). Hence $(\mathbf{\Lambda}P)(\beta,\gamma) = (\mathbf{\Lambda}Q)(\beta+\gamma)$ is a well-order, by the induction hypothesis.*

(iii) *If P is of kind $\boldsymbol{\omega}$, then write $P = \sup\limits_n P_n$; the function $n \rightsquigarrow P_n$ can be encoded by a two-variables function. In that case, it will suffice to show, by*

- *induction on γ, that $(\mathbf{\Lambda}P)(\beta,\gamma)$ is a well-order;*
- $(\mathbf{\Lambda}P)(\beta,0) = \sup\limits_n\, (\mathbf{\Lambda}P_n)(\beta,0)$.
- $(\mathbf{\Lambda}P)(\beta,\gamma+1) = \sup\limits_{n\ge n_0}\, (\mathbf{\Lambda}P_{n_0,n})(\beta,(\mathbf{\Lambda}P)(\beta,\gamma)+1)$, *where $P_n = P_{n_0}+P_{n_0,n}$, and n_0 is a sufficiently great integer, enjoying $(\mathbf{\Lambda}P_{n_0})\Big(\beta, (\mathbf{\Lambda}P)(\beta,\gamma)\Big) = (\mathbf{\Lambda}P)(\beta,\gamma)$.*
- $(\mathbf{\Lambda}P)(\beta,\gamma) = \sup\limits_{\gamma'<\gamma}\, (\mathbf{\Lambda}P)(\beta,\gamma')$ *when γ is limit.*

In fact what we have just written is nothing but the set-theoretic construction of the ordinal $(\mathbf{\Lambda}P)(\beta,\gamma)$; *it can easily (but painfully) be formalized with codes!*

(iv) If P *is of kind* Ω, *then*

$$(\mathbf{\Lambda}P)(\beta,\gamma) = (\mathbf{\Lambda}P^{\gamma})(\beta,0)$$

(this is slightly incorrect, but the difference is so slight, that this does not make any change in a sketched proof...) with $P^{\gamma} = \mathbf{SEP}(P)(\cdot,\gamma)$. *Since we obviously have:* $\omega^{1+\alpha} = \alpha$, *it follows that* $\mathbf{SEP}(P)(\alpha,\alpha) = P(\alpha)$. *Hence* $P^{\gamma}(\alpha) < P(\alpha)$, *and the induction hypothesis yields:* $(\mathbf{\Lambda}P)(\beta,0)$ *well-order.*

It remains to justify the use of transfinite induction: consider the formula which says that the predilator P *(of index* p *relative to* f, g*), has the property that* $(\mathbf{\Lambda}P)(\beta,\gamma) < \alpha$ *for all* $\gamma,\beta < \alpha$. *This can be expressed by an arithmetical formula using the parameter* $h = \{(x,y)\,;\,A(f,g,x,y)\}$, *hence this can be expressed (using* $\Sigma^0_1 - CA^*$*) by* $k(p) = 0$*: Now, assume that* $k(p) \neq 0$ *for some* p, *then:*

(i) p *is not a code of* $\underline{0}$.

(ii) If p *is a code for* $Q + \underline{1}$, *let* $p' =$ *code of* Q.

(iii) If p *is a code for* $\sup_n P_n$, *let* $p_n =$ *code of* P_n, n_0 *minimum s.t.* $k(p_n) \neq 0$; *let* $p' = p_{n_0}$.

(iv) If p *is a code for* P *of kind* Ω, *and* e *is a code for an ordinal* $\gamma < \alpha$, *let* p_e *be the resulting code for* P^{γ}. e *can be identified with an integer s.t.* $A(f,g,e,e)$; *let* $P' = P_{e_0}$, *where* e_0 *is minimum (in the sense of the usual order) s.t.* $A(f,g,e,e)$ *and* $k(p_{e_0}) \neq 0$.

(v) When $k(p) = 0$, *let* $p' = p$.

We have defined a function $(\cdot)'$ *from codes of predilators recursive in* f, g, *to themselves; this function is arithmetically defined (using* k *and* h*), hence by* $\Sigma^0_1 - CA^*$ *it can be shown to exist.*

If we introduce p_0 *being a given point such that* $P_0(\alpha)$ *is a well-order, whereas* $k(p_0) \neq 0$, $l(0) = p_0$; $l(n+1) = \big(l(n)\big)'$, *then we have explicitly constructed a descending sequence of predilators, which is indeed a descending sequence in* $P_0(\alpha)$. *Contradiction. (There is another induction used in step (iii): it is handled in the same way.)* □

11.6.2. Corollary.
The following formula is a theorem of $\mathbf{PRA}_2 + \Sigma^0_1 - CA^*$

$$(\Pi^1_1 - CA) \to \forall f \,\Big(\mathbf{dil}(f) \to \mathbf{dil}(\boldsymbol{\lambda} f)\Big) \;.$$

($\Pi^1_1 - CA$ is a formula of the form: $\forall f\ \exists g\ \forall n\ \Big(g(n) = 0 \leftrightarrow A(f,n)\Big)$, where $A(f,n)$ is Π^1_1-universal w.r.t. Π^1_1 formulas containing f; $\mathbf{dil}(f)$ is the formula which says that f is (the code of) a dilator: $\mathbf{pil}(f) \wedge \forall g\ \Big(\mathbf{WO}(g) \to \mathbf{WO}(f\,[g])\Big)$.)

Proof. *Immediate from 11.6.1.* □

11.6.3. Remark.
A more refined use of 11.6.1 is the following: show that the iteration of $\mathbf{\Lambda}$ corresponds to iteration of Π^1_1-comprehension; for instance, prove that "$\mathbf{\Lambda}$ iterated ν times" (it must be defined precisely before), say $\mathbf{\Lambda}^\nu$ enjoys something similar to 11.6.1, when the formula A is replaced by a formula in $\mathbf{ID}_\nu$, with f and g as parameters.

11.6.4. Theorem (Girard, 1979, [100]).
Let $\mathsf{Cut} - E$ *be the* Π^1_3 *formula which expresses that Theorem 11.4.1 holds for any system of inductive logic; then* $\mathbf{PRA}_2 + \Sigma^0_1 - CA^* \vdash \mathsf{Cut} - E \to (\Pi^1_1 - CA)$.

Proof. *Using* $\Sigma^0_1 - CA^*$, Π^1_1*-comprehension can be reduced to inductive definitions; more precisely, given a* Π^1_1 *formula* $A(g,x)$ *(*g *unique function parameter of* A*), then one can find a positive operator* $\mathbf{\Phi}(g,X,y)$ *such that the existence of* $\{x\,;\,A(g,x)\}$ *is formally equivalent (w.r.t.* $\Sigma^0_1 - CA^*$*) to the existence of a least fixed point for* $\mathbf{\Phi}$ *(see 11.2.7...). Now we assume that* Π^1_1*-comprehension fails for a certain parameter* g_0*; this means that* $\mathbf{\Phi}_0 = \mathbf{\Phi}(g_0,\cdot,\cdot)$ *has no least fixed point.*

We consider the following theory $\boldsymbol{T}$*: the language is* $\boldsymbol{L}^2_{\to+\{g_0\}}$*, the axiomatic is* $\mathbf{PRA}_2$ + *the axioms* $\bar{g}_0(\bar{n}) = \overline{g_0(n)}$*; the hypothesis just made, implies that* $\mathbf{ID}_1(\boldsymbol{T}, \boldsymbol{\Phi}_0)$ *has no* ω*-model of the form* $\boldsymbol{m}\,[\bar{\boldsymbol{\Phi}}_0]$ *(the concepts of (denumerable) model, ordinal ... can be expressed in* $\mathbf{PRA}^2$*), because from the interpretation of* $\boldsymbol{m}(\bar{\boldsymbol{\Phi}})$*, one would easily recover a least fixed point for* $\boldsymbol{\Phi}_0$*. By completeness, there is an inductive proof of the void sequent* $\vdash$ *in* $\mathsf{ID}_1(\boldsymbol{T}, \boldsymbol{\Phi}_0)$*. Here we use a formalized version of the* $\boldsymbol{\beta}$*-completeness theorem; there is not the slightest problem as to the formalization of* $\boldsymbol{\beta}$*-completeness in* $\mathbf{PRA}^2$*.*

Now we apply the cut-elimination theorem 11.4.1, expressed by the formula $\mathsf{Cut} - E$*; from this we obtain a proof of* $\vdash$ *in* $\boldsymbol{T} + \omega$*-logic.*

Now observe that $\boldsymbol{T}$ *has an* ω*-model: this fact is surely provable in* $\mathbf{PRA}^2 + \Sigma^0_1 - CA^*$*; but then* $\vdash$ *cannot be provable in* $\boldsymbol{T} + \omega$*-logic, contradiction.* □

11.6.5. Remark.
The $\boldsymbol{\beta}$*-completeness theorem and the* ω*-completeness are provable in* $\mathbf{PRA}^2$*, whereas the usual completeness theorem is not! The reason is simple: the two generalized theorems state the existence of specific well-founded trees, whereas in the* Σ^0_1 *case, we say more: the tree is not only well-founded, but finite. And we need a principle saying that a tree which is* ≤ 2*-branching and well-founded is finite: König's lemma is such a principle. (However, König's lemma is strictly stronger than completeness for* Σ^0_1*-logic! See 5.4.25.)*

11.6.6. Theorem *(depending on the majorations of 11.5).*
With the notations of 11.6.2 and 11.6.4:

$$\mathbf{PRA}^2 + \Sigma^0_1 - CA^* \vdash \forall f\,\Big(\mathbf{dil}(f) \to \mathbf{dil}(\boldsymbol{\lambda} f)\Big) \to \mathsf{Cut} - E\ .$$

Proof. *In fact the proof depends on a majoration of the cut-elimination procedure by* $\boldsymbol{\Lambda}$*, that has not yet been done; assuming that this has been achieved, then the only non-elementary part in the statement* $\mathsf{Cut}{-}E$ *is that the objects constructed map well-orders on well-founded structures, and if we have obtained a majoration of the cut-elimination for pre-proofs by* *means of* $\mathbf{\Lambda}$*, then all we need is to apply* $\mathbf{\Lambda}$ *to a majoration of a given proof: this yields a majoration of the corresponding cut-free proof. The existence of a majoration for any proof follows from the linearization principle, which is a consequence of* $\Sigma^0_1 - CA^*$.

Once again, I have hesitated to style "conjecture" a result like 11.6.6, which only depends on tedious, difficult, but straightforward work. □

11.6.7. Remark.
Assuming the unproblematic 11.6.6, we obtain the following analogue of 5.4.1:

In $\mathbf{PRA}_2 + \Sigma^0_1 - CA^*$*, the following are equivalent*

(i) $\Pi^1_1 - CA$.

(ii) $\forall f \left(\mathbf{dil}(f) \to \mathbf{dil}(\boldsymbol{\lambda} g)\right)$.

(iii) $\mathsf{Cut} - E$.

Observe that there is a direct proof of (i) $\to$ (iii), hence there is presumably a direct proof of (ii) $\to$ (i).

Compared with the corresponding results of 5.4.1, we are in a poorer situation; in particular, we lack combinatorial equivalents; a generalization of Ramsey's theorem was shown to be equivalent to $\Pi^1_1 - CA$ by Simpson [105].

Let us just mention the following:

11.6.8. Exercise.
Give a direct proof of $(\Pi^1_1 - CA) \to \mathsf{Cut} - E$ in $\mathbf{PRA}_2 + \Sigma^0_1 - CA^*$.

11.6.9. Exercise (Abrusci, Girard, Van der Wiele [126]).
Assume that F is a dilator; then we construct a transfinite sequence $(z_\alpha)_{\alpha \in 0n}$

(i) $z_0 = 0$.

(ii) If $z_\alpha < F(\alpha)$, $z_{\alpha+1} = F(\mathbf{E}_{\alpha\alpha+1})(z_\alpha) + 1$.

(iii) If α is limit and $z_\beta < F(\beta)$ for all $\beta < \alpha$, then $z_\alpha = \sup_{\beta<\alpha} F(\mathbf{E}_{\beta\alpha})(z_\beta)$.

(iv) In the other cases $z_\alpha = F(\alpha)$.

Show that (iv) eventually holds, i.e. $\exists\alpha\ z_\alpha = F(\alpha)$. (The first α for which (iv) holds is expressed by means of $\nu(F, 0)$, where ν is (another!) variant of $\mathbf{\Lambda}$ [126]. Hence it is likely that this result is another equivalent of $\Pi^1_1 - CA$.) Surely $\Pi^1_1 - CA$ implies 11.6.9; on the other hand, the first α for which (iv) holds seems to be bounded by $(\mathbf{\Lambda}F)(0, 0)$.

What about the variant: if $(y_\alpha)_{\alpha\in 0n}$ is such that $y_\alpha < F(\alpha)$ for all α, then there is a cofinal $X \subset 0n$ s.t. $\forall\alpha, \alpha' \in 0n$:

$$\alpha \leq \alpha' \to F(\mathbf{E}_{\alpha\alpha'})(y_\alpha) = y_{\alpha'}\ ?$$

Annex 11.A. A survey of earlier results

11.A.1. Cut-elimination in Σ^0_1-logic

The simplest thing to do with inductive definitions is to write a system of Σ^0_1-logic as follows:

$$\frac{\Gamma, \boldsymbol{\Phi}(\lambda x B(x), y) \vdash B(y), \Delta}{\Gamma, \bar{\boldsymbol{\Phi}}(t) \vdash B(t), \Delta}\ l^- \qquad \frac{\Gamma \vdash \boldsymbol{\Phi}(\bar{\boldsymbol{\Phi}}, t), \Delta}{\Gamma \vdash \bar{\boldsymbol{\Phi}}(t), \Delta}\ r^-$$

$$(y \text{ not free in } \Gamma \vdash \Delta)$$

(r^-) is the closure axiom, whereas (l^-) corresponds to $\boldsymbol{\Phi}$-induction on B. Hence this system is equivalent to $\mathsf{ID}_1(\boldsymbol{T}, \boldsymbol{\Phi})$, especially if we write the induction axioms in the same way (i.e. replacing $\boldsymbol{\Phi}$ by $\boldsymbol{N}$).

This system enjoys a cut-elimination theorem; cuts of the form

$$\boldsymbol{\pi} \left\{ \frac{\dfrac{\Gamma \vdash \boldsymbol{\Phi}(\bar{\boldsymbol{\Phi}}, t), \Delta}{\Gamma \vdash \bar{\boldsymbol{\Phi}}(t), \Delta}\ r^- \qquad \dfrac{\begin{matrix} \vdots\ \boldsymbol{\pi}' \\ \Gamma', \boldsymbol{\Phi}(\lambda x B(x), y) \vdash B(y), \Delta' \end{matrix}}{\Gamma', \bar{\boldsymbol{\Phi}}(t) \vdash B(t), \Delta'}\ l^-}{\Gamma, \Gamma' \vdash \Delta \Delta'}\ \mathsf{CUT} \right.$$

can be replaced as follows: using the proof $\boldsymbol{\pi}'$ of $\Gamma', \bar{\boldsymbol{\Phi}}(t) \vdash B(t), \Delta'$, it is easy to replace the proof $\boldsymbol{\pi}$ by a proof $\boldsymbol{\pi}_1$ of $\Gamma, \Gamma' \vdash \boldsymbol{\Phi}(\lambda x B(x), t), \Delta, \Delta'$, and then we can form the cut

$$\frac{\begin{matrix} \vdots\ \boldsymbol{\pi}_1 \\ \Gamma, \Gamma' \vdash \boldsymbol{\Phi}(\lambda x B(x), t), \Delta, \Delta' \end{matrix} \quad \begin{matrix} \vdots\ \boldsymbol{\pi}'(t) \\ \Gamma', \boldsymbol{\Phi}(\lambda x B(x), t) \vdash B(t), \Delta' \end{matrix}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \mathsf{CUT}$$

This is essentially what dit Martin-Löf in [106] (but in an intuitionistic natural deduction framework).

The obvious difficulty with the proof of cut-elimination, namely that the new cut-degree may be greater, is easily overcome by an adequate use of the *reducibility* method. (A simple way of expressing it could be to translate $\bar{\mathbf{\Phi}}$ by:

$$\bar{\mathbf{\Phi}}(t) \leftrightarrow \forall X\,(\forall z\,(\mathbf{\Phi}(X,z) \rightarrow z\varepsilon X) \rightarrow t\varepsilon X)\ .$$

Then the rule (l^-) is a consequence of $(l\forall\mathbf{s})$ in $\mathbf{PA}_2$....)

The interest of such a result is very limited, because of the absence of any reasonable subformula property. However, the subformula property holds for Σ^0_1 formulas ($\exists x\left(\bar{\boldsymbol{N}}(x) \wedge R(x)\right)$, R quantifier-free) as expected.

11.A.2. Takenti's result.

In his famous paper [98], Takenti presented a cut-elimination argument for a variant of the axiomatization of $(\Pi^1_1 - CA)$; the cut-elimination was only proved for Σ^0_1 formulas, as could be expected, but the method made us of complicated ordinal assignments, by means of *ordinal diagrams*, which are uneven systems of ordinal notations introduced also by Takenti. Jervell has shown that ordinal diagrams are bilators; see [128]. It is fair to say that, although this result was obviously a breakthrough in proof-theory, its extreme technicity (ordinal *assignments*, ordinal *diagrams*) made it a puzzle for most of proof-theorists. It is only some years later that Pohlers and, following him, other people in München, were able to extract something from this... .

11.A.3. Cut-elimination in Π^1_1-logic

We shall only describe the work of Pohlers [104], which is chronologically the first to contain an understandable approach to the "ordinal analysis" of $\Pi^1_1 - CA$, using Π^1_1-logic.

Pohlers uses notations for ordinals $< \varepsilon_{\Omega+1}$ (here Ω is ω_1^{CK}), the first $\alpha > \Omega$ s.t. $\alpha = 2^\alpha$. People familiar with dilators, denotation systems, will not be surprised to hear that all such ordinals can be effectively described, with the use of arbitrary parameters $< \omega_1^{CK} = \Omega$.

For all such α's he introduces a predicate $I\mathbf{\Phi}^\alpha$, together with the axioms:

$$\frac{\ldots\ \Gamma, \mathbf{\Phi}(I\mathbf{\Phi}^{\alpha'}, \bar{n}) \vdash \Delta\ \ldots\ \text{all } \alpha' < \alpha}{\Gamma, I\mathbf{\Phi}^{\alpha}(\bar{n}) \vdash \Delta}\ lI \qquad \frac{\Gamma \vdash \mathbf{\Phi}(I\mathbf{\Phi}^{\alpha'}, \bar{n}), \Delta}{\Gamma \vdash I\mathbf{\Phi}^{\alpha}(\bar{n}), \Delta}\ rI$$

In (rI), we require $\alpha' < \alpha$ or $\alpha = \Omega$; the case $\alpha = \Omega$ is just the way of expressing the closure axiom. That formalism is sufficient to translate $\mathbf{ID}_1(\boldsymbol{T}, \boldsymbol{\Phi})$ (with proofs "recursive enough"); the main problem is cut-elimination, the typical case being:

$$\dfrac{\dfrac{\begin{array}{c}\boldsymbol{\pi}\ \vdots\\ \Gamma \vdash \boldsymbol{\Phi}(\boldsymbol{I\Phi}^{\alpha}, \bar{n}), \Delta\end{array}}{\Gamma \vdash \boldsymbol{I\Phi}^{\Omega}(\bar{n}), \Delta} \quad \dfrac{\begin{array}{c}\vdots\ \boldsymbol{\pi}_{\lambda}\\ \ldots\ \Gamma', \boldsymbol{\Phi}(\boldsymbol{I\Phi}^{\lambda}, \bar{n}) \vdash \Delta'\ \ldots\ \text{all } \lambda < \Omega\end{array}}{\Gamma', \boldsymbol{I\Phi}^{\Omega}(\bar{n}) \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta\Delta'}\ \mathsf{CUT}$$

and here there is a fundamental asymmetry, which comes from the fact that α may be $\alpha \geq \Omega$, whereas the corresponding premises for (lI) are restricted to $\lambda < \Omega$. (Using Π^1_2-logic, we have solved this difficulty, by allowing Ω to vary, and then the Ω of the rule (lI) is changed into α; but in Pohler's method it is possible to increase Ω....) Then it is absolutely necessary to make a strong assumption: *There is no negative occurrence of* $\boldsymbol{I\Phi}$ *in* $\Gamma, \Gamma' \vdash \Delta, \Delta'$. Then, if we assume that $\boldsymbol{\pi}$ is already cut-free, since no (lI)-rule of length $\geq \Omega$ has been used in $\boldsymbol{\pi}$, we see that $\|\boldsymbol{\pi}\| < \Omega$, and the ordinal α can be replaced by a more reasonable $\alpha' < \Omega$ (α' is something like $\|\boldsymbol{\pi}\|$). It is therefore possible to replace the given cut by:

$$\dfrac{\begin{array}{c}\boldsymbol{\pi}'\ \vdots\\ \Gamma \vdash \boldsymbol{\Phi}(\boldsymbol{I\Phi}^{\alpha'}, \bar{n}), \Delta\end{array} \quad \begin{array}{c}\boldsymbol{\pi}_{\alpha'}\ \vdots\\ \Gamma', \boldsymbol{\Phi}(\boldsymbol{I\Phi}^{\alpha'}, \bar{n}) \vdash \Delta'\end{array}}{\Gamma, \Gamma' \vdash \Delta\Delta'}\ \mathsf{CUT}$$

Since the original cut seems to concentrate *unpredicativity* in it, whereas the latter seems to be perfectly predicative, Pohlers has called his method "**local predicativity**". The principal limitation of the method is that we obtain cut-elimination (with a real subformula property) for only those sequents $\Gamma \vdash \Delta$ in which $\boldsymbol{I\Phi}'$ does not occur negatively.

The method has been successfully applied to the iterated cases ($\mathbf{ID}_n, \mathbf{ID}_\nu$) by Pohlers himself, and to a theory corresponding to the first recursively inaccessible ordinal by Säger and Pohlers [107].

Since the cut-elimination results are rather limited in that framework, the essential application of this is to give "the" ordinals of the corresponding theories. In fact one can get a bit more: If f is a provably total Σ^1 function (from ω_2^{CK} to itself) of ID_2, then f can be bounded by a function $\bar{\vartheta}(\Omega_{n_1})$ ($\Omega_0 = \Omega$, $\Omega_{n+1} = \Omega^{\Omega_{n_1}}$) (private communication of W. Buchely). Compare with 22.B.2, recalling that the functions $\bar{\vartheta}(\Omega_{n_1})$ (of Buchely's system) are (essentially) dilators. For this purpose, what they do is that they give explicit bounds on α', given α (with the notations used above...). The function $\alpha' = D(\alpha)$ is called a *collapsing function.* (The idea of collapsing is natural, since in the original proof $\boldsymbol{\pi}$, only a subset of α of order type α' is actually needed!) Hence a great part of this work is devoted to the study of more and more complicated systems of collapsing functions, corresponding to bigger and bigger systems... . In general, these collapsing functions are related to the system of ordinal notations due to Buchholz [93]... .

11.A.4. Other works

The first ordinal analysis of inductive definition (besides Takenti's work, which did not use current ordinal notations) was due to Howard, and done for $\mathbf{ID}_1$: this is the origin of the terminology "Howard ordinal" for η_0 [87]. The ordinals of the $\mathbf{ID}_n$'s were found by Pohlers, as explained above.

Tait's work [108], was later improved by Sieg [109], essentially in the intuitionistic case, already treated by Pohlers.

Finally, the works of Buchholz [110], have introduced interesting variations on the main theme explained in 11.A.3.

See [3] for more details.

Annex 11.B. Applications to generalized recursion

This section requires a knowledge of admissibility and of generalized recursion.

The main property of the generalizations of recursiveness that arose in the years 1960–1975 is that they are not at all recursive! In general, people use infinitary operations on sets (or ordinals); it is still possible to call it recursion theory, because the main formal properties of recursivemess are still true... . Now observe that Π^1_2-logic proposes an obvious candidate for the concept of a recursive function from $0n$ to $0n$: $F : 0n \to 0n$ *is recursive iff* F *can be extended into a recursive dilator* (i.e. it is possible to define $F(f) \in I\Big(F(x), F(y)\Big)$ in such a way that this extension makes F a recursive dilator).

It is absolutely doubtless that a function recursive in this sense is generalized recursive in any reasonable sense ... the reason being that we have an effective process of computing $F(x)$ from x (the direct limits), and so to speak, such a F is recursive in the familiar sense of the term.

We shall try to prove the converse, namely that, roughly speaking, all generalized recursive functions are of that form; it is necessary to make some remarks which determine the obvious limits of this enterprise:

(i) Generalized recursiosn usually allows operations of the form: definition by bounded quantifiers (or bounded μ-operators); but "bounded" means bounded by some of the already computed values, and these values are infinite! There are a lot of generalized recursive functions which will not be recursive at all! But recall "bounded": it is likely that these functions, although terribly non effective, can still be bounded by recursive dilators. (Then generalized recursion can be reduced to:

   - recursive dilators.
   - "bounded" generalized recursion.

   Typically $f(z) = U\Big(\mu y < F(z)\, T)1(e, z, y)\Big)$ with U, T_1 of the "bounded" kind, and $F(\cdot)$ recursive dilator, could give a reasonable "normal form theorem".)

(ii) Generalized recursion is very often a theory of partial functions; for these partial functions, dilators bring nothing new... . Dilators answer the problem only in the context of total functions.

(iii) In fact there is a greater variety of situations than expected, since, besides generalized recursive functions from $0n$ to $0n$, it is possible to define generalized recursive functions from *admissible* ordinals to themselves; the picture already explained will still hold for a large class of *successor* admissibles, but for more general admissibles, it will be necessary to replace majoration by recursive dilators, by: majoration by means of hierarchies indexed by recursive dilators... .

It would be now the place to give the main definitions concerning admissible sets, Σ^1 functions over an admissible set, together with the basic results of this theory... . I would feel too uneasy in doing such a thing, and I prefer not to add unessential things to this too long book... . Many books on the subject are available, for instance the book of Barwise [111].

We shall proceed as follows: most of the time, we shall try to formulate the results by means of inductive definitions; when we need a relation between admissibility and inductive definability, then we shall simply admit the result... .

11.B.1. Theorem (Girard, 1979, [100]).
Let f be a total Σ^1 function from ω_1^{CK} to itself, in $L_{\omega_1^{CK}}$; then one can find a prim. rec. dilator F such that:

$$f(x) \leq F(x) \quad \text{for all } x \in [\omega, \omega_1^{CK}[\ .$$

Proof. The painful part of the proof is the translation of these set-theoretic definitions ($\Sigma^1, L_{\omega_1^{CK}}$) in terms of inductive definitions. We need the following

11.B.2. Theorem.
Under the hypotheses of 11.B.1, one can find a positive operator $\mathbf{\Phi}$, together with an arithmetical formula $A(x, y)$, with the following properties:

- $\forall x \in \mathbf{\Phi}\ \exists y \in \bar{\mathbf{\Phi}}\ A(x, y)$.

- $x \in \bar{\mathbf{\Phi}} \wedge x \notin \boldsymbol{I\Phi}^{\alpha} \wedge y \in \boldsymbol{I\Phi}^{\beta} \wedge A(x,y) \wedge \alpha < \omega_1^{CK} \rightarrow f(\alpha) \leq \beta$.
- The closure ordinal of $\mathbf{\Phi}$ is ω_1^{CK}.

Proof. It is well known that $L_{\omega_1^{CK}}$ consists of the hereditary hyperarithmetical sets (HH), that can be viewed as well-founded hyperarithmetical trees. For such trees, it is not very difficult to give an inductive definition. The rest of the proof is either clear to the reader, or impossible to explain without going too much into what is admissibility... . □

We apply now our treatment of inductive definitions (for instance the variant without ω-rule), and, since $\vdash\ \forall x \in \bar{\mathbf{\Phi}}\ \exists y \in \bar{\mathbf{\Phi}}\ A(x,y)$ is true, we obtain, by completeness a $\mathsf{Id} + \underline{1}$-proof of this sequent. Now, we apply cut-elimination to this proof, and we obtain a cut-free F-proof of the same sequent, where F is a recursive dilator; now we apply Lemmas 11.4.5 and 11.4.7, and we obtain a cut-free proof of:

$$\vdash\ \forall x \in \boldsymbol{I}\bar{\mathbf{\Phi}}^{t}\ \exists y \in \boldsymbol{I}\bar{\mathbf{\Phi}}^{u}\ A(x,y)$$

where t and u are the $F+\underline{1}$ terms: $t(\alpha)=\alpha$, $u(\alpha)=F(\alpha)$. We would like to conclude that

$$\forall x \in \boldsymbol{I\Phi}^{\alpha}\ \exists y \in \boldsymbol{I\Phi}^{F(\alpha)}\ A(x,y)$$

is true; but if α is taken arbitrarily, then some rules of the proof may become non valid for the intended interpretation (namely, when α is finite, the rule $(\bar{l})$ for $\bar{\boldsymbol{N}}$, which says that $\bar{\boldsymbol{N}} \subset \boldsymbol{IN}^{\alpha}$, is false); but this problem is eliminated if we concentrate upon infinite values of α. Recall the second condition of 11.B.2: we obtain $f(\alpha) \leq F(\alpha+1)$ for all infinite $\alpha < \omega_1^{CK}$.

This is the essence of the result; here $F \circ (\mathsf{Id}+\underline{1})$ is only recursive; if we want it to be prim. rec., consider the dendroid $D = \mathbf{BCH}\big(F \circ (\mathsf{Id}+\underline{1})\big)$; this dendroid is recursive, hence $s \in D^* \leftrightarrow \exists n\ T_1(e,s,n)$; define another dendroid D' by prolongating any $s \in D$ into $s * t$, where t is any sequence $(x_0, 0, x_1, ..., 0, x_{n-1}, 0)$ with

- $n = \mathsf{sup}\,\{m\,;\ \exists i \leq lh(s)\ T_1(e, s{\restriction} i, m)\}$.
- $x_0 < x_1 < ... < x_{n-1} \in I\!\!N$.

- $x_0 > (x)_1, (s)_3, ..., (s)_{lh(s)-2}$.

One easily checks that:

1. D' is a sh dendroid.
2. D' is prim. rec.
3. $F' = \mathbf{LIN}(D')$ is prim. rec.

Moreover, $F(x) \leq F'(x)$ when x is limit, hence $F(x) \leq F'(x+\omega) \leq F'(x+x)$ for $x \geq \omega$. This proves that F can be replaced, for infinite values, by the prim. rec. dilator $F' \circ (\mathsf{Id} + \mathsf{Id})$. □

11.B.3. Remarks.

(i) If we want the majoration to hold also at finite arguments, then observe that $F(n)$ can be infinite, hence we cannot expect F to be weakly finite. If we define the concept of a recursive (non weakly finite) dilator in the straightforward manner, then it will be true that any total Σ^1 function of ω_1^{CK} is majorized by such a dilator for all values $< \omega_1^{CK}$.

(ii) The crucial point in the proof is the use of Lemmas 11.4.5 and 11.4.7; the meaning of these lemmas has something to do with the 3-valued semantics of Chapter 3: if we replace negative occurrences of $\bar{\mathbf{\Phi}}$ by $I\mathbf{\Phi}^\alpha$, positive occurrences of $\bar{\mathbf{\Phi}}$ by $I\mathbf{\Phi}^{F(\alpha)}$, this means that we are computing the truth value of the sequent in the three-valued model $\boldsymbol{m}$ of Id_1, with

$$\begin{aligned} \boldsymbol{m}\Big(\bar{\mathbf{\Phi}}(\bar{n})\Big) &= \boldsymbol{t} && \text{if } I\mathbf{\Phi}^\alpha(\bar{n}) \\ \boldsymbol{m}\Big(\bar{\mathbf{\Phi}}(\bar{n})\Big) &= \boldsymbol{f} && \text{if } \neg I\mathbf{\Phi}^{F(\alpha)}(\bar{n}) \\ \boldsymbol{m}\Big(\bar{\mathbf{\Phi}}(\bar{n})\Big) &= \boldsymbol{u} && \text{otherwise .} \end{aligned}$$

And the theorem appears therefore as a striking example of application of three-value semantics... .

11.B.4. <u>Definition</u>.
An ordinal α is **$\boldsymbol{\beta}$-definable** iff: there exists a prim. rec. $\boldsymbol{\beta}$-theory $\boldsymbol{T}$ which has a $\boldsymbol{\beta}$-model, such that, if $\boldsymbol{m}$ is any $\boldsymbol{\beta}$-model of $\boldsymbol{T}$, then $\boldsymbol{m}(\boldsymbol{o}) = \alpha$.

11.B.5. <u>Remarks</u>.

(i) By standard manipulations, the infinite prim. rec. sequence of Def. 11.B.4 can be replaced by one single axiom... .

(ii) If α is $\boldsymbol{\beta}$-definable, then there exists a prim. rec. predilator F such that:

$$\alpha = \mathsf{inf}\{x\,;\,F(x) \text{ is not an ordinal}\}\ .$$

(<u>Proof</u>. If $\boldsymbol{T}$ is as in 11.B.4, consider the prim. rec. pre $\boldsymbol{\beta}$ proof of $\vdash$ (the void sequent) given by the $\boldsymbol{\beta}$-completeness theorem; then $\boldsymbol{\pi}_x$ is well-founded iff $\vdash$ holds in any model $\boldsymbol{m}$ of $\boldsymbol{T}$ s.t. $\boldsymbol{m}(\boldsymbol{o}) \leq x$; hence $\boldsymbol{\pi}_x$ is well-founded iff $x < \alpha$. Let $F = \mathbf{LIN}(\boldsymbol{\pi})$... . $\square$)

In general the ordinals α such that

$$\alpha = \mathsf{inf}\,\{x\,;\,F(x) \text{ is not an ordinal}\} \text{ for some prim. rec. predilator } F$$

(which form the set of **weakly $\boldsymbol{\beta}$-definable** ordinals), are not $\boldsymbol{\beta}$-definable; however, it is easily seen that α is weakly $\boldsymbol{\beta}$-definable iff one can find a prim. rec. $\boldsymbol{\beta}$-consistent $\boldsymbol{T}$ s.t.

$$\alpha = \mathsf{inf}\,\{\boldsymbol{m}(\boldsymbol{o})\,;\,\boldsymbol{m}\ \boldsymbol{\beta}\text{-model of } \boldsymbol{T}\}\ .$$

It is easily shown that the two notions of $\boldsymbol{\beta}$-definability coincide when α is admissible or a limit of admissibles; in particular α weakly $\boldsymbol{\beta}$-definable $\rightarrow$ $\alpha^+\boldsymbol{\beta}$-definable ($\alpha^+$ is the next admissible, i.e. the smallest admissible $> \alpha$).

(iii) The ordinal σ_0 which is the smallest α such that: L_α is a Σ^1 substructure of V, and which is known as the first **stable** ordinal, is related to $\boldsymbol{\beta}$-definability as follows:

$$\sigma_0 = \mathsf{sup}\,\{\alpha\,;\,\alpha \text{ is } \boldsymbol{\beta}\text{-definable}\}\ .$$

An equivalent definition of σ_0 is the following:

$$\sigma_0 = \mathsf{inf}\,\{x\,;\,\forall F \text{ prim. rec. predilator, if } F(z) < x$$
$$\text{for all } z < x, \text{ then } F \text{ is a dilator}\}\ .$$

Equivalently σ_0 is the supremum of the weakly $\boldsymbol{\beta}$-definable ordinals. The $\boldsymbol{\beta}$-definable ordinals form a proper cofinal subset of σ_0, which includes a very large initial segment I_0 of σ_0. In many applications, we remain in this initial segment... . For instance, many current admissibles such as the first recursively inaccessible, the first recursively Mahlo, belong to I_0... .

11.B.6. <u>Theorem</u> (Girard, 1979, [100]).
Assume that α and all its predecessors are $\boldsymbol{\beta}$-definable, i.e. $\alpha \in I_0$, and let f be a Σ^1 function from α^+ to itself (Σ^1 over L_{α^+}); then one can find a prim. rec. dilator F such that:

$$\forall x \left(\alpha \leq x < \alpha^+ \to f(x) \leq F(x)\right)\ .$$

<u>Proof</u>. Here too, we express α^+ as the result of an inductive definition: there exists a positive operator $\boldsymbol{\Phi}$ in the language of set-theory such that:

(i) If $\boldsymbol{m} = (L_\alpha, \varepsilon\restriction L_\alpha)$, then α^+ is the closure ordinal of $\boldsymbol{\Phi}$ w.r.t. $\boldsymbol{m}$.

(ii) There are ordinals $\xi_1, ..., \xi_n < \alpha$, and a formula $A(x,y) = A(\bar{\xi}_1, ..., \bar{\xi}_n, x, y)$ whose only ordinal parameters are $\xi_1, ..., \xi_n$, and such that

$$(\boldsymbol{m}[\bar{\boldsymbol{\Phi}} \models \forall x\ \forall y \left(A(x,y) \wedge \bar{\boldsymbol{\Phi}}(x) \wedge \neg I\boldsymbol{\Phi}^\lambda(x) \wedge I\boldsymbol{\Phi}^\mu(y)\right) \ \to$$
$$f(\lambda) \leq \mu\ .$$

(The inductive definition $\boldsymbol{\Phi}$ is that of the Σ^1 well-orders of Σ^1 subclasses of L_α... .)

When α is given, then there is a standard way to construct L_α; this means that there is a $\boldsymbol{\beta}$-theory $\boldsymbol{T}_0$ with the following properties:

1. For each α, $\boldsymbol{T}_0$ has exactly one $\boldsymbol{\beta}$ model $\boldsymbol{m}_\alpha$ s.t. $\boldsymbol{m}_\alpha(\boldsymbol{o}) = \alpha$.

2. The restriction of $\boldsymbol{m}_\alpha$ to the language of set-theory is (L_α, ε).

3. $\boldsymbol{T}_0$ consists of exactly one formula A_0.

Let $B_1, ..., B_n, B$ be formulas ($\boldsymbol{\beta}$-theories with one axiom) which define $\xi_1, ..., \xi_n, \alpha$ in the sense of $\boldsymbol{\beta}$-definability (it can be assumed without loss of generality that all these formulas, and A_0 are in the same language...). We replace the closed formulas $B_1, ..., B_n, B, A_0$, by formulas $B_1^x, ..., B_n^x, B^x, A_0^x$, depending on a variable x of type $\boldsymbol{o}$, as follows: all quantifiers of type $\boldsymbol{o}$ are relativized to x. In particular, if

B_k^x holds in a $\boldsymbol{\beta}$-model $\boldsymbol{m}$, then $x = \xi_k$

B^x holds in a $\boldsymbol{\beta}$-model $\boldsymbol{m}$, then $x = \alpha$

A_0^x holds in a $\boldsymbol{\beta}$-model $\boldsymbol{m}$, then $\boldsymbol{m}\restriction \varepsilon = (\varepsilon \restriction L_x)$.

The hypotheses of theorem and the remarks already made show that the sequent

$$\vdash \forall x_1 ... \forall x_n \ \forall x \ \forall z \ \exists z' \left(B_1^{x_1} \wedge ... \wedge B_n^{x_n} \wedge B^x \wedge A_0^x \wedge \bar{\boldsymbol{\Phi}}(z) \ \rightarrow \right.$$
$$\left. \bar{\boldsymbol{\Phi}}(z') \wedge A(x_1, ..., x_n, z, z') \right)$$

is true in all $\boldsymbol{\beta}$-models of the form $\boldsymbol{m}\,[\bar{\boldsymbol{\Phi}}]$; compared to our definitions, the only new thing here is that $\boldsymbol{m}$ is already a $\boldsymbol{\beta}$-model; but this does not change anything! By $\boldsymbol{\beta}$-completeness and cut-elimination, we find a F-proof of this sequent which is cut-free.

For the same reasons as in 11.B.1:

$$B_1^{\bar{x}_1} \wedge ... \wedge B_n^{\bar{x}_n} \wedge B^{\bar{x}} \wedge A_0^{\bar{x}} \wedge \boldsymbol{I\Phi}^\lambda(z) \quad \rightarrow$$
$$\exists z' \left(\boldsymbol{I\Phi}^{F(\lambda)}(z') \wedge A(\bar{x}_1, ..., \bar{x}_n, z, z') \right)$$

is true for all $x_1, ..., x_n, x \leq \lambda \in 0n$. In particular, take $x_1 = \alpha_1, ..., x_n = \alpha_n$, $x = \alpha$, $\lambda \geq \alpha$; we obtain $f(\lambda) \leq F(\lambda + 1)$; the majoration holds for all $\lambda \in [\alpha, \alpha^+[$. $\square$

11.B.7. Corollary.
Let Ξ_1 be the following dilator: take an enumeration of all recursive dilators, say $(F_n)_{n\in\mathbb{N}}$, and let $\Xi_1 = \sum\limits_{n<\omega} F_n$. Then, for all $\alpha \in I_0$, we have:

$$\alpha^+ = \Xi_1(\alpha)\ .$$

Proof. Consider the Σ^1 function, defined, for $x < \alpha^+$, by: $f(z) = x$; then by 11.B.6 this constant function is majorized by some F_n, for all $\lambda \geq \alpha$; hence $x \leq F_n(\alpha) < \Xi_1(\alpha)$. On the other hand remark that:

11.B.8. Lemma.
If F is a prim. rec. dilator, then the function $\lambda \to F(\lambda)$ is a Σ^1 function from α^+ to α^+.

Proof. Immediate for everybody who knows what "Σ^1" means... . □

From that, it follows that the partial sums $\sum\limits_{i<n} F_i$ map α^+ into itself, and from that $\Xi_1(\alpha) \leq \alpha^+$. □

11.B.9. Corollary.
Define a flower $\boldsymbol{\omega}$ by:

$$\begin{aligned}
&\boldsymbol{\omega}(0) = \omega && \boldsymbol{\omega}(\mathbf{E}_0) = \mathbf{E}_\omega \\
&\boldsymbol{\omega}(x+1) = \boldsymbol{\omega}(x) + 1 + \Xi_1\big(\boldsymbol{\omega}(x)\big) && \boldsymbol{\omega}(f + \mathbf{E}_1) = \boldsymbol{\omega}(f) + \mathbf{E}_1 + \Xi_1\big(\boldsymbol{\omega}(f)\big) \\
& && \boldsymbol{\omega}(f + \mathbf{E}_{01}) = \boldsymbol{\omega}(f) + \mathbf{E}_{01} + \Xi_1\big(\boldsymbol{\omega}(y)\big) \\
&\boldsymbol{\omega}(\mathsf{sup}\, x_i) = \mathsf{sup}\,\big(\boldsymbol{\omega}(x_i)\big) && \boldsymbol{\omega}\Big(\bigcup_i f_i\Big) = \bigcup_i \boldsymbol{\omega}(f_i)\ .
\end{aligned}$$

Then, for all $x \in I_0$, we have $\boldsymbol{\omega}(x) = \omega_x^{CK}$ (ω_x^{CK} is the x^{th} admissible or limit of admissibles, with $\omega_0^{CK} = \omega$).

Proof. Completely trivial... . □

11.B.10. Remarks.

(i) The conclusion of Theorem 11.B.6 fails when $\alpha \geq \sigma_0$ (because $\Xi_1 \in L_{\sigma_0} + 1$).

(ii) In fact Ressayre has proved the following result, which shows that the theorem holds cofinally in σ_0:

11.B.11. Theorem (Ressayre, 1981, [112]).
If α is a denumerably infinite ordinal, the following are equivalent:

(i) α is $\boldsymbol{\beta}$-definable and all ordinals $\lambda < \alpha$ are Σ^1-definable inside L_{α^+}, from α alone (i.e. there is a Σ^1 formula $A(\bar{\alpha}, x)$ such that λ is the only solution of $A(\bar{\alpha}, x)$ in L_{α^+} and α is the only parameter of A).

(ii) $\alpha^+ = \Xi_1(\alpha)$.

(iii) If f is a Σ^1-function from α^+ to itself, then there is a prim. rec. dilator F s.t. $f(x) \leq F(x)$ for all $x \in [\alpha, \alpha^+[$.

Proof. See Ressayre [112]. $\square$

11.B.12. Exercise.
Let $s_0 = \mathsf{sup}\, I_0$ (i.e. $I_0 = [0, s_0[= s_0)$.

(i) Show that s_0 is admissible.

(ii) Show that if x is admissible or limit of admissibles, and x is weakly $\boldsymbol{\beta}$-definable, then x is $\boldsymbol{\beta}$-definable; conclude that s_0 is not weakly $\boldsymbol{\beta}$-definable.

(iii) Consider $\Xi'_1(x) = \sum_n F'_n(x)$, where (F'_n) enumerates the set of all prim. rec. predilators s.t. $F'_n(s_0)$ is well-ordered. Show that $\Xi_1(x) \leq \Xi'_1(x)$ for all $x < s_0$. Prove that $\Xi'_1(s_0) < s_0^+$, and conclude that $s_0^+ > \Xi_1(s_0)$. (Ressayre has shown that $\Xi_1(s_0) = \Xi'_1(s_0)$; from this result, one easily gets: $s_0^+ = \Xi_1\big(\Xi_1(s_0)\big)$ ... but $s_0^{++} = \Xi_1(s_0^+)$... .)

11.B.13. Remark.
More direct proofs of the Theorems 11.B.1 and 11.B.6 have been given, which do not use the pattern of completeness and cut-elimination:

(i) Masseron [113] (1980) gives the bounding dilator by a direct construction, see 11.B.14.

(ii) Ressayre (with Harrington) [112] (1981) gives the answer by means of a Gödel sentence.

(iii) Normann [114] (1981) gives a direct proof using some kind of $\boldsymbol{\beta}$-completeness argument.

(iv) The remark of Buchholz (1982, see 11.4.9 (iii)) makes it possile to avoid the syntactic cut-elimination.

All these results are simpler than the original proofs just given; however, I still think that these original proofs give a framework in which these results are more easily understood *from a general standpoint*; moreover, the syntactic form of our proof makes it (at least theoretically!!) more informative... .

11.B.14. Exercise (Masseron, 1980, [113]).
Our purpose is to give a direct poor fo 11.B.1.

(i) If f is a Σ^1 function on $L_{\omega_1^{CK}}$, show the existence of a prim. rec. function g s.t.

$$\forall e \in O \left(g(e) \in O \wedge \|g(e)\| \geq f(\|e\|) \right) .$$

(ii) For each $e \in I\!N$, define a tree $T_e(x)$, depending on an ordinal parameter x and such that: an infinite branch in $T_e(x)$ encodes

– a s.d.s. in $g(e)$.

– a strictly increasing function from e^0 to x.

Show that $T_e(x)$ is a well-founded tree for all e and $x \in 0n$. Moreover, if $e \in O$, show that $\|g(e)\| \leq \|T_e(\|e\|)\|$. Show that the trees $T_e(\cdot)$ can be used to define dilators D_e, and conclude that $\|g(e)\| \leq \sum_n D_n(e)$ for all $e \in O$. (The end of Masseron's work proves that the dilator $\sum_n D_n$ can be replaced by a prim. rec. ladder.)

We now state a very remarkable result, due to Van de Wiele [115]; this result is connected with the theory of functions from sets to sets... .

11.B.15. Theorem (Van de Wiele, 1981, [115]).
Assume that f is a function from sets to sets which is uniformly Σ^1 over all admissibles; this means that if x is a set, and x^+ is the smallest admissible structure containing x, then $f(x)$ is the only y such that: $(x^+, \varepsilon) \models A(x, y)$ (A is a Σ^1 formula, independent of x). Then f is rank-majorized everywhere by a recursive dilator F, i.e.

$$x \in V_\alpha \to f(x) \in V_{F(\alpha)} \ .$$

Proof. We use the fact that x^+ is obtained by a uniform inductive definition from x; then we roughly proceed as follows:

- We start with the language of set theory, and we write the axiom of extensionality, and a constant C (for x).
- We add objects of type $\boldsymbol{o}$ and a binary relation between sets and ordinals $R(x, \xi)$, which says something like $rk(x) = \xi$; we write the obvious axioms for R which will make a $\boldsymbol{\beta}$-model of the theory to be essentially an arbitrary set x (the interpretation of C).
- Then we write our inductive definition of x^+ (if x is C) as usual.

The hypothesis proves that:

$$\vdash \ \exists y \Big(\bar{\boldsymbol{\Phi}}(y) \wedge A(C, y) \Big)$$

is true in all $\boldsymbol{\beta}$-models $\boldsymbol{m}\,[\bar{\boldsymbol{\Phi}}]$, hence $\mathsf{Id}+\underline{1}$-provable; after a cut-elimination, we obtain the fact that

$$\vdash \ \exists y \left(I\boldsymbol{\Phi}^{F(\alpha)}(y) \wedge A(C, y) \right)$$

is true in all models of the above theory, with $\boldsymbol{m}(\boldsymbol{o}) = \alpha$; in that case $rk\Big(\boldsymbol{m}(c)\Big) < \alpha$, and if $\boldsymbol{m}(c) = x$, we see that $rk\Big(f(x)\Big) < F(\alpha)$... . $\square$

8.B.16. Remark.
A typical corollary of 11.B.15 is that any uniformly Σ^1 function is E-*recursive* (in the sense of Normann, [116]); this fact was first proved by Van de Wiele. Of course the specialists of the field (Slaman) were soon able to give a direct proof of the same result.... But the interest of Van de Wiele's theorem is more general: recursive dilators induce functions from $0n$ to $0n$ which are truly recursive, whereas these other notions (uniformly Σ^1, E-recursive...) are not recursive in the familiar sense of the word. Furthermore, the very fact that the specialists had no real trouble in proving this equivalence by methods of their own is a further evidence of the clarifying power of Π^1_2-logic.

Annex 11.C. Π^1_n well-orders

This section is indeed the sequel and the generalization of the results of Section 11.B; we are here interested in the relationship between bigger admissibles (stable ordinals,...) and the "higher types dilators", namely the ptykes of Chapter 12.

11.C.1. Definition.
For each $n > 0$, we define π^1_n to be the smallest ordinal which is not of the form $\|R\|$, where R is a well-order of a subset of $I\!N$, given by a Π^1_n formula.

11.C.2. Remarks.

(i) A Π^1_n well-ordering of $I\!N$ would be also Σ^1_n, hence Δ^1_n (if $x \leq y \leftrightarrow A(x,y)$, then $x \leq y \leftrightarrow \neg A(y,x) \vee x = y$), hence the concept of a Π^1_n well-ordering of $I\!N$ is a priori weaker than the concept of a Π^1_n well-ordering of a subset of $I\!N$.

(ii) A more general notion is that of a Π^1_n prewell-ordering, i.e. a Π^1_n preorder relation $A(x,y)$ s.t. there is no sequence (x_n) s.t. $\forall n\, \neg A(x_n, x_{n+1})$, and which is a linear order. The ordinals associated with Π^1_n prewell-orderings are a priori greater than the ordinals associated with Π^1_n well-orders. (Ressayre gave (private communication) an explicit Π^1_2 prewell-order of height $\pi^1_2 = \sigma_0$.)

(iii) If α is the height of a Π^1_n well-order, and $\alpha' \leq \alpha$, then α' is also the height of a Π^1_n well-order: this shows that

$$\pi^1_n = \mathsf{sup}\,\{\|R\|\,;\, R \text{ a } \Pi^1_n \text{ well-order of a subset of } I\!N\}\ .$$

11.C.3. Proposition.
The ordinal π^1_1 is equal to ω_2^{CK}.

Proof. First observe that $\omega_1^{CK} < \pi^1_1$ (if we take a linear order R which is prim. rec. and which has an accessible part of order type ω_1^{CK}, then this accessible part is a Π^1_1 well-order of height ω_1^{CK}). If F is any prim. rec.

dilator, then $F(\omega_1^{CK}) < \pi_1^1$.

(Proof. if R is a Π_1^1 well-order of height ω_1^{CK}, then $F(\omega_1^{CK})$ can be described as the set of all formal denotations

$$(z_0\,;\, a_0, ..., a_{n-1}\,;\, R)_F\ , \qquad \text{with } a_0 < ... < a_{n-1} \text{ (mod } R)$$

ordered as usual: this is a Π_1^1 well-ordering... . $\Box$)

Hence $\omega_2^{CK} = \Xi_1(\omega_1^{CK}) \leq \pi_1^1$. The opposite inequality is obtained by observing that Π_1^1 well-orders can be defined by a single inductive definition, if we have a Π_1^1-universal set (that we get in $\mathbf{ID}_1$), hence by a double inductive definition; but the closure ordinal of a double inductive definition ($\mathbf{ID}_2$) is $\leq\ \omega_2^{CK}$. (We can also use the cut-elimination procedure for ID_2, which yields the bound $\Xi_1(\omega_1^{CK})$ for the closure ordinal of $\mathbf{ID}_2$... .)

11.C.4. Definition.
let n be the type defined by: $o = \boldsymbol{o}$; $p+1 = p \to \boldsymbol{o}$; we define, for each n: Ξ_n, a ptyx of type n, by: $\Xi_n = \sum_i P_i^n$ where $(P_i^n)_{i\in\mathbb{N}}$ is an enumeration of all ptyxes of type n, which are recursive and weakly finite (hence $\Xi_0 = \omega$).

11.C.5. Remark.
11.C.3 can be written as: $\Pi_1^1 = \Xi_1\Big(\Xi_1(\Xi^0)\Big)$, and we shall generalize it to: $\Pi_n^1 = \Xi_n(\Xi_{n-1})$. However, we must first say something concerning the dependence of the functors Ξ_n from the way (P_i^n) is enumerated.

11.C.6. Proposition.
Assume that (Q_i^n) is another enumeration of all prim. rec. ptyxes of type n, let $\Xi'_n = \sum_i Q_i^n$; if F is any ptyx of type $n+1$, we have

$$F(\Xi_n) = F(\Xi'_n)\ .$$

Proof. It is enough to show that $F(\Xi_n) \leq F(\Xi'_n)$; for this, we show that $I^n(\Xi_n, \Xi'_n) \neq \emptyset$: (if $T \in I^n(\Xi_n, \Xi'_n)$, then $F(T) \in I\Big(F(\Xi_n), F(\Xi'_n)\Big)$, hence $F(\Xi_n) \leq F(\Xi'_n)$...). For each p, we choose an integer $m = f(p)$, together with a natural transformation T_p from P_p to Q_m: assume that $f(0), ..., f(p-1), T_0, ..., T_{p-1}$ have been constructed, and consider the prim.

rec. ptykes $P_p + \underline{k}$ ($\underline{k}$ is a constant ptyx...); they are infinitely many, os one of them is of the form Q_m, with $m > f(0), ..., f(p-1)$. let $m = f(p)$, and if $Q_m = P_p + \underline{k}$, let $T_p = \mathbf{E}_{P_p} + \mathbf{E}_{0\underline{k}}$. The function f belongs to $I(n, m)$, hence

$$T = \sum_{i<f} T_i \qquad \textsf{belongs to } I^n(\Xi_n, \Xi'_n)\ . \qquad \square$$

11.C.7. Remarks.

(i) The meaning of 11.C.6 is that the Ξ_n's behave as "intrinsic" objects, when used in building ordinals.

(ii) But this is no longer true for functions from ordinals to ordinals: if $U \in I^{n+1}(F, G)$, then there is no reason why $U(\Xi_1)$ should equal $U(\Xi'_1)$. For instance take $F = G$ defined by $F(D) = D(\omega)$ (D dilator) $U(D) = D(f)$ where f is a given element of $I(\omega, \omega)$... .

11.C.8. Theorem.
The ordinal $\Xi_{n+1}(\Xi_n)$ is equal to: $\{F(\Xi_n)\,;\, F$ prim. rec. ptyx of type $n+1\}$.

Proof. If F is a prim. rec. ptyx of type $n+1$, then $F = P_i^{n+1}$ for some i, hence $F(\Xi_n) < \Xi_{n+1}(\Xi_n)$. Conversely, assume that $x < \Xi_{n+1}(\Xi_n)$; then $x < \Big(\sum\limits_{i<p} P_i^{n+1}\Big)(\Xi_n)$ for some p, i.e. $x < G(\Xi_n)$ for some prim. rec. G. Now, consider the *normal form* of x w.r.t. G and Ξ_n: $x = (x_0\,;\, a_0\,;\, t\,;\, \Xi_n)_G$. Recall that this implies that:

$$x = G(t)(x_0) \qquad \textsf{for some } t \in I^n\Big(a_0, \sum_{i<p} P_i^n\Big)\ .$$

a_0 is finite dimentionsal, hence it is easily checked that $t = t' + \mathbf{E}^n_{\underline{0}\Xi'_n}$, for some $t' \in I\Big(a_0, \sum\limits_{i<p} P_i^n\Big)$, for some p, with $\Xi'_n = \sum\limits_{p\le i} P_i^n$.
Let $a = \sum\limits_{i<p} P_i^n$, and consider:

$$G' = G \circ (\underline{a} + \mathsf{Id}^n\ .$$

($\underline{a} + \mathsf{Id}^n$ is the ptyx of type $n \to n$ such that: $(\underline{a} + \mathsf{Id}^n)(F) = a + F$, $(\underline{a} + \mathsf{Id}^n)(T) = \mathbf{E}_{\underline{a}} + T$.) Then $x < G'(\Xi'_n)$.

Moreover, $x = G'(\mathbf{E}^n_{\underline{0}\Xi'_n})(x_1)$ for some $x_1 \in G'(0)$. This implies that G' can be written as a sum $G' = G_1 + G_2$, with $G_1(\Xi'_n) = x$ (left to the reader). Now, by 11.C.6, $G_1 = (\Xi'_n) = G_1(\Xi_n)$; then we have succeeded in writing x under the form $G_1(\Xi_n)$. Now if we look at the way G_1 was obtained from G, it is plain that G_1 is still a prim. rec. ptyx of type $n+1$.□

11.C.9. Theorem (Girard-Ressayre, 1982, [117]).
$\pi^n_1 = \Xi_n(\Xi_{n-1})$ for $n > 1$.

Proof. Let us first establish that $\pi^n_1 \geq \Xi_n(\Xi_{n-1})$. We shall work in the particular case $n = 2$, but the argument is perfectly general: let G be a prim. rec. ptyx of type 2: we show that $G(\Xi_1) < \pi^1_2$. Ξ_1 can be expressed in a Π^1_2 way; consider an enumeration of all weakly finite prim. rec. predilators s.t. "P_p is a dilator" is a Π^1_2 formula $A(n)$. If $i_1, ..., i_p$ are integers, with $i_1 < ... < i_n$, then let $x_{i_1,...,i_n} = G(P_{i_1} + ... + P_{i_p})$, and similarly, if $\{i_1, ..., i_p\} \subset \{j_1, ..., j_q\}$, we define $f_{i_1,...,i_p\,;\,j_1,...,j_q} \in I(x_{i_1,...,i_p}, x_{j_1,...,j_q}\}$ by $f_{i_1,...,i_p\,;\,j_1,...,j_q} = G\Big(\sum\limits_{t<f} \mathbf{E}_{P_t}\Big)$, where $f \in I(p,q)$ is defined by: $i_{1+t} = j_{1+f(t)}\dots$. Let I be the set of all finite subsets of $I\!N$, ordered by inclusion; then the system (x_i, f_{ij}) is obviously a prim. rec. direct system of prim. rec. linear orders.

Let $J = \Big\{\{i_1, ..., i_p\} \in I\,;\, A(i_1) \wedge ... \wedge A(i_p)\Big\}$; then J is a Π^1_2 subset of I, and $G(\Xi_1) = \underset{J}{\underset{\longrightarrow}{\lim}}\ (x_1, f_{ij})$. We show that J is a Π^1_2 order. For this observe that $i < j \rightarrow \lceil i \rceil < \lceil j \rceil$ (with $\lceil\{i_1, ..., i_p\}\rceil = \langle i_1, ..., i_p\rangle$). Define $X = \{\langle z, \lceil i \rceil\rangle\,;\, i \in J \wedge z \in x_i \wedge \forall j < i\ z \notin rg(f_{ij})\}$. X is a Π^1_2 set, and is ordered by:

$$\langle z, \lceil i \rceil\rangle\ <_X\ \langle z', \lceil j \rceil\rangle \leftrightarrow f_{i,i\cup j}(z) < f_{j,i\cup j}(z')\ .$$

This is clearly a recursive order. We have therefore established that $G(\Xi_1)$ is a Π^1_2 well-order, hence $G(\Xi_1) < \pi^1_2$.

Conversely, let λ be Π^1_2 well-order. This means that there is a Π^1_2 formula $A(x,y)$, which defines a well-order of a subset of $I\!N$, of height λ. By Π^1_2-completeness of dilators, this relation can be replaced by:

$$f(x,y) \textit{ is the index of a prim. rec. dilator}$$

for some prim. rec. function f... .

We assume that $\boldsymbol{L}$ is a language containing a distinguished predicate $\mathbf{dil}(\cdot)$; the intended meaning of $\mathbf{dil}(n)$ is "n is the index of a prim. rec. dilator". We form a positive operator $\mathbf{\Phi}$:

$$\mathbf{\Phi}(X, x) \leftrightarrow \forall y \left(\mathbf{dil}\Big(f(y,x)\Big) \wedge y \neq x \to X(y)\right) .$$

Let $\boldsymbol{m}$ be an ω-model of $\boldsymbol{L}$; then $\boldsymbol{m}\,[\bar{\mathbf{\Phi}}] = \Big\{n\,;\,\forall g \left(g(0) = n \to \exists p\ f\Big(g(p+1), g(p)\Big) \notin \boldsymbol{m}(\mathbf{dil}) \vee g(p+1) = g(p)\right)\Big\}$. Now, assume that $\boldsymbol{m}$ enjoys the additional property:

(1) $\qquad \forall n\,(n \in \boldsymbol{m}(\mathbf{dil}) \to n \text{ is the index of a prim. rec. dilator})$.

Then $A(n,n) \to n \in \boldsymbol{m}\,[\bar{\mathbf{\Phi}}]$ is true. (If $g(0) = n$, then $\exists p \left(\neg A\Big(g(p+1), g(p)\Big) \vee g(p+1) = g(p)\right)$; but $\neg A\Big(g(p+1), g(p)\Big) \to f\Big(g(p+1), g(p)\Big)$ is not the index of a prim. rec. dilator, hence $\neg A\Big(g(p+1), g(p)\Big) \to f\Big(g(p+1), g(p)\Big) \notin \boldsymbol{m}(\mathbf{dil})$.

For the same reason, if $\boldsymbol{m}$ enjoys (1), then $\boldsymbol{m} \models \forall n \left(\mathbf{dil}\Big(f(n,n)\Big) \to \bar{\mathbf{\Phi}}(n)\right)$.

This means that $\mathbf{dil}\Big(f(n,m)\Big)$ ($= R(n,m)$) defines a well-founded relation in $\boldsymbol{m}$. It is easily checked that the closure ordinal of $\bar{\mathbf{\Phi}}$ in $\boldsymbol{m}$ ($= \|R\|$) is $\leq \lambda$, which is our Π^1_2 well-order.

Now we force the modesl we are considering to enjoy (1); the idea will be to take a dilator F, and to write an axiom saying something like: if $\mathbf{dil}(n)$, then there is a natural transformation from n to F.

We add a type $\urcorner$ to the language $\boldsymbol{L}$, and, for any $a \in \mathsf{Tr}(F)$, a constant $\bar{a}$ of type $\urcorner$; moreover, we add a 3-ary predicate letter $p(x,y,a)$ (two arguments of type $\boldsymbol{l}$, one of type $\urcorner$).

The axioms are the following:

(i) $p(x,y,a)$ defines a function (abbreviated as $g_x(y) = a$) $\forall x\,\forall y\,\exists! a\, p(x,y,a)$.

(ii) When $\mathbf{dil}(x)$, $g_x(\cdot)$ induces a natural transformation from the predilator encoded by x, to F:

1. $\mathbf{pil}(x)$. (x encodes a predilator; assume that $(X, \leq, \sigma^x_{ab})$ are the associated data, as in 8.G.10. Of course, there is no need to use this specific way of encoding predilators.)

2. g_x is a strictly increasing function from $(X, \leq)$ to $(\mathsf{Tr}(F), \leq^F)$.

3. If $a \leq b$, then $\sigma^x_{ab} = \sigma^F_{g_x(a)g_x(b)}$. (This last property is a bit difficult to write out precisely: first observe that, since we use ω-logic, x, a, b, are indeed integers $\bar{n}$, $\bar{p}$, $\bar{q}$, and by (i) there are unique points $\bar{z}$, $\bar{z}'$ in $\mathsf{Tr}(F)$ s.t. $\bar{z} = g_{\bar{n}}(\bar{p})$, $\bar{z}' = g_{\bar{m}}(\bar{q})$. Then $\sigma^F_{z,z'}$ is a uniquely determined pair $(r, \preceq_1)$ encoded by an integer $\ulcorner(r, \preceq_1)\urcorner$. We write the axiom $\overline{\ulcorner\sigma^x_{ab}\urcorner} = \overline{\ulcorner(r, \preceq_1)\urcorner}$... .)

If this theory is called $\boldsymbol{T}_F$, then it is plain that, for any $F \in \mathbf{DIL}$, any ω-model $\boldsymbol{m}$ of $\boldsymbol{T}_F$, $\boldsymbol{m}$ enjoys (1).

By a generalized $\boldsymbol{\beta}$-completeness argument (10.B), the formula $\forall n \Big(\mathbf{dil}(n) \to \bar{\boldsymbol{\Phi}}(n)\Big)$ is provable in all theories $\mathsf{ID}_1(\boldsymbol{T}_F, \boldsymbol{\Phi})$, *functorially in* F.

We can perform the cut-elimination as we did for the usual theories $\mathbf{ID}_1$, the only difference being that we must now take care of the extra functorial dependency... . The only important thing comes from 10.B.4, namely, how to "linearize" F-rules... .

We leave all details to the reader; from a cut-free proof of $\vdash \forall n \Big(\mathbf{dil}(n) \to \bar{\boldsymbol{\Phi}}(n)\Big)$, we can extract bounds as follows: First observe that the extra functorial dependency makes the cut-free proof a $H(F, \alpha)$-proof, for some recursive ptyx H of type $(\boldsymbol{o} \to \boldsymbol{o}) \to (\boldsymbol{o} \to \boldsymbol{o})$.

From this we can infer that:

$$\forall n \Big(\mathbf{dil}(n) \to I\boldsymbol{\Phi}^{H(F,0)}(n)\Big)$$

is true in any model $\boldsymbol{m}$ of $\boldsymbol{T}_F$. In particular, this formula is true in the model $\boldsymbol{m}_0$ of $\boldsymbol{T}_{\Xi_1}$, s.t.:

– $\boldsymbol{m} \models \mathbf{dil}(\bar{n}) \leftrightarrow n$ is the index of a prim. rec. dilator.

– g_n is the natural transformation from the dilator P^1_n encoded by n to $\Xi_1 = \sum_i P^1_i$, defined to be $\sum_{i<f} T_i$, with $f \in I(1, \omega)$: $f(0) = n$, $T_0 = \mathbf{E}_{P^1_n}$... .

Hence we obtain:

$$\boldsymbol{m}_0 \models \forall n \Big(\mathbf{dil}(n) \to I\boldsymbol{\Phi}^{H(\Xi_1,0)}(n)\Big)\text{... .}$$

But in $\boldsymbol{m}_0$, the relation $f(x, y) \in \mathbf{dil}$ corresponds to our order $A(x, y)$; the closure ordinal of $\mathbf{\Phi}$ must be λ, so

$$\lambda \leq H(\Xi_1, 0) \ .$$

If $H'(F) = H(F, 0)$, $H'(T) = H(T, \mathbf{E}_0)$, we obtain $\lambda \leq H'(\Xi_1)$.

(I confess that I have not checked the possibility of the replacement of H' by a prim. rec. ptyx. Anyway this is a very inessential problem... .) We obtain $\pi^1_2 \leq \Xi_2(\Xi_1)$. □

11.C.10. Remarks.

(i) The theorem cannot be extended to prewell-orderings: if $f(x, y) \in \mathbf{dil}$ defines a prewell-ordering (typically $x \leq y$ for all x and y), and $\boldsymbol{m}$ enjoys (1), then the relation $f(x, y) \in \mathbf{dil}$ in $\boldsymbol{m}$ is not necessarily well-founded any more... .

(ii) We have concentrated our attention upon π^1_2, because it is the most important case, and also because the specific knowledge we have of dilators, makes some part of the proof more immediate; but there is nothing in this proof that cannot be immediately generalized to π^1_n... . The inconvenience is that we work with abstract ptykes, whose structure is more puzzling than the structure of the now familiar dilators... .

(iii) Of course, alternative proofs of 11.C.9 can be given... .

11.C.12. Corollary.
$\sigma_0 = \Xi_2(\Xi_1)$.

Proof. It is well known that $\sigma_0 = \delta^1_2$, the first non Δ^1_2 ordinal, hence $\sigma_0 \leq \pi^1_2$. We shall prove $\Xi_2(\Xi_1) \leq \sigma_0$: let H be a prim. rec. ptyx of type 2; if x is any ordinal, we can define Ξ^x_1 to be the sum of all prim. rec. predilators P s.t. $P(x)$ is well-ordered. In particular $\Xi^{\sigma_0}_1 = \Xi_1$, hence $\exists x \ H(\Xi^x_1)$ well-ordered. If this Σ^1 formula is true, it must be true in $\boldsymbol{L}_{\sigma_0}$, hence $\exists x < \sigma_0 \ H(\Xi^x_1)$ is a well-order $< \sigma_0$. But $I^1(\Xi_1, \Xi^x_1) \neq \emptyset$, hence

$$H(\Xi_1) \leq H(\Xi^x_1) < \sigma_0 \ .$$

From this we obtain $\Xi_2(\Xi_1) \leq \sigma_0 \ldots$. □

11.C.13. Remark.
It is not possible to give a majoration for all Σ^1 functions from σ_0 to σ_0; for instance the function $x \rightsquigarrow x^+$, a typical such function, cannot be majorized by any function of th form

$$x \to H(\Xi_1, x)$$

for some prim. rec. ptyx H of type $(\boldsymbol{o} \to \boldsymbol{o}) \to (\boldsymbol{o} \to \boldsymbol{o})$.

(Proof. One easily checks that $H(\Xi_1, \sigma_0) < \sigma_0^+$. (In fact, our methods would show that $\sigma_0^+ = \Xi_{2'}(\Xi_1, \sigma_0)$ with $2' = (\boldsymbol{o} \to \boldsymbol{o}) \to (\boldsymbol{o} \to \boldsymbol{o})$. But σ_2^1, the first non Σ_2^1 ordinal, is not equal to σ_0^+; in fact $\sigma_2^1 = \Xi_1(\sigma_0)$, see [117].) Hence $\exists x\, (H(\Xi_x^1, x) < x^+)$; hnce such an x can be found $< \sigma_0$, and $H(\Xi_1, x) \leq H(\Xi_x^1, x) < x^+$. □)

However, it can be shown that there is a prim. rec. H such that: $f(x) \leq H(\Xi_1, x)$ for all $x \in X$, where X is a cofinal subset of σ_0, independent of the Σ' function f.

The proof of 11.C.9 cannot be directly used to give bounds for functions, because of the bad behaviour of the formalism we use in that case w.r.t. negative occurrences ... (models enjoying (1) make it easier to be a Π_2^1 well-order, hence more difficult to be a Π_2^1 non well-order...).

Let us look at 11.C.9, in the special case $n = 2$; the result says (if combined with 11.C.8) that every ordinal $< \pi_2^1 = \sigma_0$ can be expressed as $F(\Xi_1)$ for some prim. rec. ptyx of type 2. Hence the problem will be to find, given $\alpha < \pi_2^1$, the specific expression:

$$\alpha = F(\Xi_1) \ .$$

When doing this, we get as close as possible to a recurive description of α; if we know that $P_0, \ldots, P_n$ are prim. rec. dilators, then we can imagine that $\Xi_1 = P_0 + \ldots + P_n + \ldots$, hence we can approximate α by $F(P_0 + \ldots + P_n) = \alpha_n$, which is a recursive ordinal; if at a later stage, we recognize that P_{n+1} is a prim. rec. dilator too, then the approximation must be replaced by $\alpha_{n+1} = F(P_0 + \ldots + P_{n+1})$, and α_n must be embedded into α_{n+1} by means of $F(\mathbf{E}_{P_0}+...+\mathbf{E}_{P_n}+\mathbf{E}_{\underline{0}P_{n+1}})$; the embeddings are recursive as well. Finally, what is not recursive in this is the choice of an enumeration (P_n) of all prim. rec. dilators... .

In practice, the expression $\alpha = F(\Xi_1)$ can be replaced by an expression of α by induction on dilators: typically, a hierarchy indexed by (prim. rec.) dilators. Let us give a few examples:

11.C.14. Example.

(i) $\Theta_{\underline{0}}(x) = x$

(ii) $\Theta_{F+\underline{1}}(x) = \Theta_F(\omega_x^{CK})$

(iii) $\Theta_{\sup_i F_i}$ enumerates $\bigcap_i rg(\Theta\, F_i)$

(iv) $\Theta_F(x) = \Theta_{\mathbf{SEP}(F)(\cdot,x)}(0)$

defines a hierarchy of functions from $0n$ to $0n$. This hierarchy (if one approximates ω_x^{CK} by $\boldsymbol{\omega}(x)$, what is legitimate since we are only interested in arguments $<$ the first recursively inaccessible), is idential to a variant $\mathbf{\Lambda}'$ of $\mathbf{\Lambda}$, where the only change is the value $\mathbf{\Lambda}'_1$: $(\mathbf{\Lambda}'\underline{1})(x,y) = \boldsymbol{\omega}(y)$... .

11.C.15. Conjcture.
Let i_0^{CK} be the first recursively inaccessible, i.e. the first admissible limit of admissibles; then

(i) $i_0^{CK} = \Theta_{\Xi_1}(0)$; in other terms, any $x < i_0^{CK}$ can be majorized by an ordinal $\Theta_F(0)$ for some prim. rec. F.

(ii) If f is a total Σ^1 function from i_0^{CK} to itself, then $f(x) \leq \Theta_F(x+1)$ for all $x < i_0^{CK}$, for a certain prim. rec. F.

This conjecture has now been established; see [118].

11.C.16. Remarks.

(i) Under the assumption of 11.C.15, i_0^{CK} can be constructed "from below"; more precisely, given a prim. rec. F, the computation of Θ_F only makes use of previously computed initial segments of $i_0^{CK}, \ldots$.

(ii) We have seen that Θ is a variant of $\mathbf{\Lambda}$; more precisely one can write $\Theta_F(x) = H(F, \boldsymbol{\omega}, x)$, for a certain prim. rec. ptyx $H\ldots$. Moreover, one can replace the argument $\boldsymbol{\omega}$ by Ξ_1, using the expressiosn of $\boldsymbol{\omega}$ in terms of Ξ_1: $\Theta_F(x) = H'(F, \Xi_1, x)$. Hence from $i_0^{CK} = H'(\Xi_1, \Xi_1, 0)$, we obtain, with $H''(F) = H'(F, F, 0)$...

$$i_0^{CK} = H''(\Xi_1) \ ,$$

hence we clearly see that the hierarchy expression of i_0^{CK} (11.C.15 (i)) implies an expression of the kind implied by 11.C.9; conversely any expression of the form $H''(\Xi_1)$ can be converted practically into a hierarchy.

(iii) The bound $\Theta_F(x+1)$ (and not $\Theta_F(x)$) in 11.C.15 (ii) is due to the fact that the functions Θ_F are continuous, whereas f need not to be continuous! There are plenty of points where $\Theta_F(x) = x$, but in general $f(x) > x$ for all $x < i_0^{CK}$.

11.C.17. Example.

$$\boldsymbol{\pi}_{\underline{0}}(x) = x \ .$$

$\boldsymbol{\pi}_{F+\underline{1}}$ enumerates the closure (in the order topology) of the set of admissible fixed points of $\boldsymbol{\pi}_F$.

$$\boldsymbol{\pi}_{\underset{i}{\mathsf{sup}}\,(F_i)} \text{ enumerates } \bigcap_i rg(\boldsymbol{\pi}_{F_i})$$

$$\boldsymbol{\pi}_F(x) = \boldsymbol{\pi}_{\mathbf{SEP}(F)(\cdot,x)}(0) \ .$$

11.C.18. Conjecture.
Let μ_0^{CK} be the first recursively Mahlo, i.e. the first admissible α s.t. if f

is any Σ^1 function on $\boldsymbol{L}_\alpha$, then there is $x < \alpha$, x admissible s.t. $f(z) < x$ for all $z < x$; then

(i) $\mu_0^{CK} = \boldsymbol{\pi}_{\Xi_1}(0)$; in other terms, any $x < \mu_0^{CK}$ can be majorized by an ordinal $\boldsymbol{\pi}_F(0)$ for some prim. rec. F.

(ii) If f is a total Σ^1 function from μ_0^{CK} to itself, then $f(x) \leq \boldsymbol{\pi}_F(x+1)$, for all $x < \mu_0^{CK}$, for a certain prim. rec. F.

This conjecture has now been established; see [119].

11.C.19. Remark.
$\boldsymbol{\pi}$ can also be expressed as a functor; typically if $\boldsymbol{\pi}_{F+\underline{1}} = H(\boldsymbol{\pi}_F)$, for some H, then $\boldsymbol{\pi}_F$ will be a $\boldsymbol{\Lambda}$-like object, obtained by iterating H. The main question is therefore how to express $\boldsymbol{\pi}_{F+\underline{1}}$ under the form $H(\boldsymbol{\pi}_F)$. A typical example is when $F = \underline{1}$; then $\boldsymbol{\pi}_F(x) = \omega_x^{CK}$, whereas $\boldsymbol{\pi}_{F+\underline{1}}(x) = i_x^{CK}$, the x^{th} recursively inaccessible or limit of recursively inaccessibles... . Now an immediate extension of the conjecture 11.C.15 yields:

$$i_x^{CK} = \Theta_{\Xi_1}(x) \ .$$

More generally, one can surmise ([118] establishes the result) that $\boldsymbol{\pi}_{F+1}(x) = \boldsymbol{\Lambda}'_{\Xi_1}(x)$, where $\boldsymbol{\Lambda}'$ is the variant of $\boldsymbol{\Lambda}$ corresponding to $(\boldsymbol{\Lambda}'\underline{1})(x,y) = \boldsymbol{\pi}_F(y)$. From this:

(i) One gets the expression of $\boldsymbol{\pi}_{F+\underline{1}}$:

$$\boldsymbol{\pi}_{F+\underline{1}} = H(\boldsymbol{\pi}_F, \Xi_1) \ , \qquad \text{and } H \text{ prim. rec.}$$

(ii) Moreover, since H is indeed already a hierarchy (as a variant of $\boldsymbol{\Lambda}$, and $\boldsymbol{\pi}$ is built from H by means of a hierarchy, it is likely that the one-indexed hierarchy $\boldsymbol{\pi}$ can be replaced by a two-indexed one, extending Θ... .

This means that Θ is constructed by recursion up to Ξ_1; but $\boldsymbol{\pi}$ (as a two-indexed hierarchy) is constructed by recursion up to $(\Xi_1)^2$; in general, this process can be continued as long as we have "fundamental sequences of length $\leq \Xi_1$"; the general way of obtaining such fundamental sequences is by taking type 2 ptykes, and performing some separation-of-variables process in them. The hierarchy we are building appears therefore as a $\mathbf{\Lambda}$ of the next type, presumably closely related to the fucntor which performs the cut-elimination in $(\Pi^1_2 - CA)$.

CHAPTER 12
PTYKES

Originally, ptykes were a generalization of dilators to finite types; however, many dilator-like objects which are not ptykes of finite type occur rather natually, and the term "ptyx" now covers this immense area. The variety of situations is so large that, at this early stage of development of the theory, there is no evidence that the main lines of the general concept of ptyx have been drawn.... However, it is possible to focus on specific types for instance ptykes of type 2, of finite type,.... (see [117] for a reformulation of the theory using indiscernibles.)

12.1. Ptykes of type 2

12.1.1. Definition.

(i) A **ptyx** of type 2 is a functor from **DIL** to **ON** preserving direct limits and pull-backs.

(ii) If F and G are ptyxes of type 2, $I^2(F,G)$ is the set of all natural transformations from F to G.

(iii) The category $\mathbf{PT}^2$ is defined by:

- *object*: ptyxes of type 2.
- *morphisms*: $I^2(F,G)$.

12.1.2. Example.
The functor:

$$F \rightsquigarrow (\mathbf{\Lambda}\, F)(0,0)$$

$$T \rightsquigarrow (\mathbf{\Lambda}\, T)(0,0)$$

is a typical ptyx of type 2.

12.1.3. Remark.
$I^2(F,G)$ is a set because a natural transformation from F to G is uniquely determined by its restriction to finite dimensional dilators, which form a denumerable set... .

12.1.4. Theorem (normal form theorem).
Assume that F, D, x are of respective type 2, 1, 0 (i.e. ptyx of type 2, type 1 (dilator), type 0 (ordinal)), and that $x < F(D)$; then is is possible to find a finite dimensional D_0, $T \in I^1(D_0, D)$ such that

(i) $\quad x \in \mathsf{rg}\Big(F(T)\Big)$.

Furthermore, assume that $\mathsf{rg}\Big(\mathsf{Tr}(T)\Big)$ is minimal for inclusion among all solutions of (i); then D_0, T, and x_0 such that $x = F(T)(x_0)$ are uniquely determined.

Proof. Express D as a direct limit: $(D, T_i) = \varinjlim\, (D_i, T_{ij})$ with all D_i's finite dimensional; then $\Big(F(D), F(T_i)\Big) = \varinjlim\, \Big(F(D_i), F(T_{ij})\Big)$, hence $\exists i \in I$ s.t. $x \in \mathsf{rg}\Big(F(T_i)\Big)$. (i) is therefore satisfied. Moreover, if $x \in \mathsf{rg}\Big(F(T')\Big) \cap \mathsf{rg}\Big(F(T'')\Big)$, then $x \in F(T' \wedge T'')$, and this shows that, if $\mathsf{rg}\Big(\mathsf{Tr}(T)\Big)$ is minimal s.t. (i) holds, it is also minimum ... from that unicity follows. $\square$

12.1.5. Exercise.
Assume that F is a functor from **DIL** to **ON** enjoying Theorem 12.1.4; then show that F is a ptyx of type 2.

12.1.6. Notation.
Assume that $x = F(T)(x_0)$, and that $\mathsf{rg}\Big(\mathsf{Tr}(T)\Big)$ is minimum with this property; then we denote this situation by:

$$x = (x_0\,;\, D_0\,;\, T\,;\, D)_F\ .$$

(This notation is analogous to the familiar $z = (z_0\,;\, x_0, ..., x_{n-1}\,;\, x)_F$ of dilators. $x_0, ..., x_{n-1}$ encode an element of $I(n, x)$; the source n of this morphism is very conspicuous, and we have no need to indicate it more specifically; in the type 2 case, it is better to record $D_0 \ldots$.)

12.1.7. Definition.
The following data define a category $\mathbf{SET}_i$:

(i) *objects*: sets.

(ii) *morphisms*: injective functions.

12.1.8. Proposition.

(i) If (x, f_i) enjoys 8.1.11 (i)–(iii) w.r.t. (x_i, f_{ij}) in $\mathbf{SET}_i$, then

$$(x, f_i) = \lim_{\longrightarrow} (x_i, f_{ij}) \leftrightarrow x = \bigcup_{i \in I} \mathsf{rg}(f_i) \ .$$

(ii) If f_1, f_2, f_3 enjoy 8.1.24 (i) in $\mathbf{SET}_i$, then

$$f_1 \wedge f_2 = f_3 \leftrightarrow \mathsf{rg}(f_1) \cap \mathsf{rg}(f_2) = \mathsf{rg}(f_3) \ .$$

Proof. Left to the reader. □

12.1.9. Definition.
We define a functor Tr from $\mathbf{PT}^2$ to $\mathbf{SET}_i$:

$$\mathsf{Tr}(F) = \{(x_0, D_0)\,;\, x_0 = (x_0\,;\, D_0\,;\, \mathbf{E}^1_{D_0}\,;\, D_0)_F\}$$

$$\mathsf{Tr}(T)\left((x_0, D_0)\right) = (T(D_0)(x_0).D_0) \ .$$

12.1.10. Lemma.
$\mathsf{Tr}(T)$ is an injective function from $\mathsf{Tr}(F)$ to $\mathsf{Tr}(G)$, when $T \in I^2(F, G)$.

Proof. The non trivial part of the lemma is the fact that $\mathsf{Tr}(T)$ maps $\mathsf{Tr}(F)$ into $\mathsf{Tr}(G)$: but observe that given $U \in I^1(D, D')$, it is possible to find D'' together with $U_1, U_2 \in I^1(D', D'')$ such that: $U_1 \wedge U_2 = U_1 U = U_2 U$.

(Proof. Let $D''(x) = 2D'(x)$, $D''(f) = \mathbf{E}_2 D'(f)$, and define $U_1(x)(z) = 2z$; $U_2(x)(z) = 2z$ if $z \in \mathsf{rg}\Big(T(x)\Big)$, $=\ 2z+1$ otherwise.... $\square$)

Consider the diagram

$$\begin{tikzcd}
F(D) \arrow[r, "F(U)"] \arrow[d, "T(D)"'] & F(D') \arrow[r, shift left, "F(U_1)"] \arrow[r, shift right, "F(U_2)"'] \arrow[d, "T(D')"'] & F(D'') \arrow[d, "T(D'')"] \\
G(D) \arrow[r, "G(U)"'] & G(D') \arrow[r, shift left, "G(U_1)"] \arrow[r, shift right, "G(U_2)"'] & G(D'')
\end{tikzcd}$$

and assume that $z' \in \mathsf{rg}\Big(T(D')\Big) \cap \mathsf{rg}\Big(G(U)\Big)$; then $G(U_1)(z') = G(U_2)(z')$ (because $U_1U = U_2U$), hence the point $z'' = G(U_1)(z')$ belongs to $\mathsf{rg}\Big(T(U_1)\Big)$ $\cap \mathsf{rg}\Big(T(U_2)\Big)$: if $z'' = T(D'')(z''_1)$, we can write $z''_1 = F(U_1)(z'_1) = F(U_2)(z'_1)$, where z'_1 is s.t. $z' = T(D')(z'_1)$. Now, since F preserves pull-backs, $F(U_1) \wedge F(U_2) = F(U_1U) = F(U_1)F(U)$: the pint z''_1 belongs to $\mathsf{rg}\Big(F(U_1) \wedge F(U_2)\Big)$, hence it is of the form $F(U_1)F(U)(z_1)$: $z'_1 = F(U)(z_1)$. This establishes the lemma because, assume that $z'_1 \in F(D')$ is such that $z'_1 \notin \mathsf{rg}\Big(F(U')\Big)$ for all U' whose target is D', $U' \neq \mathbf{E}_{D'}$, i.e. $(z'_1, D') \in \mathsf{Tr}(F)$; then $z' \in G(D')$. If $z' \in \mathsf{rg}\Big(G(U)\Big)$, then we know that $z'_1 = F(U)(z_1)$, hence $U = \mathbf{E}_{D'}$. So $(z', D') \in \mathsf{Tr}(G)$. $\square$

12.1.11. Theorem.
The functor **trace** has the following properties:

(i) Assume that (a, t_i) enjoys 8.1.11 (i)–(iii) w.r.t. (a_i, t_{ij}); then

$$(a, t_i) = \underset{\longrightarrow}{\lim}\ (a_i, t_{ij}) \leftrightarrow \Big(\mathsf{Tr}(a), \mathsf{Tr}(t_i)\Big) = \underset{\longrightarrow}{\lim}\ \Big(\mathsf{Tr}(a_i), \mathsf{Tr}(t_{ij})\Big)\ .$$

(ii) Assume that t_1, t_2, t_3 enjoy 8.1.24 (i), then

$$t_1 \wedge t_2 = t_3 \leftrightarrow \mathsf{Tr}(t_1) \wedge \mathsf{Tr}(t_2) = \mathsf{Tr}(t_3)\ .$$

(iii) If $X \subset \mathsf{Tr}(a)$, there exist *unique* b and $t \in I^2(b, a)$ s.t. $X = \mathsf{rg}\Big(\mathsf{Tr}(t)\Big)$.

<u>Proof</u>. (iii): Assume $X \subset \mathsf{Tr}(a)$; if D is a dilator, we consider the subsets $X_D \subset a(D)$ consisting of those $z \in a(D)$ of the form

$$(z_0\,;\,D_0\,;\,T\,;\,D)_a\ , \qquad \text{for some } (z_0, D_0) \in X\ ,\quad T \in I^1(D_0, D)\ .$$

Then define $b(D) = \|X_D\|$, and if $U \in I^1(D, D')$, $b(U)$ to be the function making the diagram:

$$\begin{tikzcd} b(D) \arrow[r, "T(D)"] \arrow[d, "b(U)"'] & a(D) \arrow[d, "a(U)"] \\ b(D') \arrow[r, "t(D')"'] & a(D') \end{tikzcd}$$

commutative (with $\mathsf{rg}\Big(t(D)\Big) = X_D ...$). One easily checks that b is a functor from **DIL** to **ON**, and that t is a natural transformation from b to a. In order to show that b is a ptyx, it suffices by 12.1.5, to show that b enjoys a normal form theorem; this is immediate. The unicity of b and t is trivial.

(i): Assume that $(a, t_i) = \underset{\longrightarrow}{\lim}\ (a_i, t_{ij})$, and consider the subset $X = \bigcup_{i \in I} \mathsf{rg}\Big(\mathsf{Tr}(t_i)\Big)$ of $\mathsf{Tr}(a)$; define b and $u \in I^2(b, a)$ by $\mathsf{rg}\Big(\mathsf{Tr}(u)\Big) = X$. Now, if we apply condition (iii) to $Y_i = \mathsf{Tr}(u)^{-1}\Big(\mathsf{rg}\Big(\mathsf{Tr}(t_i)\Big)\Big)$, we get $u_i \in I(a'_i, b)$ s.t. $\mathsf{rg}\Big(\mathsf{Tr}(u_i)\Big) = Y_i$. By unicity, we obtain $a'_i = a_i$ and $uu_i = t_i$. Since $(a, t_i) = \underset{\longrightarrow}{\lim}\ (a_i, t_{ij})$ one can find $v \in I^2(a, b)$ s.t. $vt_i = u_i$.

(<u>Proof</u>. Show, using (iii), that $u_j t_{ij} = u_i$. $\square$

Hence, $uvt_i = t_i$ for all i. uv must be the identity: $\mathsf{Tr}(u)\mathsf{Tr}(v)(z) = z$ for all $z \in \mathsf{Tr}(a)$. Since u and v are injective, this forces $\mathsf{Tr}(u)$ and $\mathsf{Tr}(v)$ to be bijections: but necessarily $X = \mathsf{Tr}(a)$ and we have therefore shown that $\Big(\mathsf{Tr}(a), \mathsf{Tr}(t_i)\Big) = \underset{\longrightarrow}{\lim}\ \Big(\mathsf{Tr}(a_i), \mathsf{Tr}(t_{ij})\Big)$. Conversely, assume that $\Big(\mathsf{Tr}(a), \mathsf{Tr}(t_i)\Big) = \underset{\longrightarrow}{\lim}\ \Big(\mathsf{Tr}(a_i), \mathsf{Tr}(t_{ij})\Big)$; if D is any dilator, then

$$\Big(a(D), t_i(D)\Big) = \underset{\longrightarrow}{\lim}\ \Big(a_i(D), t_{ij}(D)\Big).$$

(<u>Proof</u>. If $z \in a(D)$, write $z = (z_0\,;\,D_0\,;\,T\,;\,D)_a$, and choose i such that

$(z_0\,;\,D_0) \in \mathsf{rg}\Big(\mathsf{Tr}(t_i)\Big)$; then $z \in \mathsf{rg}\Big(t_i(D)\Big)$. $\square$)

Now observe that:

12.1.12. <u>Lemma</u>.
If $\Big(a(D), t_i(D)\Big) = \underset{\longrightarrow}{\lim}\ \Big(a_i(D), t_{ij}(D)\Big)$ for all $D \in \mathbf{DIL}$, then $(a, t_i) = \underset{\longrightarrow}{\lim}\ (a_i, t_{ij})$.

<u>Proof</u>. If (b, u_i) is any family enjoying 8.1.11 (i)–(iii) w.r.t. (a_i, t_{ij}), then given any D, D' and $T \in I^1(D, D')$, the family $\Big(b(D'), u_i(T)\Big)$ enjoys 8.1.11 (i)–(iii) w.r.t. $\Big(a_i(D), t_{ij}(D)\Big)$. By 8.1.11 (iv), there is a unique morphism $u(T) \in I\Big(a(D), b(D')\Big)$ such that $u_i(T) = u(T)t_i(T)$ for all $i \in I$. Using the unicity of $u(T)$, we easily get: $u(T)a(U) = b(T)a(U)$ when $U \in I^1(D'', D)$. From that $\Big(u(\mathbf{E}^1_D)\Big)_{D \in \mathbf{DIL}}$ defines an element of $I^2(a, b)$, which is the only solution of $u_i = ut_i$ $(i \in I)$. $\square$

The lemma concludes the proof of (i).

(ii): Assume that $t_i \in I(a_i, b)$ $(i = 1, 2, 3)$, and that t_{31} and t_{32} are such that: $t_3 = t_1 t_{31} = t_2 t_{32}$; in particular $\mathsf{Tr}(t_3) = \mathsf{Tr}(t_1)\mathsf{Tr}(t_{31}) = \mathsf{Tr}(t_2)\mathsf{Tr}(t_{32})$; hence $\mathsf{rg}\Big(\mathsf{Tr}(t_3)\Big) \subset \mathsf{rg}\Big(\mathsf{Tr}(t_1)\Big) \cap \mathsf{rg}\Big(\mathsf{Tr}(t_2)\Big) = X$. We apply (iii) and we obtain a'_3 and $t'_3 \in I^2(a'_3, b)$ such that $\mathsf{rg}\Big(\mathsf{Tr}(t'_3)\Big) = X$. Now, if we consider $Y_1 \subset t_i(a_1)$: $Y_1 = \mathsf{Tr}(t_1)^{-1}(X)$, by (iii) we obtain $t'_{31} \in I^2(a''_3, a_1)$ such that $\mathsf{rg}\Big(\mathsf{Tr}(t'_{31})\Big) = Y_1$; but $\mathsf{rg}\Big(\mathsf{Tr}(t_1)\mathsf{Tr}(t'_{31})\Big) = X$, hence $a''_3 = a'_3$ and $t_1 t'_{31} = t'_3$; in the same way, we get $t'_{32} \in I^2(a'_3, a_2)$ s.t. $t_2 t'_{32} = t'_3$.

Now assume that $t_1 \wedge t_2 = t_3$; by 8.1.24 (ii), one can find $u \in I^2(a'_3, a_3)$ s.t. $t'_{31} = t_{31}u$, $t'_{32} = t_{32}u$; hence $t'_3 = t_3 u$; if $a'_3 \neq a_3$, the injective function $\mathsf{Tr}(u)$ cannot be surjective, if $z \notin \mathsf{rg}\Big(\mathsf{Tr}(u)\Big)$, then $\mathsf{Tr}(t_3)(z) \notin \mathsf{rg}\Big(\mathsf{Tr}(t'_3)\Big) = X$, but $\mathsf{rg}\Big(\mathsf{Tr}(t_3)\Big) = X$, contradiction. Hence $X = \mathsf{Tr}(b)$. We have therefore established that $\mathsf{Tr}(t_1) \wedge \mathsf{Tr}(t_2) = \mathsf{Tr}(t_3)$.

Conversely assume that $\mathsf{Tr}(t_1) \wedge \mathsf{Tr}(t_2) = \mathsf{Tr}(t_3)$. If D is a dilator, it is immediate (using the normal form theorem) that $t_1(D) \wedge t_2(D) = t_3(D)$; we use now the

12.1.13. <u>Lemma</u>.

If $T_1(D) \wedge T_2(D) = T_3(D)$ for all $D \in \mathbf{DIL}$, then $T_1 \wedge T_2 = T_3$.

Proof. Easy result, similar to 12.1.12. $\square$

This concludes the proof of (ii) and establishes the theorem. $\square$

12.1.14. Corollary.
Assume that (A_i, T_{ij}) is a direct system in $\mathbf{PT}^2$, and that (A, T_i) enjoys 8.1.11 (i)–(iii) w.r.t. (A_i, T_{ij}); then the following are equivalent:

(i) $(A, T_i) = \varinjlim (A_i, T_{ij})$.

(ii) $\big(A(D), T_i(D)\big) = \varinjlim \big(A_i(D), T_{ij}(D)\big)$ for all dilators D.

(iii) $\big(A(D), T_i(D)\big) = \varinjlim \big(A_i(D), T_{ij}(D)\big)$ for all finite dimensional $D \in \mathbf{DIL}$.

(iv) $\big(A(D), T_i(U_i)\big) = \varinjlim \big(A_i(D_i), T_{ij}(U_{ij})\big)$ for all direct systems (D_i, U_{ij}) and all $D, U_i)$ in $\mathbf{DIL}$ s.t. $(D, U_i) = \varinjlim (D_i, U_{ij})$.

Proof. The implications (iv) $\rightarrow$ (ii) $\rightarrow$ (iii) are trivial.

(iii) $\rightarrow$ (i): If $z \in \mathsf{Tr}(A)$, write $z = (z_0, D_0)$; hence $z_0 \in A(D_0)$; hence for some $i \in I$, $z_0 \in \mathsf{rg}\big(T_i(D)\big)$, i.e. $(z_0, D_0) \in \mathsf{rg}\big(\mathsf{Tr}(T_i)\big)$.

(i) $\rightarrow$ (iv): If $z \in A(D)$, write $z = (z_0\,;\, D_0\,;\, U_0\,;\, D)_A$; choose i such that:

– $(z_0\,;\, D_0) \in \mathsf{rg}\big(\mathsf{Tr}(T_i)\big)$, i.e. $z_0 = T_i(D_0)(z_0')$.

– $\mathsf{rg}\big(\mathsf{Tr}(U_0)\big) \subset \mathsf{rg}\big(\mathsf{Tr}(U_i)\big)$; in particular, $U_0 = U_i U_0'$.

Hence $z = (z_0\,;\, D_0\,;\, U_i U_0'\,;\, D)_A = A(U_i)\big((z_0\,;\, D_0\,;\, U_0'\,;\, D_i)\big)_A$ and $(z_0\,;\, D_0\,;\, U_0'\,;\, D_i)_A = T_i(D_0)\big((z_0'\,;\, D_0\,;\, U_0'\,;\, D_i)_{A_i}$, and $z = T_i(U_i)\big((z_0'\,;\, D_0\,;\, U_0'\,;\, D_i)_{A_i}$. $\square$

12.1.15. Remark.
It is of some interest of give explicit conditions ensuring the existence of a direct limit for (A_i, T_{ij}) in $\mathbf{PT}^2$.

(i) The existence of the direct limit of $\big(A_i(D), T_{ij}(D)\big)$ for all dilators D is a sufficient condition; the limit functor $\varinjlim^* \big(A_i(D), T_{ij}(D)\big) = A(D)$, $\varinjlim \big(A_i(U)\big) = A(U)$ is easily shown to enjoy the normal form theorem, and then it is the direct limit by the preceding results... .

(ii) In particular, as soon as 8.1.11 (i)–(iii) is satisfied by some (A, T_i), then the system has a direct limit.

12.1.16. Corollary.
Assume that $T_i \in I^2(A_i, B)$ $(i = 1, 2, 3)$; then the following are equivalent:

(i) $T_1 \wedge T_2 = T_3$.

(ii) $T_1(D) \wedge T_2(D) = T_3(D)$ for all dilators D.

(iii) $T_1(D) \wedge T_2(D) = T_3(D)$ for all finite dimensional dilators D.

(iv) $T_1(U_1) \wedge T_2(U_2) = T_3(U_3)$ for all morphisms U_1, U_2, U_3 in **DIL** such that $U_1 \wedge U_2 = U_3$.

Proof. (iv) $\to$ (ii) $\to$ (iii) is trivial.

(iii) $\to$ (i): If $(z_0, D_0) \in \mathsf{rg}\big(\mathsf{Tr}(T_1)\big) \cap \mathsf{rg}\big(\mathsf{Tr}(T_2)\big)$, then $z_0 \in \mathsf{rg}\big(T_1(D_0)\big) \cap \mathsf{rg}\big(T_2(D_0)\big) = \mathsf{rg}\big(T_3(D_0)\big)$, since D_0 is finite dimensional. Hence $(z_0, D_0) \in \mathsf{rg}\big(\mathsf{Tr}(T_3)\big)$. Conversely, from $(z_0, D_0) \in \mathsf{rg}\big(\mathsf{Tr}(T_3)\big)$, we get $z_0 \in \mathsf{rg}\big(T_1(D_0)\big) \cap \mathsf{rg}\big(\mathsf{Tr}(D_0)\big)$, hence $(z_0, D_0) \in \mathsf{rg}\big(\mathsf{Tr}(T_1)\big) \cap \mathsf{rg}\big(\mathsf{Tr}(T_2)\big)$.

(i) $\to$ (iv): Assume that $z \in B(D)$ $(U_i \in I^1(D_i, D))$; write $z = (z_0\,;\, D_0\,;\, U\,;\, D)_B$; then it is immediate that $z \in \mathsf{rg}\big(T_i(U_i)\big) \leftrightarrow (z_0, D_0) \in \mathsf{rg}\big(\mathsf{Tr}(T_i)\big) \wedge \mathsf{rg}\big(\mathsf{Tr}(U)\big) \subset \mathsf{rg}\big(\mathsf{Tr}(U_i)\big)$. From this, we easily obtain, from $T_1 \wedge T_2 = T_3$, $U_1 \wedge U_2 = U_3$, that $T_1(U_1) \wedge T_2(U_2) = T_3(U_3)$. $\square$

12.1.17. Remark.
Of course, in $\mathbf{PT}^2$, pull-backs are always defined.

12.1.18. Theorem.
Assume that $T_1 \in I^2(A, B_1)$, $T_2 \in I^2(A, B_2)$; then it is possible to find B, together with $U_1 \in I^2(B_1, B)$, $U_2 \in I^2(B_2, B)$ rendering the diagram

$$\begin{tikzcd} & B_1 \arrow[dr, "U_1"] & \\ A \arrow[ur, "T_1"] \arrow[dr, "T_2"'] & & B \\ & B_2 \arrow[ur, "U_2"'] & \end{tikzcd}$$

commutative. We use the notation:

$$(T, B) = (T_1, B_1) + (T_2, B_2)\ , \quad \text{with } T = U_1T_1 = U_2T_2\ .$$

(In fact we have slightly more:

(i) $\mathsf{rg}\Big(\mathsf{Tr}(U_1)\Big) \cup \mathsf{rg}\Big(\mathsf{Tr}(U_2)\Big) = \mathsf{Tr}(B)$.

(ii) $\mathsf{rg}\Big(\mathsf{Tr}(U_1)\Big) \cap \mathsf{rg}\Big(\mathsf{Tr}(U_2)\Big) = \mathsf{rg}\Big(\mathsf{Tr}(T)\Big)$, i.e. $U_1 \wedge U_2 = T$.)

Proof, If D is a dilator, we define $B(D)$ as follows: we introduce the disjoint union X of $B_1(D)$ and $B_2(D)$: $X = \{(i, z)\,;\, i = 1 \wedge z < B_1(D) \vee i = 2 \wedge z < B_2(D)\}$. We preorder X as follows: $(i, z) \ \leq^1_D \ (i', z')$ iff $(i = i' \wedge z \leq z') \vee \exists z_0 \in A(D)\, (z \leq T_i(D)(z_0) \wedge T_{i'}(D)(z_0) \leq z')$. The transitivity of $\leq^1_D$ is immediate. The associated equivalence relation $\sim_D$ is defined by: $(i, z) \ \sim_D \ (i', z')$ iff $(i = i' \wedge z = z') \vee \exists z_0 \in A(D) \Big(z = T_i(D)(z_0) \wedge z' = T_{i'}(D)(z_0)\Big)$. A first trivial remark is that $\leq^1_D / \sim_D$ is a well-founded order relation, but not a linear order in general... .

Now we consider the preorder $\leq^2_D$, defined by: $(i, z) \ \leq^2_D \ (i', z') \leftrightarrow \exists D'\ \exists t' \in I^1(D, D')\, \Big(i, B_i(t')(z)\Big) \ \leq^1_{D'} \ \Big(i', B_{i'}(t')(z')\Big)$.

Observe that $\leq^2_D$ is transitive: iff $(i', B_{i'}(t'')(z')\Big) \leq^1_{D''} \Big(i'', B_{i''}(t'')(z'')\Big)$, choose D''', together with $u' \in I^1(D', D''')$, $u'' \in I^1(D'', D''')$ s.t. $u't' = u''t'' = t$ (8.G.13). Then $\Big(i, B_i(t)(z)\Big) \leq^1_{D'''} \Big(i', B_{i'}(t)(z')\Big) \leq^1_{D'''} \Big(i'', B_{i''}(t)(z'')\Big)$ hence $(i, z) \leq^2_D (i'', z'')$. $\leq^2_D$ has another interesting property, namely

$(i,z)\ \leq^2_D\ (i',z') \leftrightarrow \Big(i, B_i(t)(z)\Big)\ \leq^2_{D'}\ \Big(i', B_{i'}(t)(z')\Big)$ for all D' and $t \in I^1(D.D')$; it is precisely in order to get this property that we replace $\leq^1_D$ by $\leq^2_D$ ... . Finally, observe that $\leq^2_D$ is a preorder with associated equivalence $\sim_D$, and that $\leq^2_D / \sim_D$ is well-founded (trivial).

We define linear preorders $\leq^3_D$ and $\leq^4_D$ as follows:

$$(i,z)\ \leq^3_D\ (i',z') \leftrightarrow (i.z)\ \leq^1_D\ (i',z') \vee \Big((i',z')\ \not\leq^1_D\ (i,z) \wedge i < i'\Big)$$

$$(i,z)\ \leq^4_D\ (i',z') \leftrightarrow (i.z)\ \leq^2_D\ (i',z') \vee \Big((i',z')\ \not\leq^2_D\ (i,z) \wedge i < i'\Big)\ .$$

12.1.19. <u>Lemma</u>.

(i) $\leq^3_D / \sim_D$ is a well-order.

(ii) $\leq^4_D / \sim_D$ is a well-order.

<u>Proof</u>. (i): If we show that $\leq^3_D$ is transitive, then it will be a linear preorder extending $\leq^1_D$, and $\leq^3_D / \sim_D$ will be a well-order. Assume therefore that $(i,z)\ \leq^3_D\ (i',z')\ \leq^3_D\ (i'',z'')$:

1. If $(i,z)\ \leq^1_D\ (i',z')\ \leq^1_D\ (i'',z'')$, then $(i,z)\ \leq^1_D\ (i'',z'')$.

2. If $(i,z)\ \leq^1_D\ (i',z')$, $(i'',z'')\ \not\leq^1_D\ (i',z')$ and $i' < i''$: $(i'',z'')\ \leq^1_D\ (i',z')$ is impossible, so the result is immediate when $i = i'$; if $i \neq i'$, choose z_0 s.t. $z \leq T_i(D)(z_0)$, $T_{i'}(D)(z_0) \leq z'$; necessarily $T_{i''}(D)(z_0) \leq z''$, since $(i'',z'')\ \not\leq^1_D\ (i',z')$; but then $(i,z)\ \leq^1_D\ (i'',z'')$.

3. If $(i',z')\ \not\leq^1_D\ (i,z)$, $i < i'$, and $(i',z')\ \leq^1_D\ (i'',z'')$: symmetric to 2.

4. $(i'',z'')^1\ \not\leq^1_D\ (i',z')\ \not\leq^1_D\ (i,z)$ and $i < i' < i''$ impossible.

(However, let us make the proof when i varies over a linear order: assume that $(i'',z'')\ \leq^1_D\ (i,z)$: if $z'' \leq T_{i''}(D)(z_0)$, $T_i(D)(z_0) \leq z$, then since z' is comparable with $T_{i'}(D)(z_0)$, we obtain

$$z' \leq T_i(D)(z_0) \rightarrow\ (i',z')\ \leq^1_D\ (i,z) \quad \text{impossible}$$

$$T_i(D)(z_0) \leq z' \rightarrow\ (i'',z'')\ \leq^1_D\ (i',z')\ \text{impossible}\,.\,)$$

(ii): Assume that $(i,z) \ \leq^4_D \ (i',z') \ \leq^4_D \ (i'',z'')$; and choose (t_1,D_1), (t_2,D_2), (t_3,D_3) such that $\Big(i, B_i(t_1)(z)\Big) \leq^3_{D_1} \Big(i', B_{i'}(t_1)(z')\Big)$, $\Big(i', B_{i'}(t_2)(z')\Big) \leq^3_{D_2} \Big(i'', B_{i''}(t_2)(z'')\Big)$ and $(i,z) \leq^4_D (i'',z'') \leftrightarrow \Big(i, B_i(t_3)(z)\Big) \leq^3_{D_3} \Big(i'', B_{i''}(t_3)(z'')\Big)$. Define D' and t'_1, t'_2, t'_3 s.t. $t'_1 t_1 = t'_2 t_2 = t'_3 t_3 = t = \mathsf{constant}$. Then $\Big(i, B_i(t)(z)\Big) \leq^3_{D'} \Big(i', B_{i'}(t)(z')\Big) \leq^3_{D'} (i'', B_{i''}(t)(z'')\Big)$, and by case (i) above, $\Big(i, B_i(t)(z)\Big) \ \leq^3_{D'} \ \Big(i'', B_{i''}(t)(z'')\Big)$. Now the choice of D' ("after D_3") is such that one can infer that $(i,z) \ \leq^4_D \ (i'',z'')$. □

We define $B(D) = \leq^4_D / \sim_D$, if $t \in I^1(D,D')$, we define $B(t)$ as follows:

$$B(t)\left(\overline{(i,z)}\right) = \left(\overline{i, B_i(t)(z)}\right) .$$

This defines a strictly increasing function since

$$(i,z) \ \leq^4_D \ (i',z') \leftrightarrow \Big(i, B_i(t)(z)\Big) \ \leq^4_{D'} \ \Big(i', B_{i'}(t)(z')\Big) \ ,$$

which is a consequence of the similar property of $\leq^2_D$ ....

B is a ptyx: it is a functor from **DIL** to **OW** ($\sim$ **ON**), which obviously enjoys the normal form theorem... .

Finally the conditions (i) and (ii) on traces are trivially checked... . □

12.1.20. Exercise.
Extend the result 12.1.18 t a family $T_i \in I^2(A, B_i)$, $i < \alpha$, where $\alpha \in 0n$. (*Hint. The non trivial part is the well-foundedness of the orders* $\leq^1_D$, $\leq^2_D$, $\leq^3_D$, $\leq^4_D$. *The most delicate case is well-foundedness of* $\leq^2_D$: *if* (i_n, z_n) *is a s.d.s. in* $\leq^2_D$, *take* (t_n, D_n) *s.t.* $\Big(i_{n+1}, B_{i_{n+1}}(t_n)(z_{n+1})\Big) \ \leq^1_{D_n} \ \Big(i_n, B_{i_n}(t_n)(z_{n+1})\Big)$; *then apply the analogue of 12.1.20 for dilators... .*)

12.1.21. Example.
A typical example of 12.1.8 is when A is the ptyx $\underline{0}$: $\underline{0}(D) = 0$, $O(t) = \mathbf{E}_0$. Then a pair (T,B) with $T \in I^2(\underline{0}, B)$ can be identified with B, since T is uniquely determined by $\mathsf{rg}\Big(\mathsf{Tr}(T)\Big) = \emptyset$. Then we use the notation $B_1 + B_2$, in such a situation, to denote in fact $(T_1, B_1) + (T_2, B_2)$... .

One would easily show that $B_1 + B_2$ is exactly the composition of B_1, B_2 with the functor sum: $(B_1 + B_2)(D) = B_1(D) + B_2(D)$; $(B_1 + B_2)(t) = B_1(t) + B_2(t)$.

By the way, let us introduce the notation: $\mathbf{E}^2_{BB'}$, when B' is of the form $B + B''$: $\mathbf{E}^2_{BB'}(D) = \mathbf{E}_{B(D)B'(D)} \cdot \mathbf{E}^2_{BB'} \in I^2(B, B')$. As usual we shall denote by $\mathbf{E}^2_B = \mathbf{E}^2_{BB}$ the identity of B. Finally, if $T_1 \in I^2(A_1, B_1)$, $T_2 \in I^2(A_2, B_2)$, $T_1 + T_2 \in I^2(A_1 + A_2, B_1 + B_2)$ can be defined by $(T_1 + T_2)(D) = T_1(D) + T_2(D)$.... .

12.1.22. Definition.
If D, D' are dilators, then $I^1(D, D')$ is ordered by: $T \leq T' \leftrightarrow \forall x\ T(x) \leq T'(x)$ (i.e. $\forall x\ \forall z < D(x)\ T(x)(z) \leq T'(x)(z)$).

12.1.23. Theorem.
Assume that $T, U \in I^1(D, D')$; then $T \leq U$ iff $\exists D'' \in$ **DIL** $\exists V \in I^1(D', D'')$
$\exists W \in I^1(D'', D'')$: $VU = WVT$.

Proof. The condition is obviously sufficient, for if $VU = WVT$, then $(VU)(x)$
$(z) = (WVT)(x)(z)$, i.e. $V(x)U(x)(z) = W(x)V(x)T(x)(z)$, and since $W(x)(z') \geq z'$ for all z', we get: $V(x)U(x)(z) \geq V(x)T(x)(z)$, hence $U(x)(z) \geq T(x)(z)$. The converse is not easy; we follow [5], Chapter 4. Let x be an ordinal; we consider the set $D'(x) \times I\!N$, with the following preorder $\leq^1_x$: $(a, n) \leq^1_x (b, m) \leftrightarrow \exists (a_0, n_0), ..., (a_{p-1}, n_{p-1})$ s.t. $\forall i < p - 1$:

1. either $n_i = n_{i+1}$ and $a_i < a_{i+1}$
2. or $n_i = n_{i+1} + 1$ and for some z, $a_{i+1} = U(x)(z)$, $a_i = T(x)(z)$.
3. or $n_{i+1} = n_i + 1$ and for some z, $a_i = U(x)(z)$, $a_{i+1} = T(x)(z)$.

One checks without difficulty that

– if $n = m$ then only step 1 is needed (once!).

– if $n \geq m$ then only steps 1 and 3 are needed.

– if $n \leq m$ then only steps 1 and 3 are needed.

And that the associated equivalence $\sim_x$ corresponds to the identification of $(U(x)(z), n)$ with $(T(x)(z), n + 1)$.

$\leq_x^1 / \sim_x$ is well-founded.

(Proof. $\leq_x^1$ has the following essential property: if $(a,n) \leq_x^1 (b,m)$, if $(b,m) \not\leq_x^1 (a,n)$, if $n \geq m$, then $a < b$. This is the only point were $T \leq U$ is used!

Then given a s.d.s. (a_p, n_p) in $\leq_x^1$,

- either the values n_p are unbounded, and we can assume as well that $n_{p+1} > n_p$ for all p; then by the remark just made, (a_p) is a s.d.s. in $D'(x)$.

- or the values n_p are bounded, and we can assume as well that $n_p = \mathsf{constant}$; then (a,p) is still a s.d.s. in $D'(x)$. $\square$

We introduce now a new preorder $\leq_x^2$ on the same set:

$$(a,n) \leq_x^2 (b,m) \leftrightarrow \exists y \, \exists f \in I(x,y) \, (D'(f)(a), n) \leq_y^1 (D'(f)(b), m) \; .$$

This order is shown to be transitive, by using the methods of 12.1.18 (for the order $\leq_x^2$ of this proof); this uses the analogue of 12.1.18 for ordinals, which is trivial. Observe that $\leq_x^2$ is a preorder with associated equivalence $\sim_x$, and that $\leq_x^2 / \sim_x$ is well-founded.

(Proof. As for $\leq_x^1 / \sim_x$. $\square$)

We introduce $\leq_x^3$ and $\leq_x^4$ by:

$$(a,n) \leq_x^3 (b,m) \leftrightarrow (a,n) \leq_x^1 (b,m) \vee \Big((b,m) \not\leq_x^1 (a,n) \wedge n < m\Big)$$

$$(a,n) \leq_x^4 (b,m) \leftrightarrow (a,n) \leq_x^2 (b,m) \vee \Big((b,m) \not\leq_x^2 (a,n) \wedge n < m\Big) \; .$$

12.1.24. Lemma.

(i) $\leq_x^3 / \sim_x$ is a well-order.

(ii) $\leq_x^4 / \sim_x$ is a well-order.

Proof. (i): We check transitivity:

1. $(a,n) \leq_x^1 (b,m) \leq_x^1 (c,p)$ trivial: $(a,n) \leq_x^1 (c,p)$.

2. $(a,n) \ \leq^1_x \ (b,m)$, $(c,p) \ \not\leq^1_x \ (b,m)$, $m < p$; then $(c,p) \ \not\leq^1_x \ (a,n)$, and either $n < p$ and we are done, or $n \geq p$l in that case $m < n$, hence $\exists(d,p)$ s.t. $(a,n) \ \leq^1_x \ (d,p) \ \leq^1_x \ (b,m)$; necessarily $c > d$ (otherwise $(c,p) \ \leq^1_x \ (b,m)$). Hence $(a,n) \ \leq^1_x \ (d,p)$.

3. $(b,m) \ \leq^1_x \ (c,p)$, $(b,m) \ \not\leq^1_x \ (a,n)$, $n < m$: symmetric to 2.

4. $(c,p) \not\leq^1_x (b,m) \not\leq^1_x (a,n)$ and $n < m < p$; we show that $(c,p) \not\leq^1_x (a,n)$: otherwise, there would be a (d,m) s.t. $(c,p) \ \leq^1_x \ (d,m) \ \leq^1_x \ (a,n)$, and:

$$d \leq b \to (c,p) \ \leq^1_x \ (b,m) \quad \text{absurd}$$

$$b \leq d \to (b,m) \ \leq^1_x \ (a,n) \quad \text{absurd} .$$

$\leq^3_x \ / \sim -x$ is a linear order; in order to check its well-foundedness, consider a s.d.s. (a_p, n_p) and partition $\{(p,q)\,;\, p < q\}$ into X_0 and X_1:

$$(p.q) \in X_0 \leftrightarrow (a_q, n_q) \ \leq^1_x \ (a_p, n_p) \ .$$

By Ramsey's theorem for pairs, we get an infinite homogeneous subset Y: if for all $p, q \in Y$, $p < q$, then $(p,q) \in X_0$, we contradict the well-foundedness of $\leq^1_x \ / \sim -x$; if for all $p, q \in Y$, $p < q$, then $(p,q) \in X_1$, we obtain a s.d.s. of integers... .

(ii): The transitivity of $\leq^4_x$ is derived from the transitivity of $\leq^3_x$ exactly as in 12.1.19 (ii). The well-foundedness of $\leq^4_x$ is exactly proved as in (i).□

The construction is such that

$$(a,n) \ \leq^4_x \ (b,m) \leftrightarrow (D'(f)(a), n) \ \leq^4_y \ (D'(f)(b), m)$$

see 12.1.18 for a justification in a close context...). Hence define $D''(x) = \leq^4_x \ / \sim_x$, and $D''(f)\big(\overline{(a,n)}\big) = \overline{(D'(f)(a), n)}$. It is immediate that D'' is a dilator. Finally define

$$V(x)(a) = \overline{(a,0)}$$

$$W(x)\big(\overline{(a,n)}\big) = \overline{(a, n+1)} \ .$$

Then $V(x)U(x)(z) = \overline{(U(x)(z), 0)} = \overline{(T(x)(z), 1)}$ whereas $W(x)V(x)T(x)(z) = W(x)\,\overline{(T(x)(z), 0)} = \overline{(T(x), (z), 1)}$. Hence $VU = WVT$. □

12.1.25. Theorem.
If A is a ptyx of type 2, then A preserves the ordering of morphisms: $T \leq U \to A(T) \leq a(U)$.

Proof. Trivial consequence of 12.1.23. □

12.1.26. Exercise.

(i) Define an ordering between morphisms in $I^2(A, B)$; show that if $T, U \in I^2(A, B)$, $t, u \in I^1(A, b)$, $T \leq U$, $t \leq u$, then $T(t) \leq U(u)$.

(ii) Prove the analogue of 12.1.23 for the category $\mathbf{PT}^2$.
(*The proof 12.1.23 can be extended without any problem.*)

12.1.27. Comment.
The importance of 12.1.25 lies in technical applications to the characterization of finite dimensional ptykes of type 2. However, there is another interest, located in the normal form theorem. It is possible to rewrite the normal form theorem (of type 2) as follows:

$$z = (C\,;\, x_0, ..., x_{n-1}\,;\, D)_A \quad \text{instead of} \quad z = (z_0\,;\, D_0\,;\, T\,;\, D)_A\ .$$

This means that: $C = (z_0\,;\, D_0)$, and that the ordinals $x_0, ..., x_{n-1}$ are defined as follows: let $i_0, ..., i_{n-1}$ be the enumeration of $\mathsf{Tr}(D_0)$ in increasing order modulo $\leq^{D_0}$ (8.4.22); then $\mathsf{Tr}(T)(i_0) = (x_0, p_0), ..., \mathsf{Tr}(T)(i_{n-1}) = (x_{n-1}, p_{n-1})$, for some integers $p_0, ..., p_{n-1}$. These integers are not really part of the dats, since $i_0 = (y_0, p_0), ..., i_{n-1} = (y_{n-1}, 0_{n-1})$ for appropriate $y_0, ..., y_{n-1}$.

We have therefore obtained a denotation which is close to the familiar dilatoral denotations... . There are, however, important differences, essentially:

(i) $x_0, ..., x_{n-1}$ are not necessarily a strictly increasing sequence of ordinals.

(ii) The condition for $(C\,;\,x_0,...,x_{n-1}\,;\,D)_A$ to be a denotation is complicated.

However, a natural question is "are denotations increasing in their coefficients?" In other terms if $(C\,;\,x_0,...,x_{n-1}\,;\,D)_A$ and $(C\,;\,x'_0,...,x'_{n-1}\,;\,D)_A$ are denotations, and $x_0 \leq x'_0,...,x_{n-1} \leq x'_{n-1}$, then do we have: $(C\,;\,x_0,..., x_{n-1}\,;\,D)_A \leq (C\,;\,x'_0,...,x'_{n-1}\,;\,D)_A$? The answer is positive, since, if $x_0,..., x_{n-1}$ encode T, $x'_0,...,x'_{n-1}$ encode T', then $T \leq T' \leftrightarrow x_0 \leq x'_0 \wedge ... \wedge x_{n-1} \leq x'_{n-1}$.

12.1.28. Definition.
A ptyx A of type 2 is **finite dimensional** iff $\mathsf{dim}(A) = \mathsf{card}\Big(\mathsf{Tr}(A)\Big)$ is finite; $\mathsf{dim}(A)$ is the **dimension** of A. $\mathbf{PT}^2_{fd}$ denotes the category of finite dimensional ptyxes of type 2, a full subcategory of $\mathbf{PT}^2$.

12.1.29. Theorem.
Every ptyx of type 2 is a direct limit of finite dimensional ptykes of type 2.

Proof. Let $I = \{i\,;\,i \subset \mathsf{Tr}(A), i \text{ finite}\}$. If $i \in I$, define A_i and $T_i \in I^2(A_i, A)$ by $\mathsf{rg}\Big(\mathsf{Tr}(T_i)\Big) = i$. A_i is finite dimensional, of dimension $\mathsf{card}(i)$. Consider, if $i \subset j$, the set $Y = \mathsf{Tr}(T_j)^{-1}(i) \subset \mathsf{Tr}(A_j)$. Define A'_i and $T_{ij} \in I^1(A'_i, A_j)$ by $\mathsf{rg}\Big(\mathsf{Tr}(T_{ij})\Big) = Y$; then by a unicity argument, $A'_i = A_i$, $T_jT_{ij} = T_i$. (A_i, T_{ij}) is easily shown to be a direct system, moreover, $\mathsf{Tr}(A) = \bigcup_{i\in I} \mathsf{rg}\Big(\mathsf{Tr}(T_i)\Big)$, hence $A = \underset{\longrightarrow}{\lim}{}^*\,(A_i, T_{ij})$. □

12.1.30. Definition.
A **preptyx** of type 2 is a functor A from **PIL** to **OL** enjoying the following properties:

(i) A preserves direct limits.

(ii) A preserves pull-backs.

(iii) A preserves the ordering of morphisms: $t \leq u \rightarrow A(t) \leq A(u)$.

The category of preptyxes of type 2 is denoted by $\mathsf{p}\mathbf{PT}^2$.

12.1.31. Theorem.
Preptykes of type 2 are exactly the direct limits of ptykes of type 2 (or finite dimensional ptykes...).

Proof. The theorem rests upon a non trivial property, namely

12.1.32. Theorem.
If A is a finite dimensional preptyx, then A is (isomorphic to) a ptyx.

Proof. Assume that D is a dilator, and that $(z_n)_{n\in\mathbb{N}}$ is a s.d.s. in $A(D)$; we use the normal forms as in 12.1.27: $z_n = (C_n\,;\, x_0^n, ..., x_{p_n-1}^n\,;\, D)_A$. Now, C_n varies through a finite set, hence, one can assume that C_n is constant: $C_n = C$, so $p_n = p$. We define a partition of $\{(n,m)\,;\, n < m\}$ as follows: $(n,m) \in X_i$ if $x_i^m < x_i^n$ and $x_j^m \geq x_j^n$ $\forall j < i < p$. This is a partition, because $x_i^m \geq x_i^n$ for all $i < p$ implies by property (iii) of preptykes $z_m \geq z_n$. Now, by Ramsey's theorem, there is an infinite homogeneous subset, and we obtain $x_{i_0}^m < x_{i_0}^n$ for all $m, n \in Y$, $m < n$; but the points $x_{i_0}^m$ are ordinals (they belong to some $D(n)$!). □

12.1.31 easily follows now at once: if we express A as a direct limit of finite dimensional preptykes, they can be replaced by finite dimensional ptykes. Conversely, 12.1.25 shows that ptykes are preptykes; moreover, preptykes are trivially closed under direct limits... . □

Our next goal is to find a characterization of finite dimensional ptykes. By 12.1.32, it suffices to look for finite dimensional preptykes, i.e. we have reduced the problem to a purely algebraic question.

We shall try to answer the question in the following way: given a functor A from a finite subcategory $\mathcal{C}$ of $\mathbf{DIL}_{fd}$ to $\mathbf{ON} < \omega$ (such a functor can obviously be encoded by an integer, since it consists of finitely many finitistic data), one can ask the questions:

(i) Is A the restriction of a finite dimensional ptyx of type 2?

(ii) Is the extension of A into a f.d. ptyx unique?

We shall answer condition (i) by giving explicit conditions on $\mathcal{C}$ and A; these conditions will be decidable; moreover, if $\mathcal{C}$ is generated in a certain finitary way from the finite set $\mathsf{Tr}(A)$, (ii) will be fulfilled.

This kind of answer is sufficient for the usual purposes; in particular, this is easily extended to ptykes of finite type... . However, we are far from the characterizations obtained for finite dimensional dilators. It is highly improbable that a very simple construction of these f.d. ptykes using permutations ... or something close to it, can be found... . But it may be worthwhile to try.

12.1.33. Proposition.
If A and D are finite dimensional, then $A(D)$ is finite.

Proof. If $n < A(D)$, then the number of possible choices for $n = (z_0\,;\,D_0\,;\,T_0\,;\,D)_A$ is bounded by $N \cdot M$, where N is the number of choices for $(z_0\,;\,D_0)$, M is the number of morphisms with target D, i.e. $\mathsf{dim}(A) \cdot 2^{\mathsf{dim}(D)}$. $\square$

12.1.34. Theorem.
Assume that X is a finite set of finite dimensional dilators; define another finite set X' by (see Exercixe 8.G.14): $D \in X' \leftrightarrow \exists D_1, D_2, D_3 \in X\ \exists T_1, T_2, T_3,\ T_i \in I^1(D_i, X)\ (i = 1, 2, 3)$ s.t. $\mathsf{Tr}(D) = \mathsf{rg}\Big(\mathsf{Tr}(T_1)\Big) \cup \mathsf{rg}\Big(\mathsf{Tr}(T_2)\Big) \cup \mathsf{rg}\Big(\mathsf{Tr}(T_3)\Big)$.

$$\begin{array}{ccccc} D_1 & & & & \\ & & T_1 & & \\ D_2 & T_2 & & & D \\ & & & & \\ & & T_1 & & \\ D_3 & & & & \end{array}$$

Finally define X'' by: $D \in X'' \leftrightarrow \exists D' \in X'\ I^1(D, D') \neq \emptyset$. We introduce the category $\mathcal{C}_{X''}$ by:

objects: elements of X''.
morphisms: $I^1(D, D')$.

Now, assume that A is a functor from $\mathcal{C}_{X''}$ to **ON**, with the following properties:

(i) If $D \in X''$ and $a \in A(D)$, one can express $a = A(T)(z_0)$ for some $D_0 \in X$, $T \in I^1(D_0, D)$, $z_0 \in A(D_0)$; furthermore the condition "$\mathsf{rg}\Big(\mathsf{Tr}(T)\Big)$ minimal" renders T uniquely determined. (In other terms, $a = (z_0\,;\,D_0\,;\,T\,;\,D)_A$, with $D_0 \in X$.

(ii) $A(D)$ is finite for all $d \in X$.

(iii) If $D, D' \in X''$, if $T, U \in I^1(D, D')$, then

$$T \leq U \to A(T) \leq A(U) \ .$$

Then there exists one and only one ptyx F of type 2 such that:

1. $F\lceil \mathcal{C}_{X''} = A$.

2. $\mathsf{Tr}(F) \subset 0n \times x$.

<u>Proof</u>. Assume that D is a dilator; we define $F(D)$ to consist of all formal denotations

$$(z_0\,;\, D_0\,;\, T\,;\, D)_F \ , \qquad \text{with } (z_0\,;\, D_0) \in \mathsf{Tr}(A) \ .$$

In order to compare $(z_0\,;\, D_0\,;\, T\,;\, D)$ with $(z_1\,;\, D_1\,;\, T'\,;\, D)$, we proceed as follows: consider $\mathsf{rg}\Big(\mathsf{Tr}(T)\Big) \cup \mathsf{rg}\Big(\mathsf{Tr}(T')\Big) = Y$; then define D' and $U \in I^1(D', D)$ s.t. $\mathsf{rg}\Big(\mathsf{Tr}(U)\Big) = Y$, then T_1 and T_1' s.t. $T = UT_1$, $T' = UT_1'$:

$$\begin{array}{ccccccc}
D_0 & & & T & & & \\
 & T_1 & & & & & \\
 & & & D' & U & & D \\
 & T_1' & & & & & \\
 & & T_1' & T' & & & \\
D_1 & & & & & &
\end{array}$$

hence $\mathsf{rg}\Big(\mathsf{Tr}(T_1')\Big) \cup \mathsf{rg}\Big(\mathsf{Tr}(T_2')\Big) = \mathsf{Tr}(D')$ so $D' \in X' \subset X''$. By definition: $(z_0\,;\, D_0\,;\, T\,;\, D)_F \leq (z_1\,;\, D_1\,;\, T'\,;\, D)_F$ iff $(z_0\,;\, D_0\,;\, T_1\,;\, D)_A \leq (z_1\,;\, D_1\,;\, T_1'\,;$
$D)_A$. Finally, we define, when $t \in I^1(D, D')$, $F(t) \in I\Big(F(D), F(D')\Big)$ by:

$$F(t)(z_0\,;\, D_0\,;\, T\,;\, D)_F = (z_0\,;\, D_0\,;\, tT\,;\, D')_F \ .$$

This is obviously the only way if defining F, if we want F to enjoy 1 and 2; from that, we get unicity. But is this a ptyx? Preservation of direct limits and pull-backs is immediate from the normal forms, the functions $F(t)$ preserve the order.... But we do not even know that $\leq$ defined on $F(D)$

is an order relation. We must check reflexivity, antisymmetry, transitivity and linearity; these four properties are all handled in the same way, and let us verify transitivity: assume that

$$(z_0\,;\,D_0\,;\,T\,;\,D)_F \leq (z_1\,;\,D_1\,;\,T'\,;\,D)_F \leq (z_2\,;\,D_2\,;\,T''\,;\,D)_F$$

then choose D', together with U, T_1, T_1', T_1'', s.t.

$$\begin{array}{ccccccc}
 & & & D_0 & & & \\
 & T_1 & & & & T & \\
 & & T_1' & & T' & & \\
D' & & & D_1 & & & D \\
 & T_1'' & & & & T'' & \\
 & & & D_2 & & & \\
U & & & & & &
\end{array}$$

is commutative, and $\mathsf{Tr}(D') = \mathsf{rg}\Big(\mathsf{Tr}(T_1)\Big) \cup \mathsf{rg}\Big(\mathsf{Tr}(T_2)\Big) \cup \mathsf{rg}\Big(\mathsf{Tr}(T_3)\Big)$; then $D' \in X' \subset X''$, hence we have:

$$(z_0\,;\,D_0\,;\,T_1\,;\,D')_A \leq (z_1\,;\,D_1\,;\,T_1'\,;\,D')_A \leq (z_2\,;\,D_2\,;\,T_1''\,;\,D')_A\ ,$$

and since $A(D')$ is linearly ordered:

$$(z_0\,;\,D_0\,;\,T_1\,;\,D')_A \leq (z_2\,;\,D_2\,;\,T_1''\,;\,D')_A\ ,$$

from which we conclude that

$$(z_0\,;\,D_0\,;\,T\,;\,D')_F \leq (z_2\,;\,D_2\,;\,T''\,;\,D)_F\ .$$

Finally F is a finite dimensional preptyx, since $\mathsf{Tr}(F) \subset N \times X$, with $N = \mathsf{sup}\,\{A(D)\,;\,D \in X\}$, and F enjoys (iii). By 12.1.32, F is (isomorphic to) a ptyx. □

12.1.35. Remarks.

(i) 12.1.34 extends to natural transformations: assume that A and A' enjoy the conditions of 12.1.34 w.r.t. X, and that T is a natural transformation from A to A', then T can be uniquely extended into a natural transformation T' from F to F', by:

$$T'(D)\Big((z_0\,;\,D_0\,;\,t\,;\,D)_F\Big) = (T(D_0)(Z_0)\,;\,D_0\,;\,t\,;\,D)_{F'}\ .$$

(ii) The question of the characterization of finite dimensional ptykes of type 2 by means of a finite amount of information, is perfectly solved by 12.1.34; moreover, if one encodes these finitary data by integers, then the predicate $P(n)$ which says "n encodes a finite dimensional ptyx of type 2", is prim. rec.

12.1.36. Exercise.

(i) Show that 12.1.33 can be improved into: assume that A is a preptyx, that $\mathsf{Tr}(A) \subset 0n \times X$ for a certain finite $X \subset \mathbf{DIL}_{fd}$, then A is a ptyx.

(ii) Conclude that 12.1.34 holds when condition (ii) is removed. (Of course the dilator F constructed is no longer finite dimensional.)

12.1.37. Discussion.
We now discuss the following concepts:

– **recursive** ptykes of type 2.

– **weakly finite** ptykes of type 2.

(i) As soon as the ptyx A sends finite dimensional dilators on recursive ordinals, it can be expressed as a denumerable direct limit:

$$A = \underset{\substack{\longrightarrow \\ I\!N}}{\lim}{}^{*} (A_n, T_{nm})$$

with all A_n finite dimensional. We shall therefore say that A is **recursive** when we can find such a direct system (A_n, T_{nm}), and when this system can be encoded by a recursive function. This is the widest acceptation of "recursive"; one easily sees that, in this acceptation, there is no distinction between recursive and prim. rec. ptykes!

A more restrictive notion of recursiveness is when A maps finite dimensional dilators on integers; then A is completely determined from its restriction A':

$$A' : \ \mathbf{DIL}_{fd} \to \mathbf{ON} < \omega$$

and A' itself can be uniquely encoded (given specific encodings of the objects and morphisms of $\mathbf{DIL}_{fd}$ and $\mathbf{ON} < \omega$ by integers) by a function f from $I\!N$ to $I\!N$. Of course we can now say that

A is **recursive** iff f is recursive.

A is **prim. rec.** iff f is prim. rec.

This concept is far more natural than the more general one first introduced; when we shall need a concept of recursive ptyx of type 2, we shall therefore use the second version... .

(ii) This choice seems to indicate that we shall define weakly finite ptykes of type 2 as being those A sending $\mathbf{DIL}_{fd}$ into $\mathbf{ON} < \omega$. In fact this notion does not suit very well the practice; we prefer the following definition:
A is **weakly finite** iff A sends weakly finite dilators on integers.

It is easy to produce ptykes sending finite dimensional dilators on integers, but which are not weakly finite. (Example: Consider the functor $A(D) = a_0 + a_1 + ... + a_n + ...$, with $a_n =$ the order type of the $\{z \in D(n) \,;\, (z \,;\, n) \in \mathsf{Tr}(D)\}$, together with $A(T)$ defined to make it a functor. Then A is a ptyx sending finite dimensional dilators on integers, but $A(D)$ is infinite when D is infinite dimensional.)

In practice we shall therefore mainly be concerned with recursive weakly finite ptykes.

(iii) However, I do not want to hide the fact that these definitions are not necessarily the best ones; they are good in a lot of cases, but there is still room for discussion. A typical example of the limitation of our choice of concept is that the functor $A(D) = (\mathbf{\Lambda} D)(0,0)$ is recursive, but not weakly finite. This kind of inadequacy between concepts and practice is very limited but real, and seems to indicate the possibility of further improvements of the notions. For my part, I would surmise that the systematic reformulation of the theory of ptykes by means of ideas similar to the regularity of Boquin (Exercise 9.B.8), will wipe out all these small inconsistencies... .

The "algebraic part" of the theory of ptykes of type 2 has just been devloped; let us now have a look at the non algebraic part, i.e. the question of the definition of a predecessor relation between ptyxes, which makes them some kind of well-founded classes. We shall leave to the reader the straightforward definitions of the sums $\sum\limits_{i<x} A_i$, $\sum\limits_{i<f} T_i$, and of *connectedness.*

12.1.38. Theorem.

(i) If A is a ptyx of type 2, then A can be uniquely written as $\sum\limits_{i<x} A_i$, where the A_i's are *connected* ptykes. The ordinal x is the **length** of A: $x = \mathbf{LH}(A)$.

(ii) If B is another ptyx of type 2, and its decomposition is $\sum\limits_{j<y} B_j$, then $T \in I^2(A, B)$ can be uniquely written as $\sum\limits_{i<f} T_i$. The function f is the **length** of T, and is denoted by $\mathbf{LH}(T)$.

Proof. This is not at all different from the well-known property of dilators; this is therefore offered as an exercise to the reader! □

12.1.39. Definition.

(i) The ptyx A is of **kind**

0 if $\mathbf{LH}(A) = 0$.

$\boldsymbol{\omega}$ if $\mathbf{LH}(A)$ is limit.

1 if $\mathbf{LH}(A) = x' + 1$, and $A_{x'} = \underline{1}$
(the constant ptyx $\underline{1}(D) = 1$, $\underline{1}(T) = \mathbf{E}_1$.

$\boldsymbol{\Omega}$ if $\mathbf{LH}(A) = x' + 1$, and $A_{x'} \neq \underline{1}$.

(ii) The morphisms $T \in I^2(A, B)$ is said to be **deficient** when $T = T' + \mathbf{E}^2_{\underline{0}B'}$, $B' \neq \underline{0}$; when T is not deficient, then A and B are of the same kind: this common kind is by definition the **kind** of T.

12.1.40. Theorem.
If D is a dilator, let us denote by $\underline{D}+\mathsf{Id}^1$ the functor from **DIL** to **DIL** defined by:

$$(\underline{D}+\mathsf{Id}^1)(F)=D+F\ , \qquad (\underline{D}+\mathsf{Id}^1)(T)=\mathbf{E}^1_D+T\ .$$

Similarly, if $T\in I^1(D,D')$, let us denote by $\underline{T}+\mathsf{Id}^1$ the natural transformation frin $\underline{D}+\mathsf{Id}^1$ to $\underline{D}'+\mathsf{Id}^1$, defined by:

$$(\underline{T}+\mathsf{Id}^1)(U)=T+U\ .$$

(i) If A is a ptyx of type 2, and D is a dilator, then the ptykes A and $A\circ(\underline{D}+\mathsf{Id}^1)$ are of the same kind.

(ii) If $T\in I^2(A,B)$ and $U\in I^1(D,D')$, then the morphisms T and $T\circ(\underline{U}+\mathsf{Id}^1)$ are of the same kind.

Proof. Something implicit in the theorem is that $A\circ(\underline{D}+\mathsf{Id}^1)$ is a ptyx: this follows from the fact that $\underline{D}+\mathsf{Id}^1$ preserves direct limits and pull-backs.

(i) – If A is of kind **0**, then $A\circ(\underline{D}+\mathsf{Id}^1)=\underline{0}$ is of kind **0**.

– If A is of kind $\boldsymbol{\omega}$, write $A=\sum_{i<x}A_i$, with $A_i\neq\underline{0}$ for all i, and x limit; then $A\circ(\underline{D}+\mathsf{Id}^1)=\sum_{i<x}A_i\circ(\underline{D}+\mathsf{Id}^1)$ is of kind $\boldsymbol{\omega}$ as well.

– If A is of kind **1**, then $A=A'+\underline{1}$, and $A\circ(\underline{D}+\mathsf{Id}^1)=A'\circ(\underline{D}+\mathsf{Id}^1)+\underline{1}$ is of kind **1**.

– If A is of kind $\boldsymbol{\Omega}$, write $A=A'+A''$, with A'' connected, $A''\neq\underline{1}$;

let $B=A\circ(\underline{D}+\mathsf{Id}^1)$; observe that:

+ B cannot be of kind **0** (since $B\neq\underline{0}\,!$).

+ B cannot be of kind **1**: otherwise, one would have: $A''\circ(\underline{D}+\mathsf{Id}^1)=B''+\underline{1}$, and this means that one can find $(z_0,D_0)\in\mathsf{Tr}(A'')$, $T_0\in I^1(D_0,D)$, such that $x_{D'}=(z_0\,;D_0\,;T_0+\mathbf{E}^1_{0D'}\,;D+D')_{A''}$ is the topmost element of $A''(D+D')$, for all D'. But the point $x=(z_0\,;D_0\,;\mathbf{E}^1_{0D}+T_0\,;D+D)_{A''}$ is $\geq x_D$ because of 12.1.25; also D_0 is $\neq\underline{0}$, since A'' is connected and $\neq\underline{1}$, hence $x_D<x$, contradiction.

+ B cannot be of kind $\boldsymbol{\omega}$: we establish first the

12.1.41. Lemma.
If A'' is connected, then

$$(z_0\,;\,D_0\,;\,T+\mathbf{E}^1_{\underline{0}D'}\,;\,D+D')_{A''} \leq (z_1\,;\,D_1\,;\,\mathbf{E}^1_{\underline{0}D}+U\,;\,D+D')_{A''}\ .$$

Proof. Assume that the conclusion if false; then one easily sees (using 12.1.25) that: $(z_0\,;\,D_0\,;\,T'\,;\,D')_{A''} > (z_1\,;\,D_1\,;\,T''\,;\,D^1)_{A''}$ for all D' and T', T''. This fact can be used to decompose A'' into a sum $A_1 + A_2 \dots$. □

Using the lemma, assuming that $A''_0(\underline{D}^1_{\mathsf{Id}}) = B''$ is of kind $\boldsymbol{\omega}$, we obtain: write $B'' = \sum_{i<x} B''_i$, with x limit and $B''_i \neq \underline{0}$. for all i, and choose D' such that $B''_i(D') \neq 0$ for all i. (For instance D' = a sum of all the finite dimensional dilators.) Choose a cofinal sequence $(z_i)_{i<x}$ in $B''(D')$, with $\sum_{j<i} B''_j(D') < z_i$ for all $i < x$. Then if $T \in I^1(D', D'')$ and $z < B''(D'')$, one can find $i < x$ such that $z < B''(T)(z_i)$. But consider $D'' = D' + D + D'$, $T = \mathbf{E}^1_{D'D''}$, and z the point $A''(\mathbf{E}^1_{\underline{0}D+D'} + \mathbf{E}^1_{D+D'})(z_0)$. The lemma yields $z > B''(T)(z_i)$, contradiction.

+ by elimination B must be of kind $\boldsymbol{\Omega}$.

(ii) Amounts to showing that T is deficient if $T \circ (\underline{U} + \mathsf{Id}^1)$ is deficient; but:

- If $T = T' + \mathbf{E}^2_{\underline{0}B'}$, $B' \neq \underline{0}$, then

  $$T \circ (\underline{U} + \mathsf{Id}^1) = T' \circ (\underline{U} + \mathsf{Id}^1) + \mathbf{E}^2_{\underline{0}B' \circ (\underline{D}' + \mathsf{Id}^1)}$$

  and $B' \circ (\underline{D}' + \mathsf{Id}^1)$ is $\neq \underline{0}\dots$.
- Assume conversely that $T \circ (\underline{U} + \mathsf{Id}^1) = T' + \mathbf{E}^2_{\underline{0}B'}$, $B' \neq \underline{0}$; then we successively check that T cannot be of kind $\mathbf{0}$, $\mathbf{1}$, $\boldsymbol{\omega}$ or $\boldsymbol{\Omega}$: T must be deficient.... □

12.1.42. Outline of the construction.
Since this chapter can only be read by people who have a good understanding of the concept of Chapter 9, it is not necessary to carry the construction in full details....

(i) Separation of variables is obtained as follows: if A is connected and of kind $\mathbf{\Omega}$, one can consider the functor B_0 from $\mathbf{DIL}^2$ to $\mathbf{ON}$ defined by:

The functors $\mathbf{UN}^2$ and $bfSEP^2$ are easily shown to be reciprocal. Observe that the construction is the obvious generalization of 9.3.18 and 9.3.22 to ptykes of type 2.

(iii) The **predecessors** are defined as follows:

– A is a predecessor of $A + A'$ when $A' \neq \underline{0}$.

– If A is a predecessor of A' and A' is a predecessor of A'', then A is a predecessor of A''.

– If A is of kind $\mathbf{\Omega}$ and D is a dilator, then $\mathbf{SEP}^2(A)(\cdot, D)$ is a predecessor of A.

Since dilators are not linearly ordered, the set of predecessors of a given dilator is not linearily ordered; but the order is still a well-founded class, hence it is possible to prove a principle of induction on ptykes of type 2.

(iv) A typical application of induction on ptykes of type 2 would be to define an analogue of $\mathbf{\Lambda}$ in this new context. The choice of the precise variant of this construction is a delicate problem; but the general pattern of the definition of $\mathbf{\Lambda}$ can obviously be transferred to ptykes of type 2... .

12.1.43. Exercise.

(i) Assume that A is a 1-dimensional ptyx of type 2, and that $\mathsf{Tr}(A) = \{(0, D)\}$, and let $n = \mathbf{LH}(D)$. Assume that $D = D_0 + ... + D_{n-1}$ (D_i connected), let $D' = D_0 + D_0 + D_1 + D_1 + ... + D_{n-1} + D_{n-1}$, and define, for $i < n$, $T_i \in I^1(D, D')$ by:

$$T_i = \mathbf{E}^1_{D_0} + bfE^2_{\underline{0}D_0} + ... + \mathbf{E}^1_{D_{i-1}} + \mathbf{E}^1_{\underline{0}D_{i-1}} + \mathbf{E}^1_{\underline{0}D_i} + \mathbf{E}^1_{D_i} + ... + \mathbf{E}^1_{D_{n-1}} + ... + \mathbf{E}^1_{\underline{0}D_{n-1}}$$

and consider $z_i = (0\,;\, D\,;\, T_i\,;\, D')$, and define σ by: $\sigma(i) < \sigma(j) \leftrightarrow z_i > z_j$. Show that the knowledge of σ enables us to compare any two points $(0\,;\, D\,;\, T\,;\, D_1)_A$ and $(0\,;\, D\,;\, T'\,;\, D_1)_A$, when T and T' are of the form $\sum\limits_{i<f} T_i$, $\sum\limits_{i<g} T'_i$, and s.t. for all $i < n$:

– either $f(i) = g(i)$ and $T_i = T'_i$.

– or $\quad f(i) \in \mathsf{rg}(g)$, $f(j) \neq g(i)$ for all $j < n$.

(ii) Assume that A is a connected pttx of type 2, $A \neq \underline{1}$; if $(z\,;\, D) \in \mathsf{Tr}(A)$, define a permutation $\sigma_{z,D}$ of $\mathbf{LH}(D)$. Prove the following property: Consider two points $w = \Big(z\,;\, D\,;\, \sum\limits_{i<f} T_i\,;\, D'\Big)_A$ and $w' = \Big(z'\,;\, D'\,;\, \sum\limits_{i<f'} T'_i\,;\, D'_i\Big)_A$, let $m = \sigma_{z,D}(0)$, $m' = \sigma_{z',D'}(0)$ and assume that $f(m) < f'(m')$: show that $w < w'$.

(iii) Assume that A is as in (ii); if D and D' are dilators, consider $\overline{A(D,D')} \subset S(D'+D)$: $(z\,;\, D_0\,;\, T_0\,;\, D'+D)_A \in \overline{A(D,D')}$ iff T_0 can be written $T_0 = T' + T$, $T \in I^1(D'_0, D)$, $T' \in I^1(D''_0, D')$ and $\mathbf{LH}(D'_0) = \sigma_{z,D_0}(0){+}!$. Using the sets $\overline{A(D,D')}$ define a biptyx A' of type 2.

(iv) Prove that $A' = \mathbf{SEP}^2(A)$; from this deduce a way of constructing $\mathbf{SEP}^2$ from the algebraic structure of the denotations... .

12.1.44. <u>Exercise</u>.

(i) Let A be a connected biptyx of type 2; if D is a dilator, consider $\overline{A(D)} \subset A(D,D)$: $(z_0\,;\, D'_0\,;\, D''_0\,;\, T'\,;\, T''\,;\, D\,;\, D)_A \in \overline{A(D)} \leftrightarrow \exists D'\ \exists D''\ \exists T'_1 \in I^1(D'_0, D'')\ \exists T''_1 \in I^1(D''_0, D')\,(T' = \mathbf{E}^1_{\underline{0}D'} + T'_1 \wedge T'' + \mathbf{E}^1_{\underline{0}D''})$. Using the sets $\overline{A(D)}$, construct a ptyx A' of type 2; show that A' is connected, and of kind $\mathbf{\Omega}$.

(ii) Prove that $A' = \mathbf{UN}^2(A)$.

(iii) Using 13.1.44 (ii) together with 13.1.43, prove that the functors $\mathbf{UN}^2$ and $\mathbf{SEP}^2$ are reciprocal.

(iv) Prove that:

$$\mathbf{UN}^2(A)(\underline{\omega}^{1+F}) = A(\underline{\omega}^{1+F}, \underline{\omega}^{1+F})$$

$$\mathbf{UN}^2(A)(\underline{\omega}^{1+T}) = A(\underline{\omega}^{1+T}, \underline{\omega}^{1+T})$$

$$\mathbf{UN}^2(U)(\underline{\omega}^{1+F}) = U(\underline{\omega}^{1+F}, \underline{\omega}^{1+F}) \ .$$

12.2. Ptykes

Ptykes are objects which generalize ordinals, dilators,...; for instance ptykes of type 2 were at the heart of the preceding section. Many types of ptykes can be imagined, and, according to the degree of generalization that we shall allow, various portions of the result of Section 12.1 can be transferred in the new context.

The difficulty of our task is to find a compromise between the generality of the concepts and the results following from the use of the concept, since it is clear that, the more general our concept, the less results we can expect.... We have finally chosen a particular definition for "ptyx" which is a rather good compromise. No doubt, however, that more general concepts can be produced. Our principal motivation for this concept was (besides its simplicity), its direct relationship to **ON**, which makes it explicit that ptykes have something to do with ordinals.... A more general concept of ptyx could perhaps be given by means of a list of requirements for a given category $\mathcal{C}$ to be a category of ptykes (for instance existence of a functor trace, existence of sums...), in the same way as one defines, say, abelian categories.

In this section we shall try to follow 12.1 as far as possible: we shall indicate systematically the analogues of the definitions, theorems, remarks ... of Sec. 12.1.

12.2.1. Definition.

(i) Let $\mathcal{C}$ be a category; $\mathcal{C}$ has the **sum property** when: given $x \neq 0$, A, $(B_i)_{i<x}$, and morphisms $A \xrightarrow{T_i} B_i$ it is possible to find C, together with morphisms $B_i \xrightarrow{U_i} C$ such that the diagrams:

$$\begin{tikzcd} & B_i \arrow[dr, "U_i"] & \\ A \arrow[ur, "T_i"] \arrow[dr, "T_j"'] & & C \\ & B_j \arrow[ur, "U_j"'] & \end{tikzcd}$$

are commutative. $\mathcal{C}$ has the **dimension property** when the class

$|\mathcal{C}_{fd}| = \{a \in |\mathcal{C}|\,;\, \{(t,b)\,;\, t \in \mathsf{Mor}_{\mathcal{C}}(b,a)\}\ \mathsf{finite}\}$ is a set.

(ii) A **ptyx** is any functor F from some category $\mathcal{C}$ with the sum and the dimension properties to **ON**, and enjoying the **normal form property**: if $z < F(a)$ $(a \in |\mathcal{C}|)$, then it is possible to find unique $b \in |\mathcal{C}_{fd}|$, f from b to a, $z_0 < F(b)$ such that:

1. $z = F(t)(z_0)$.
2. If $z = F(t')(z_1)$, $z_1 < F(b')$, t' from b' to a, then there is unique t'' from b to b'' s.t. $z_1 = F(t'')(z_0)$ (notation: $z = (z_0\,;\, b\,;\, t\,;\, a)_F$).

(iii) If F and G are ptykes, then we define $I(F,G)$ as follows: assume that F and G are defined on $\mathcal{C}$ and $\mathcal{D}$ respectively; then

– if $\mathcal{C} \neq \mathcal{D}$, $I(F,G) = \emptyset$.

– if $\mathcal{C} = \mathcal{D}$, then $I(F,G)$ consists of all natural transformations T from F to G enjoying the normal form property such that

$$T(a)\Big((z_0\,;\, b\,;\, t\,;\, a)_F\Big) = (T(b)(z_0)\,;\, b\,;\, t\,;\, a)_G$$

for all $z = (z_0\,;\, b\,;\, t\,;\, a)_F$ in $F(a)$, and all $a \in |\mathcal{C}|$.

(iv) The ptyx F of (ii) is said to be **of type** $\mathcal{C} \to \mathbf{O}$. In general, we define the concept of type by: a type is a category of the form $\mathcal{C} \to \mathbf{O}$: this means that $\mathcal{C}$ is a category with the sum property, and the objects of $\mathcal{C} \to \mathbf{O}$ are the ptykes of type $\mathcal{C} \to \mathbf{O}$, whereas the morphisms from F to G in $\mathcal{C} \to \mathbf{O}$ are given by the set $I^{\mathcal{C}\to\mathbf{O}}(F,G) = I(F,G)$.

We shall usually denote types by the letters $\boldsymbol{\sigma}$,$\boldsymbol{\tau}$,$\boldsymbol{\rho}$,... .

12.2.2. <u>Examples</u>.

(i) The smallest category enjoying the property is the void category $\emptyset$. The type $\emptyset \to \mathbf{O}$, denoted by $(\,)$, consists of one element, and one morphism.

(ii) The type $(\,) \to \mathbf{O}$, denoted by $\mathbf{O}$, can be identified with **ON**.

(iii) In general, if $\boldsymbol{\sigma}$ and $\boldsymbol{\tau} = \mathcal{C} \to \boldsymbol{O}$ are types, then we can define a new type $\boldsymbol{\sigma} \to \boldsymbol{\tau}$, by $\boldsymbol{\sigma} \to \boldsymbol{\tau} = \boldsymbol{\sigma} \times \mathcal{C} \to \boldsymbol{O}$. Here we use the familiar product of categories. We have two definitions of $\boldsymbol{\sigma} \to \boldsymbol{O}$, which coincide, up to a trivial isomorphism. Of course, it will be necessary to verify somewhere that $\boldsymbol{\sigma} \times \mathcal{C}$ has the sum property, and the dimension property.

(iv) If $\boldsymbol{\sigma} = \mathcal{C} \to \boldsymbol{O}$ and $\boldsymbol{\tau} = \mathcal{D} \to \boldsymbol{O}$ are types, then we can define a new type $\boldsymbol{\sigma} \times \boldsymbol{\tau}$, by $\boldsymbol{\sigma} \times \boldsymbol{\tau} = (\mathcal{C} + \mathcal{D}) \to \boldsymbol{O}$, where $\mathcal{C} + \mathcal{D}$ is the familiar disjoint sum of the categories $\mathcal{C}$ and $\mathcal{D}$.

(v) The categories **DIL**, $\mathbf{PT}^2$ can therefore be identified with $\boldsymbol{O} \to \boldsymbol{O}$ and $(\boldsymbol{O} \to \boldsymbol{O}) \to \boldsymbol{O}$.

12.2.3. <u>Definition</u> (12.1.9).
We define the functor **trace** from $\boldsymbol{\sigma}$ to $\mathbf{SET}_i$:

(i) $\mathsf{Tr}(F) = \{(z, a)\,;\, z = (z\,;\, a\,;\, \mathsf{id}_a\,;\, a)_F\}$.

(ii) $\mathsf{Tr}(T)\Big((z, a)\Big) = (T(a)(z)\,;\, a)$.

12.2.4. <u>Lemma</u> (12.1.10).
Definition 12.2.3 is sound.

<u>Proof</u>. (i): $\mathsf{Tr}(F)$ is a set, because $\mathsf{Tr}(F) \subset \{(z, a)\,;\, z \in F(a) \wedge a \in |\mathcal{C}_{fd}|\}$.

(ii): $\mathsf{Tr}(T)$ maps $\mathsf{Tr}(F)$ into $\mathsf{Tr}(G)$ because $T(a)\Big((z\,;\, a\,;\, \mathsf{id}_a\,;\, a)_F\Big) = (T(a)(z)\,;\, a\,;\, \mathsf{id}_a\,;\, a)_G$. If $(z, a) \neq (z', a')$, then either $a \neq a'$ (then $(T(a)(z), a) \neq (T(a')(z'), a')$) or $a = a'$ and $z \neq z'$ (then $T(a)(z) \neq T(a)(z')$): this proves that $\mathsf{Tr}(T)$ is an injective function. $\square$

12.2.5. <u>Theorem</u> (12.1.11).
The functor Trace has the following properties:

(i) Assume that (F, T_i) enjoys 8.1.11 (i)–(iii) w.r.t. (F_i, T_{ij}); then

$$(F, T_i) = \varinjlim (F_i, T_{ij}) \leftrightarrow \Big(\mathsf{Tr}(F), \mathsf{Tr}(T_i)\Big)$$

$$= \varinjlim \Big(\mathsf{Tr}(F_i), \mathsf{Tr}(T_{ij})\Big) \ .$$

(ii) Assume that t_1, t_2, t_3 enjoy 8.1.24 (i); then

$$t_1 \wedge t_2 = t_3 \leftrightarrow \mathsf{Tr}(t_1) \wedge \mathsf{Tr}(t_2) = \mathsf{Tr}(t_3) \ .$$

(iii) Given a subset $X \subset \mathsf{Tr}(F)$, there are unique G and $T \in I(G, F)$ s.t. $X = \mathsf{rg}\Big(\mathsf{Tr}(T)\Big)$.

Proof. (iii): If $X \subset \mathsf{Tr}(F)$, and $a \in |\mathcal{C}|$, we consider the subsets $X_a \subset F(a)$, consisting of those $z \in F(a)$ of the form

$$(z_0\,;\,a_0\,;\,t_0\,;\,a)_F \ , \qquad \textsf{for some } (z_0\,;\,a_0) \in X \ .$$

If t: $a \to a'$, we can define a function X_t from X_a to $X_{a'}$, by $X_t\Big((z_0\,;\,a_0\,;\,t_0\,;\,a)_F\Big) = (z_0\,;\,a_0\,;\,tt_0\,;\,a)_F$. One easily checks that X_t is the restriction of $F(t)$, hence is strictly increasing. We define G by:

$$G(a) = \|X_a\| \qquad G(t) = \|X_t\| \ .$$

We define $T \in I(G, F)$ by the condition $\mathsf{rg}\Big(\mathsf{Tr}\Big(T(a)\Big)\Big) = X_a$; the diagrams

$$\begin{tikzcd} G(a) \arrow[r, "T(a)"] \arrow[d, "G(t)"'] & F(a) \arrow[d, "F(t)"] \\ G(a') \arrow[r, "T(a')"'] & F(a') \end{tikzcd}$$

are clearly commutative.

G has the normal form property: if $z \in G(a)$, write $T(a)(z) = (z_0\,;\,a_0\,;\,t_0\,;\,a)_F$, and observe that $(z_0\,;\,a_o) \in X$. This implies that $z_0 \in X_{a_0}$, hence $z_0 = T(a_0)(z_1)$; then we can write $z = (z_1\,;\,a_0\,;\,t_0\,;\,a)_G$. This is a normal form since:

1. $z = G(t_0)(z_1)$.

2. If $z = G(t'_0)(z'_1)$, t'_0: $a'_0 \to a$, then $T(a)(z) = F(t'_0)\Big(T(a'_0)(z'_1)\Big)$, hence it is possible to find a unique form t'' from a_0 to a'_0 s.t. $F(t'')(z_0) = T(a'_0)(z'_1)$. But then $G(t'')(z_1) = z'_1 \ldots$.

It is immediate from that, that

$$T(a)\Big((z_0\,;\,a_0\,;\,t\,;\,a)_G\Big) = \Big(T(a_0(z_0)\,;\,a_0\,;\,t\,;\,a)_F$$

hence T is a morphism from G to F. Obviously $X = \mathsf{rg}\Big(\mathsf{Tr}(T)\Big)$. The unicity of G and T is immediate.

(i): Assume that $(F, T_i) = \varinjlim (F_i, T_{ij})$; if we consider the subset X of $\mathsf{Tr}(F)$: $X = \bigcup_{i \in I} \mathsf{rg}\Big(\mathsf{Tr}(T_i)\Big)$, then the argument of 12.1.11 (i) can be used to show that $X = \mathsf{Tr}(F)$, hence $\Big(\mathsf{Tr}(F), \mathsf{Tr}(T_i)\Big) = \varinjlim \Big(\mathsf{Tr}(F_i), \mathsf{Tr}(T_{ij})\Big)$. Conversely, the hypothesis $\Big(\mathsf{Tr}(F), \mathsf{Tr}(T_i)\Big) = \varinjlim \Big(\mathsf{Tr}(F_i), \mathsf{Tr}(T_{ij})\Big)$ implies that $\Big(F(a), T_i(a)\Big) = \varinjlim \Big(F_i(a), T_{ij}(a)\Big)$ for all $a \in |\mathcal{C}|$.

(<u>Proof</u>. If $z \in F(a)$, write $z = (z_0\,;\,a_0\,;\,t\,;\,a)_F$ and choose i such that $(z_0\,;\,a_0) \in \mathsf{rg}\Big(\mathsf{Tr}(T_i)\Big)$; then $z \in \mathsf{rg}\Big(T_i(a)\Big)$. □)

12.2.6. <u>Lemma</u> (12.1.12).
If $\Big(F(a), T_i(a)\Big) = \varinjlim \Big(F_i(a), T_{ij}(a)\Big)$ for all $a \in |\mathcal{C}|$, then $(F, T_i) = \varinjlim (F_i, T_{ij})$.

<u>Proof</u>. We argue exactly as in 12.1.12; we construct, given (G, U_i) enjoying 8.1.11 (i)–(iii) w.r.t. (F_i, T_{ij}), a unique natural transformation T from F to G such that $U_i = TT_i$ for all $i \in I$. In fact $T \in I(F, G)$, since, given $z = (z_0\,;\,a_0\,;\,t\,;\,a)_F$, one can choose i and z_i such that $z = (T_i(a_0)(z_i)\,;\,a_0\,;\,t\,;\,a)_F$. Hence

$$\begin{aligned} T(a)(z) =\ & U_i(a)\Big((z_i\,;\,a_0\,;\,t\,;\,a)_{F_i}\Big) = (U_i(a_0)(z_i)\,;\,a_0\,;\,t\,;\,a)_F\ = \\ =\ & (TT_i(a_0)(z_i)\,;\,a_0\,;\,t\,;\,a)_F = (T(a_0)(z_0)\,;\,a_0\,;\,t\,;\,a)_F\ .\ \square \end{aligned}$$

The lemma concludes the proof of (i).

(ii): Assume that $T_i \in I(F_i, G)$ $(i = 1, 2, 3)$, and that T_{31} and T_{32} are such that $T_3 = T_1T_{31} = T_2T_{32}$; hence $\mathsf{Tr}(T_3) = \mathsf{Tr}(T_1)\mathsf{Tr}(T_{31}) = \mathsf{Tr}(T_2)\mathsf{Tr}(T_{32})$, and $\mathsf{rg}\Big(\mathsf{Tr}(T_3)\Big) \subset \mathsf{rg}\Big(\mathsf{Tr}(T_1)\Big) \cap \mathsf{rg}\Big(\mathsf{Tr}(T_2)\Big) = X$. We show exactly as in 12.1.11 that the hypothesis $T_1 \wedge T_2 = T_3$ implies $X = \mathsf{rg}\big(\mathsf{Tr}(T_3)\big)$, i.e. that $\mathsf{Tr}(T_1) \wedge \mathsf{Tr}(T_2) = \mathsf{Tr}(T_3)$. Conversely the hypothesis $\mathsf{Tr}(T_1) \wedge \mathsf{Tr}(T_2) = \mathsf{Tr}(T_3)$ implies $T_1(a) \wedge T_2(a) = T_3(a)$ for all $a \in \mathcal{C}$ (use the normal form property), and the

12.2.7. Lemma (12.1.13).
If $T_1(a) \wedge T_2(a) = T_3(a)$ for all $a \in \mathcal{C}$, then $T_1 \wedge T_2 = T_3$.

Proof. Left to the reader. □

concludes the proof. □

12.2.8. Corollary (12.1.14).
Assume that (A_i, T_{ij}) is a direct system in $\mathcal{C} \to \boldsymbol{O}$, and that (A, T_i) enjoys 8.1.11 (i)–(iii) w.r.t. (A_i, T_{ij}); then the following are equivalent:

(i) $(A, T_i) = \varinjlim (A_i, T_{ij})$.

(ii) $\big(A(a), T_i(a)\big) = \varinjlim \big(A_i(a), T_{ij}(a)\big)$ for all $a \in |\mathcal{C}|$.

(iii) $\big(A(a), T_i(a)\big) = \varinjlim \big(A_i(a), T_{ij}(a)\big)$ for all $a \in |\mathcal{C}_{fd}|$.

(iv) $\big(\mathsf{Tr}(A), \mathsf{Tr}(T_i)\big) = \varinjlim \big(\mathsf{Tr}(A_i), \mathsf{Tr}(T_{ij})\big)$.

Proof. Immediate, left to the reader. □

12.2.29. Remark (12.1.15).

(i) The existence of the direct limit $\varinjlim \big(A_i(a), T_{ij}(a)\big)$ for all $a \in |\mathcal{C}|$ ensures the existence of $\varinjlim (A_i, T_{ij})$.

(ii) The existence of (B, U_i) enjoying 8.1.11 (i)–(iii) w.r.t. (A_i, T_{ij}) ensures the existence of $\varinjlim (A_i, T_{ij})$.

12.2.10. Corollary (12.1.16).
Assume that $T_i \in I(A_i, B)$, $(i = 1, 2, 3)$; then the following are equivalent:

(i) $T_1 \wedge T_2 = T_3$.

(ii) $T_1(a) \wedge T_2(a) = T_3(a)$ for all $a \in |\mathcal{C}|$.

(iii) $T_1(a) \wedge T_2(a) = T_3(a)$ for all $a \in |\mathcal{C}_{fd}|$.

(iv) $\mathsf{Tr}(T_1) \wedge \mathsf{Tr}(T_2) = \mathsf{Tr}(T_3)$.

Proof. Left to the reader. □

12.2.11. Remark (12.1.17).
Pull-backs always exist in $\mathcal{C} \to \boldsymbol{O}$.

12.1.12. Remark.
Observe that we have no analogues for the parts (iv) of 12.1.14 and 12.1.16: simply because objects of $\mathcal{C} \to \boldsymbol{O}$ do not necessarily preserve direct limits and pull-backs: we know too little about $\mathcal{C}$....

12.1.13. Theorem (12.1.18).
Assume that, for $i < x$, $T_i \in I(A, B_i)$; then it is possible to find C, together with $U_i \in I(B_i, C)$ rendering the diagrams

$$\begin{tikzcd} & B_i \arrow[dr, "U_i"] & \\ A \arrow[ur, "T_i"] \arrow[dr, "T_j"'] & & C \\ & B_j \arrow[ur, "U_j"'] & \end{tikzcd}$$

commutative; furthermore for $i \neq j$, $U_i \wedge U_j = U_i T_i = U_j T_j$.

Proof. We essentially follow the argument of 12.1.18:

(i): In a first step we introduce, given $a \in |\mathcal{C}|$, the disjoint union $X = \{(i, z)\,;\, i < x \wedge z < B_i(a)\}$, preordered by: $(i, z) \;\leq^1_a\; (i', a')$ iff $(i = i' \wedge z \leq z') \vee \exists z_0 \in A(a)\,(z \leq T_i(a)(z_0) \wedge T_{i'}(a)(z_0) \leq z')$. The associated equivalence relation $(i, z) \;\sim_a\; (i', z')$ iff $(i = i' \wedge z = z') \vee \exists z_0 \in$

$A(D)\big(z = T_i(a)(z_0) \wedge z' = T_{i'}(a)(z_0)\big)$ plays an essential role in the proof. Our task will be to extend the well-founded order $\leq_a^1 / \sim_a$ into a well-order.

(ii): $(i,z) \leq_a^2 (i',z') \leftrightarrow \exists a' \, \exists t'\colon a \to a' \big(i, B_i(t')(z)\big) \leq_{a'}^1 \big(i', B_{i'}(t')(z')\big)$. $\leq_a^2$ is transitive since $\mathcal{C}$ has the sum property, $\leq_a^2 / \sim_a$ is a well-founded order (the sum property is needed here: see 12.1.10) and we have, when t: $a \to a'$, $(i,z) \leq_a^2 (i',z') \leftrightarrow \big(i, B_i(t)(z)\big) \leq_{a'}^2 \big(i', B_{i'}(t)(z')\big)$.

The linear preorders $\leq_a^3$ and $\leq_a^4$ are defined by:

$$(i,z) \leq_a^3 (i',z') \leftrightarrow (i,z) \leq_a^1 (i',z') \vee \Big((i',z') \not\leq_a^1 (i,z) \wedge i < i'\Big)$$

$$(i,z) \leq_a^4 (i',z') \leftrightarrow (i,z) \leq_a^2 (i',z') \vee \Big((i',z') \not\leq_a^2 (i,z) \wedge i < i'\Big) \, .$$

12.2.14. <u>Lemma</u> (12.1.19, 12.1.20).

(i) $\leq_a^3 / \sim_a$ is a well-order.

(ii) $\leq_a^4 / \sim_q$ is a well-order.

<u>Proof</u>. See 12.1.19. □

We define $C(a) = \leq_a^4 / \sim_a$, and when t: $a \to a'$, $C(t)\big(\overline{(i,z)}\big) = \overline{\big(i, B_i(t)(z)\big)}$.

C is (isomorphic to) a functor from $\mathcal{C}$ to $\mathbf{ON}$, which is easily shown to enjoy the normal form property. We can define U_i by:

$$U_i(a)(z) = \overline{(i,z)}$$

and it is immediate that $U_i \in I(B_i, C)$. The property $U_i \wedge U_j = U_i T_i = U_j T_j$ for $i \neq j$ is immediate. □

12.2.15. <u>Theorem</u>.
Under the hypotheses of 12.2.13, assume that x', A, T'_j, B'_j is another family, and that the function $f \in I(x, x')$ and the morphisms $V_i \in I(V_i, f'_{f(i)})$ render the diagrams:

$$\begin{tikzcd} & B_i \arrow[dd, "V_i"] \\ A \arrow[ur, "T_i"] \arrow[dr, "T'_{f(i)}"'] & \\ & B'_{f(i)} \end{tikzcd}$$

commutative. Then if U'_j and C' are the data obtained by 12.2.13 from x', A, T'_j, B'_j, it is possible to construct a morphism $V \in I(C, C')$ rendering the diagrams:

$$\begin{tikzcd}
 & B_i \arrow[dd, "V_i"'] \arrow[r, "U_i"] & C \arrow[dd, "V"] \\
A \arrow[ur, "T_i"] \arrow[dr, "T'_{f(i)}"'] & & \\
 & B'_{f(i)} \arrow[r, "U'_{f(i)}"'] & C'
\end{tikzcd}$$

commutative; if $x \neq 0$, then V is unique.

Proof. Immediate property of the construction. □

12.2.16. Notations.

(i) In 12.2.13, let $T = U_i T_i$; then $(T, C) = \sum_{i<x} (T_i, B_i)$ (when $x \neq 0$!; if $x = 0$ let $C = A$, $T = \mathbf{E}_A$).

In 12.2.15, we use the notation: $V = \sum_{i<f} V_i$ (when $x \neq 0$; if $x = 0$, V is the morphism T' such that $(T', C') = \sum_{i<x} (T'_i, B'_i)$).

(ii) The category $\mathcal{C} \to \boldsymbol{O}$ contains an initial element, denoted by $\underline{0}$: apply 12.2.13 with $x = 0$; we obtain $\underline{0}$. Apply now 12.2.15 with: $x' = 1$, $B_0 = A$, $T_0 =$ the identity of A; we get $V \in I(\underline{0}, A)$. In fact we easily check that:

1. $\underline{0}$ is the only element of $\mathcal{C} \to \boldsymbol{O}$ with a void trace: $\mathsf{Tr}(\underline{0}) = \emptyset$.
2. There is one and only one morphism $\mathbf{E}_{\underline{0}A}$ from $\underline{0}$ to A, when $A \in |\mathcal{C}|$.

(iii) If $A_i \in |\mathcal{C}|$ $(i < x)$, we can apply 12.2.13 to x, $\underline{0}$, A_i: we obtain B, together with $U_i \in I(A_i, B)$; the notation for B is $\sum_{i<x} A_i$. If $(A'_j)_{j \in x'}$ is another family, if $T_i \in I(A_i, A_{f(i)})$ for all $i < x$, and $f \in I(x, x')$, then by 12.2.15 we can introduce

$$T = \sum_{i<f} T_i \in I\left(\sum_{i<x} A_i\,,\, \sum_{i<x} A'_j\right)\,.$$

(iv) The notations (i) and (iii) have a special case when $x = 2$, $f = \mathbf{E}_2$:

$$\begin{array}{lll} (T_1, B_1) + (T_2, B_2) & , & V_1 + V_2 \\ A_1 + A_2 & , & T_1 + T_2 \ . \end{array}$$

(v) One checks the following property: $A + B = A + B' \to B = B'$; from this it follows that we can always define, when B is of the form $A + B'$, a morphism $\mathbf{E}_{AB} \in I(A, B)$:

$$\mathbf{E}_{AB} = \mathbf{E}_A + \mathbf{E}_{\underline{0}B'}$$

where $\mathbf{E}_A$ denotes as usual the identity of A; hence $\mathbf{E}_A = \mathbf{E}_{AA}$.

(vi) Consider the order relation:, "$B = A + B'$ for some B'"; then if we are given a family $(A_i)_{i<x}$, which is increasing w.r.t. this order, it is possible to define a supremum:

$$\underset{i<x}{\mathsf{sup}}\ A_i = A_0 + \sum_{i<x} A'_i$$

where A'_i is defined by $A_{i+1} = A_i + A'_i$.

In the same way, if we consider the following order relation between morphisms: "$U = T + U'$ for some U'"; then given a family $(T_i)_{i<x}$, increasing in this sense, we shall be able to introduce the union

$$\bigcup_{i<x} T_i = T_0 + \sum_{i<\mathbf{E}_x} T'_i$$

with $T_{i+1} = T_i + T'_i$.

12.2.17. Definition (12.1.22).
The set $I(A, A')$ is order by:

$$T \leq U \leftrightarrow \forall a \in |\mathcal{C}|\ \forall x \in A(a) \Big(T(a)(z) \leq U(a)(z) \Big) \ .$$

12.2.18. <u>Theorem</u> (12.1.23).
$T \le U$, when $T, U \in I(A, A')$, is equivalent to the existence of A'', $V \in I(A', A'')$, $W \in I(A'', A'')$ such that:

$$VU = WVT \ .$$

<u>Proof</u>. See 12.1.23. □

12.2.19. <u>Definition</u> (12.1.28).
A is **finite dimensional** iff $\mathsf{dim}(A) = \mathsf{card}\big(\mathsf{Tr}(A)\big)$ is finite; $\mathsf{dim}(A)$ is the **dimension** of A. $(\mathcal{C} \to \boldsymbol{O})_{fd}$ denotes the full subcategory of $\mathcal{C} \to \boldsymbol{O}$ consisting of the finite dimensional ptykes of that type.

12.2.20. <u>Theorem</u> (12.1.29).
Every ptyx is a direct limit of finite dimensional ptykes.

<u>Proof</u>. See 12.1.29. □

12.2.21. <u>Definition</u> (12.1.30).
A **preptyx** of type $\mathcal{C} \ \to_p \ \boldsymbol{O}$ is a functor A from $\mathcal{C}$ to **OL**, enjoying the following properties:

(i) A has the normal form property (straightforward adaptation of 12.2.1 (ii)).

(ii) For all $(z_0\,;\,a_0) \in \mathsf{Tr}(A)$ and $a \in |\mathcal{C}|$, the set of denotations $(z_0\,;\,a_0\,;\,t\,;\,a)_A$ is well-ordered.

12.2.22. <u>Theorem</u> (12.1.31).
Preptykes are exactly the direct limits of ptykes in the category $\mathcal{C} \ \to_p \ \boldsymbol{O}$ of preptykes of type $\mathcal{C} \to \boldsymbol{O}$.

<u>Proof</u>. The morphisms in $\mathcal{C} \ \to_p \ \boldsymbol{O}$ are exactly the natural transformations enjoying 12.2.1 (iii). Now the theorem is rather trivial, since condition (ii) of 12.2.21 ensures that finite dimensional preptykes are (isomorphic to) ptykes. □

12.2.23. Remark.
We know too little concerning $\mathcal{C}$ to be able to replace 12.2.21 (ii) by a more natural condition... .

12.2.24. Discussion (12.1.37).
Is it possible to define in a general setting the notions of recursive and weakly finite ptykes? Obviously not; these concepts are twoo much linked to the knowledge of $\mathcal{C}$, which can be a very uneven category object.

The algebraic part of the theory has been developed to some extent, and we turn our attention towards the well-foundedness properties of $\mathcal{C} \to \boldsymbol{O}$; now we stumble on an essential difficulty: we lose the unicity of the decomposition into sums. For instance, if $\mathcal{C}$ consists of two objects a, b, with no morphism from a to b or b to a, then the constant functor $\underline{1}$ can be written as $A + B$, or $B + A$ with $A(a) = 1$, $A(b) = 0$, $B(a) = 0$, $B(b) = 1$. This is due to a lack of connectedness of $\mathcal{C}$; we shall recover unicity by restricting our attention to *connected* types:

12.2.25. Definition.
A type $\mathcal{C} \to \boldsymbol{O}$ is **connected** when $\mathcal{C}$ has the following property: $\mathcal{C}$ is non void and for all $a, b \in |\mathcal{C}|$ $\exists c \in |\mathcal{C}|$ $\mathsf{Mor}_{\mathcal{C}}(a, c) \neq \emptyset \wedge \mathsf{Mor}_{\mathcal{C}}(b, c) \neq \emptyset$.

12.2.26. Theorem.
Every type is isomorphic to the product of a set of connected types. This product is unique (up to isomorphism).

Proof. Given $a, b \in |\mathcal{C}|$, we define $a \sim b$ by $\exists x \in |\mathcal{C}|$ ($\mathsf{Mor}_{\mathcal{C}}(a, c) \neq \emptyset$ and $\mathsf{Mor}_{\mathcal{C}}(b, c) \neq \emptyset$). The sum property ensures that $\sim$ is transitive. We can consider the equivalence classes $|\mathcal{C}_i|$ of $\sim$, and define categories $\mathcal{C}_i$ accordingly. Then $\mathcal{C}$ appears as the disjoint union of the $\mathcal{C}_i$'s. The category $\mathcal{C} \to \boldsymbol{O}$ therefore appears as the product of the categories $\mathcal{C}_i \to \boldsymbol{O}$. Now, by the *dimension property* $I = \{i\,; \mathcal{C}_i \to \boldsymbol{O} \neq \boldsymbol{O}\}$ is a set, hence $\mathcal{C} \to \boldsymbol{O}$ is isomorphic to $\prod_{i \in I} \mathcal{C}_i \to \boldsymbol{O}$. □

12.2.27. Theorem (12.1.38).

Assume that $\boldsymbol{\sigma}$ is connected; then

(i) Every ptyx A of type $\boldsymbol{\sigma}$ can be uniquely written as a sum $A = \sum_{i<x} A_i$, where the A_i's are connected. (B is **connected** iff $B \neq \underline{0}$ and $B = B' + B'' \rightarrow B' = \underline{0} \vee B'' = \underline{0}$.) The ordinal x is the **length** of A, and is denoted by $\mathbf{LH}(A)$.

(ii) If B is another ptyx of the same type, whose decomposition is $\sum_{j<y} B_j$, and if $T \in I(A, B)$, then T can be written in a unique way as a sum: $T = \sum_{i<f} T_i$, $T_i \in I(A_i, B_{f(i)})$. The function f is the length of T, and is denoted by $\mathbf{LH}(T)$.

Proof. (i): We consider the class X of all pairs (z, a), with $z < A(a)$; we preorder this class by $(z, a) \leq (z', a') \leftrightarrow \exists b \in |\mathcal{C}|\ \exists t\ \exists t' \Big(a \xrightarrow{t} b \wedge a' \xrightarrow{t'} b \wedge A(t)(z) \leq A(t')(z')\Big)$. The relation is transitive because of the sum property; moreover, since $\boldsymbol{\sigma}$ is connected, it is always possible to find b and morphisms t and t' from a and a' to b, hence we shall get $A(t)(z) \leq A(t')(z')$ or $A(t')(z') \leq A(t)(z)$, and so $(z, a) \leq (z', a') \vee (z', a') \leq (z, a)$: our preorder is linear. Now we again use the sum property, together with the connectedness of $\mathcal{C}$, to yield a point $a_0 \in |\mathcal{C}|$ such that for all $b \in |\mathcal{C}_{fd}|$, $\mathsf{Mor}_{\mathcal{C}}(b, a_0) \neq \emptyset$. Clearly, given $(z, a) \in X$, it is possible to find $z' \in A(a_0)$ s.t. $(z, a) \sim (z', a_0)$, $\sim$ being the equivalence of $\leq$.

(Proof. Write $z = (z_0\,; a'\,; t\,; a)_A$, with $a' \in |\mathcal{C}_{fd}|$, and let $z' = (z_0\,; a'\,; t'\,; a_0)_A$, for some $t' \in \mathsf{Mor}_{\mathcal{C}}(a', a_0)$.... $\square$

This proves that the linear order $\leq / \sim$ is equal to its restriction to $A(a_0) \times \{a_0\}$; but if $z \leq z'$, then $(z, a_0) \leq (z', a_0)$, hence the equivalence classes X_i modulo $\sim$ are determined by their "traces" $Y_i = \{z\,; (z, a_0) \in X_i\}$ on $A(a_0)$, which are intervals. From that we obtain that $\leq / \sim$ is a well-order, isomorphic to a quotient of $A(a_0)$. Let x be this ordinal (well-order), and let X_i be the enumeration of the classes: $(X_i)_{i<x}$. Define $X'_i = X_i \cap \mathsf{Tr}(A)$, and apply 12.2.5 (iii) t X'_i: we obtain A_i together with $U_i \in I(A_i, A)$ s.t. $X'_i = \mathsf{rg}\Big(\mathsf{Tr}(U_i)\Big)$. The fact that the A_i's are connected, and that $A = \sum_{i<x} A_i$ is left to the reader.

(ii): Left to the reader. □

12.2.8. Corollary (splitting lemma).
If $T \in I(A,B)$, and $B = B' + B''$, then T can be uniquely written as $T' + T''$, $T' \in I(A', B')$, $T'' = I(A'', B'')$ for some T', T'', A', A''.

Proof. The proof is an easy corollary of 12.2.27; observe that the result still holds when $\boldsymbol{\sigma}$ is not connected, since it can be proved first for the components of $\boldsymbol{\sigma}$, then extended to $\boldsymbol{\sigma}$. □

12.2.29. Definition (12.1.39).

(i) The ptyx A is of **kind**

- $\mathbf{0}$ if $\mathbf{LH}(A) = 0$.
- $\boldsymbol{\omega}$ if $\mathbf{LH}(A)$ is limit.
- $\mathbf{1}$ if $\mathbf{LH}(A) = x' + 1$, and $A_{x'}$ is the constant $\underline{1}$.
- $\mathbf{\Omega}$ if $\mathbf{LH}(A) = x' + 1$, and $A_{x'} \neq \underline{1}$.

(ii) The morphism T is said to be **deficient** when $\mathbf{LH}(T) = f + \mathbf{E}_{0x'}$ for some $x' \neq 0$; when T is not deficient, then the source and the target of T are of the same kind, which is by definition the **kind** of T.

12.2.30. Remark (12.1.42).
It would be now the time to carry out an argument of *separation of variables* in the case of kind $\mathbf{\Omega}$; unfortunately, our hypotheses on $\mathcal{C}$ are too limited for that.

So essentially we have obtained a decomposition theorem, and a classification for connected types.

We shall now study more specific types.

12.3. <u>Arrows</u>

12.3.1. <u>Definition</u>.
Assume that $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ $(= \mathcal{C} \to \boldsymbol{O})$ are types; then we define the type $\boldsymbol{\sigma} \to \boldsymbol{\tau}$ to be

$$(\boldsymbol{\sigma} \times \mathcal{C}) \to \boldsymbol{O} \ .$$

This definition makes sense, since:

– $\boldsymbol{\sigma}$ enjoys the sum property (12.2.13).

– the sum property is preserved by $\times$.

12.3.2. <u>Notations</u>.
It is more convenient to consider $\boldsymbol{\sigma} \to \boldsymbol{\tau}$ as a category of functors from $\boldsymbol{\sigma}$ to $\boldsymbol{\tau}$, namely, via the isomorphism between

– functors from $\boldsymbol{\sigma} \times \mathcal{C}$ to $\mathbf{ON}$.

– functors from $\boldsymbol{\sigma}$ to $\boldsymbol{\tau}$.

This identification is perfectly trivial, but may induce some ambiguity, hence we make the following conventions:

1. When we speak of a ptyx of type $\boldsymbol{\sigma} \to \boldsymbol{\tau}$, we view it as a functor from $\boldsymbol{\sigma}$ to $\boldsymbol{\tau}$.

2. We shall denote by A^* the corresponding functor from $\boldsymbol{\sigma} \times \mathcal{C}$ to $\mathbf{ON}$: $A^*(a \otimes a') = A(a)(a')$, $A^*(t \otimes t') = A(t)(t')$.

3. We make similar conventions for morphisms.

The definitions of 12.2 enable us to define a trace for A, namely $\mathsf{Tr}(A^*)$; however, the following definition (a trivial variant) is far more flexible:

$$\mathsf{Tr}(A) = \Big\{\Big((z,b),a\Big) \,;\, (z, a \otimes b) \in \mathsf{Tr}(A^*)\Big\}$$
$$\mathsf{Tr}(T)\Big(\Big((z,b),a\Big)\Big) = \Big((T^*(a \otimes b)(z), b), a\Big) \ ;$$

observe that

$$\mathsf{Tr}(T)\Big((z', a)\Big) = \Big(\mathsf{Tr}\Big(T(a)\Big)(z'), a\Big) \ ,$$

since

$$\mathsf{Tr}\Big(T(a)\Big)\Big(z\,;\,b)\Big) = (T(a)(b)(z), b) = (T^*(a \otimes b)(z), b) \ .$$

12.3.3. Theorem (Normal Form Theorem).
If $A \in |\boldsymbol{\sigma} \to \boldsymbol{\tau}|$ and $a \in |\boldsymbol{\sigma}|$, if $z \in \mathsf{Tr}\Big(A(a)\Big)$, then z can be uniquely expressed as

$$z = (z_0\,;\,a_0\,;\,t\,;\,a)_A$$

where $a_0 \in |\boldsymbol{\sigma}_{fd}|$, $t \in I(a_0, a)$ and $z_0 \in \mathsf{Tr}\Big(A(a_0)\Big)$; the meaning of this **denotation** (**normal form**) is:

(i) $z = \mathsf{Tr}\Big(A(t)\Big)(z_0)$.

(ii) $\mathsf{rg}\Big(\mathsf{Tr}(t)\Big)$ is minimal for inclusion among solutions of (i).

Proof. Assume that $z = (z', b_0)$; then the normal form property w.r.t. A^* yields a normal form $z' = (z'_0\,;\,a_0 \otimes b_0\,;\,t \otimes \mathsf{id}_{b_0}\,;\,a \otimes b_0)_{A^*}$, and a_0 is such that $\{(b, t)\,;\,t \in I(a, b)\}$ is finite; this forces a_0 to be finite dimensional. Now we propose the normal form $(z', b_0) = \Big((z'_0, b_0)\,;\,a_0\,;\,t\,;\,a\Big)_A$: certainly $(z', b_0) = \mathsf{Tr}\Big(A(t)\Big)\Big((z'_0, b_0)\Big)$, and the fact that $\mathsf{rg}\Big(\mathsf{Tr}(t)\Big)$ is minimum for inclusion among solutions of (i) comes from the normal form property of A^*. □

12.3.4. Proposition.

(i) Assume that $A \in |\boldsymbol{\sigma} \to \boldsymbol{\tau}|$, $t \in I(a, b)$; then $z = (z_0\,;\,a_0\,;\,t'\,;\,b)_A \in \mathsf{rg}\Big(\mathsf{Tr}\Big(A(t)\Big)\Big) \leftrightarrow \mathsf{rg}\Big(\mathsf{Tr}(t')\Big) \subset \mathsf{rg}\Big(\mathsf{Tr}(t)\Big)$.

(ii) Assume that $T \in I(A, B)$, $a \in |\boldsymbol{\sigma}|$; then $z = (z_0\,;\,a_0\,;\,t'\,;\,a)_B \in \mathsf{rg}\Big(\mathsf{Tr}\Big(T(a)\Big)\Big)$ iff $(z_0, a_0) \in \mathsf{rg}\Big(\mathsf{Tr}(T)\Big)$.

Proof. (i): Obviously $A(t)\Big((z_0\,;\,a_0\,;\,t''\,;\,a)_A\Big) = (z_0\,;\,a_0\,;\,tt''\,;\,b)_A$, hence $z \in \mathsf{rg}\Big(\mathsf{Tr}\Big(A(t)\Big)\Big)$ iff t' is of the form tt'':

– if $t' = tt''$, then $\mathsf{Tr}(t') = \mathsf{Tr}(t)\mathsf{Tr}(t'')$ hence $\mathsf{rg}\Big(\mathsf{Tr}(t')\Big) \subset \mathsf{rg}\Big(\mathsf{Tr}(t)\Big)$.

– if $\mathsf{rg}\Big(\mathsf{Tr}(t')\Big) \subset \mathsf{rg}\Big(\mathsf{Tr}(t)\Big)$, consider the subset $X \subset \mathsf{Tr}(a)$ defined by $X = \mathsf{Tr}(t)^{-1}\Big(\mathsf{rg}\Big(\mathsf{Tr}(t')\Big)\Big)$, and apply 12.2.5 (iii): we obtain a' and $t'' \in I(a', a)$ with $\mathsf{rg}\Big(\mathsf{Tr}(t'')\Big) = X$; since $\mathsf{rg}\Big(\mathsf{Tr}(tt'')\Big) = \mathsf{rg}\Big(\mathsf{Tr}(t')\Big)$, we obtain $tt'' = t'$.

(ii): Immediate, left to the reader... . □

12.3.5. Theorem.
$|\boldsymbol{\sigma} \to \boldsymbol{\tau}|$ consists of all functors from $\boldsymbol{\sigma}$ to $\boldsymbol{\tau}$ preserving direct limits and pull-backs.

Proof. (i): Let A be an element of $|\boldsymbol{\sigma} \to \boldsymbol{\tau}|$; we prove that A preserves direct limits: if (a_i, t_{ij}) is a direct system in $\boldsymbol{\sigma}$,and $(a, t_i) = \varinjlim\ (a_i, t_{ij})$, then by 12.2.5. $\Big(\mathsf{Tr}(a), \mathsf{Tr}(t_i)\Big) = \varinjlim\ \Big(\mathsf{Tr}(a_i), \mathsf{Tr}(t_{ij})\Big)$; if $z \in \mathsf{Tr}\Big(A(a)\Big)$, then we have the normal form

$$z = (z_0\,;\, a_0\,;\, t\,;\, a)_A$$

with a_0 finite dimensional; hence the finite set $\mathsf{rg}\Big(\mathsf{Tr}(t)\Big)$ is included in $\mathsf{Tr}(a) = \bigcup\limits_i \mathsf{rg}\Big(\mathsf{Tr}(t_i)\Big)$; by 12.3.4 (i) this implies that $z \in \mathsf{rg}\Big(\mathsf{Tr}\Big(A(t_i)\Big)\Big)$ for some i. Hence $\Big(\mathsf{Tr}\Big(A(a)\Big), \mathsf{Tr}\Big(A(t_i)\Big)\Big) = \varinjlim\ \Big(\mathsf{Tr}\Big(A(a_i)\Big), \mathsf{Tr}\Big(A(t_{ij})\Big)\Big)$. We now establish that A preserves pull-backs: if $t_1 \wedge t_2 = t_3$, then, by 12.2.5, $\mathsf{Tr}(t_1) \wedge \mathsf{Tr}(t_2) = \mathsf{Tr}(t_3)$; if $z = (z_0\,;\, a_0\,;\, t\,;\, a)_A$ belongs to $\mathsf{rg}\Big(\mathsf{Tr}\Big(A(t_1)\Big)\Big)$ $\cap\ \mathsf{rg}\Big(\mathsf{Tr}\Big(A(t_2)\Big)\Big)$, this means that $\mathsf{rg}\Big(\mathsf{Tr}\Big(A(t)\Big)\Big) \subset \mathsf{rg}\Big(\mathsf{Tr}\Big(A(t_1)\Big)\Big) \cap \mathsf{rg}\Big(\mathsf{Tr}\Big(A(t_2)\Big)\Big)$ $(= \mathsf{rg}\Big(\mathsf{Tr}\Big(A(t_3)\Big)\Big)$) by 12.3.4 (i); hence $z \in \mathsf{rg}\Big(\mathsf{Tr}\Big(A(t_3)\Big)\Big)$. This establishes that $A(t_1) \wedge A(t_2) = A(t_3)$.

(ii): Assume now that A is a functor from $\boldsymbol{\sigma}$ to $\boldsymbol{\tau}$ preserving direct limits and pull-backs; then one easily shows that A enjoys the normal form theorem 12.3.3 (left to the reader). The question is how to obtain the normal form property for A^*; but assume that $z \in A^*(a \otimes b)$, i.e. $z \in A(a)(b)$; the normal form property for $A(a)$ yields:

$$z = (z_0\,;\, b_0\,;\, u\,;\, b)_{A(a)}$$

and the normal form theorem yields

$$(z_0, b_0) = \Big((z_0', b_0)\,;\, a_0\,;\, t\,;\, a\Big)_A$$

from which we deduce

$$z = (z'_0 \,;\, a_0 \otimes b_0 \,;\, t \otimes u \,;\, a \otimes b)_{A^*} \; .$$

This is a normal form, because if $z = A^*(t' \otimes u')(z')$, i.e. $z = A(t')(u')(z')$, with $t' \in I(a', a)$, $u' \in I(b', b)$, then, if $z'' = A(t')(\mathbf{E}_{b'})(z')$, since $z = A(a)(u')(z'') = A(a)(u)(z_0)$, we obtain a unique u_1 such that $z'' = A(a)(u_1)$ (z_0), $u_1 \in I(b_0, b')$, i.e. $z'' = (z_0 \,;\, b_0 \,;\, u_1 \,;\, b')_{A(a)}$. The unicity condition in the normal form implies $u'u_1 = u$. Now recall that $A(t')$ is a morphism from $A(a')$ to $A(a)$ in the sense of 12.2.1; hence $z_0 = A(t')(b_0)(z''_0)$ for some $z''_0 \in A(a')(b_0)$. The normal form theorem yields $z''_0 = A(t_1)(b_0)(z'_0)$ for some $t_1 \in I(a_0, a')$, and $t'_1 t_1 = t$.

Finally $A(t_1)(u_1)(z'_0) = A(a')(u_1)(z''_0) = y$ is such that $A(t')(u')(y) = A(t't_1)(u'u_1)(z'_0) = A(t)(u_1)(z'_0) = z$, hence $y = z'$: from that we get $z' = A^*(t_1 \otimes u_1)(z'_0)$. The fact that t_1 and u_1 are unique is immediate. $\square$

12.3.6. Theorem.
Assume that $A, B \in |\boldsymbol{\sigma} \to \boldsymbol{\tau}$; then $I(A, B)$ is the set of natural transformations from A to B.

Proof. (i): Assume that $T \in I(A, B)$; giving back to the definition, this means that $T^* \in I(A^*, B^*)$, which in turn implies that the diagrams

$$\begin{tikzcd} A(a) \arrow[r, "T(a)"] \arrow[d, "A(t)"'] & B(a) \arrow[d, "B(t)"] \\ A(b) \arrow[r, "T(b)"'] & B(b) \end{tikzcd}$$

are commutative, hence T is a natural transformation from A to B.

(ii): Conversely, if T is a natural transformation from A to B, it is immediate that T^* is a natural transformation from A^* to B^*, hence it suffices to look at the action of T^* on denotations, or equivalently the action of T on denotatins: it suffices to establish that

$$\mathsf{Tr}\Big(T(a)\Big)\Big((z_0 \,;\, a_0 \,;\, t \,;\, a)_A\Big) = \Big(\mathsf{Tr}\Big(T(a_0)\Big)(z_0) \,;\, a_0 \,;\, t \,;\, a\Big)_B$$

and the non-trivial part of the proof consists in showing that $(z_0 \,;\, a_0) \in \mathsf{Tr}(A) \to \Big(\mathsf{Tr}\Big(T(a_0)\Big)(z_0) \,;\, a_0\Big) \in \mathsf{Tr}(B)$. This is established on the model of Lemma 12.1.10: if $t \in I(a_1, a_0)$, apply the property 12.2.13 to $A = a_1$, $B_0 = B_1 = a_0$, $T_0 = T_1 = t$; we obtain a_2 together with $t_0, t_1 \in I(a_0, a_2)$ such that $t_0 \wedge t_1 = t_0 t = t_1 t$.

$$\begin{tikzcd}
A(a_1) \arrow[r, "A(t)"] \arrow[d, "T(a_1)"'] & A(a_0) \arrow[r, shift left, "A(t_0)"] \arrow[r, shift right, "A(t_1)"'] \arrow[d, "T(a_0)"'] & A(a_2) \arrow[d, "T(a_2)"] \\
B(a_1) \arrow[r, "B(t)"'] & B(a_0) \arrow[r, shift left, "B(t_0)"] \arrow[r, shift right, "B(t_1)"'] & A(a_2)
\end{tikzcd}$$

Now, assume that $(z_0, a_0) \in \mathsf{Tr}(A)$, and that $x = \mathsf{Tr}\big(T(a_0)\big)(z_0) \in \mathsf{rg}\Big(\mathsf{Tr}\big(B(t)\big)\Big)$, say $\mathsf{Tr}\big(T(a_0)\big)(z_0) = \mathsf{Tr}\big(B(t)\big)(y)$; then $\mathsf{Tr}\big(B(t_0)\big)(x) = \mathsf{Tr}\big(B(t_1)\big)(x) = \mathsf{Tr}\big(B(t_0 t)\big)(y) = \mathsf{Tr}\big(B(t_1 t)\big)(y)$. From that we obtain: $\mathsf{Tr}\big(A(t_0)\big)(z_0) = \mathsf{Tr}\big(A(t_1)\big)(z_0)$, since the function $\mathsf{Tr}\big(T(a_2)\big)$ is injective. This proves that $\mathsf{Tr}\big(A(t_0)\big)(z_0) \in \mathsf{rg}\Big(\mathsf{Tr}\big(A(t_0) \wedge A(t_1)\big)\Big)$ by 12.2.5 (ii); but A preserves pullbacks, so $A(t_0) \wedge A(t_1) = A(t_0 \wedge t_1) = A(t_0 t)$, $\mathsf{Tr}\big(A(t_0)\big)(z_0) = \mathsf{Tr}\big(A(t_0)\big)$ $\mathsf{Tr}\big(A(t)\big)(z'_0)$, so we obtain $z_0 = \mathsf{Tr}\big(A(t)\big)(z'_0)$ for some $z'_0 \in \mathsf{Tr}\big(A(a_1)\big)$. Now, the hypothesis $z_0 \in \mathsf{rg}\Big(\mathsf{Tr}\big(A(t)\big)\Big)$ entails, since $(z_0, a_0) \in \mathsf{Tr}(A)$ that $a_1 = a_0$ and $t = \mathbf{E}_{a_0}$. We have therefore shown that the pair $\Big(\mathsf{Tr}\big(T(a_0)\big)(z_0)$,
$a_0\Big)$ belongs to $\mathsf{Tr}(B)$. □

We have implicitly used the

12.3.7. Proposition.
$\mathsf{Tr}(A)$ is the set of all pairs (z_0, a_0) with:

1. a_0 finite dimensional, $z_0 \in \mathsf{Tr}\big(A(a_0)\big)$.

2. if $z_0 \in \mathsf{rg}\Big(\mathsf{Tr}\big(A(t)\big)\Big)$ for some $t \in I(a_1, a_0)$, then $a_1 = a_0$ and $t = \mathbf{E}_{a_0}$.

Proof. Immediate from the normal form theorem. □

12.3.8. Theorem.

Assume that (A, T_i) enjoys 8.1.11 (i)–(iii) w.r.t. (A_i, T_{ij}) in $\boldsymbol{\sigma} \to \boldsymbol{\tau}$; then the following are equivalent:

(i) $(A, T_i) = \underset{\longrightarrow}{\lim}\ (A_i, T_{ij})$.

(ii) $\big(A(a), T_i(a)\big) = \underset{\longrightarrow}{\lim}\ \big(A_i(a), T_{ij}(a)\big)$ for all $a \in |\boldsymbol{\sigma}|$.

(iii) $\big(A(a), T_i(a)\big) = \underset{\longrightarrow}{\lim}\ \big(A_i(a), T_{ij}(a)\big)$ for all $a \in |\boldsymbol{\sigma}_{fd}|$.

(iv) For any direct system (a_i, t_{ij}) with a direct limit (a, t_i) in $\boldsymbol{\sigma}$, $\big(A(a), T_i(t_i)\big) = \underset{\longrightarrow}{\lim}\ \big(A_i(a_i), T_{ij}(t_{ij})\big)$.

(v) $\big(\mathsf{Tr}(A), \mathsf{Tr}(T_i)\big) = \underset{\longrightarrow}{\lim}\ \big(\mathsf{Tr}(A_i), \mathsf{Tr}(T_{ij})\big)$.

Proof. This is very close to 12.2.8 and 12.1.14; left to the reader. The equivalent (iv) is made possible by 12.3.5... . □

12.3.9. Theorem.
Assume that $T_i \in I(A_i, B)$ enjoy 8.1.25 (i) in the category $\to$; then the following are equivalent:

(i) $T_1 \wedge T_2 = T_3$.

(ii) $T_1(a) \wedge T_2(a) = T_3(a)$ for all $a \in |\boldsymbol{\sigma}|$.

(iii) $T_1(a) \wedge T_2(a) = T_3(a)$ for all $a \in |\boldsymbol{\sigma}_{fd}|$.

(iv) $T_1(t_1) \wedge T_2(t_2) = T_3(t_3)$ for all t_1, t_2, t_3 in $\boldsymbol{\sigma}$ such that $t_1 \wedge t_2 = t_3$.

(v) $\mathsf{Tr}(T_1) \wedge \mathsf{Tr}(T_2) = \mathsf{Tr}(T_3)$.

Proof. This is very close to 12.2.10, 12.1.16; the equivalent (iv) is made possible by 12.3.5. The details of the proof are left to the reader. □

12.3.10. Theorem.
The objects of $\boldsymbol{\sigma} \to \boldsymbol{\tau}$ are increasing on morphisms: $t \leq u \to A(t) \leq A(u)$.

(In fact $T \leq U$, $t \leq u \to T(t) \leq U(u)$.)

Proof. $t \leq u \to \exists v\ \exists w\ vu = wvt$, hence $A(v)A(u) = A(w)A(v)A(t)$, hence $A(t) \leq A(u)$. $\Box$

12.3.11. Theorem.

Let A be a functor from $\boldsymbol{\sigma}$ to ${}_p\boldsymbol{\tau}$ preserving direct limits and pull-backs; assume furthermore that A is finite dimensional, and that A is increasing on morphisms. Then A is (isomorphic to) a ptyx of type $\boldsymbol{\sigma} \to \boldsymbol{\tau}$. (We use the notation ${}_p\boldsymbol{\tau}$ for the category of preptykes of type $\boldsymbol{\tau}$.)

Proof. It suffices to show that A maps (up to isomorphism) $\boldsymbol{\sigma}$ in $\boldsymbol{\tau}$, or equivalently that: $A^*(a \otimes b)$ is a well-order for all $b \in |\mathcal{C}|$ (if $\boldsymbol{\tau} = \mathcal{C} \to \boldsymbol{O}$) and $a \in |\boldsymbol{\sigma}|$.

So let us assume for contradiction that $z_n = (z_0^n\,;\, a_0^n \otimes b_0^n\,;\, t_n \otimes u_n\,;\, a \times b)_{A^*}$ is a s.d.s. in $A^*(a \otimes b)$; the hypothesis A^* finite dimensional forces z_0^n, a_0^n, b_0^n to vary through finite sets, and we may as well suppose that $z_0^n = z_0$, $a_0^n = a_0$, $b_0^n = b_0$; now two cases occur:

- Assume that there is an infinite $X \subset I\!N$ such that $n \leq m$, $n, m \in X \to t_n \leq t_m$; then if we consider $z'_n = (z_0\,;\, a_0 \otimes b_0\,;\, \mathbf{E}_{a_0} \otimes u_n\,;\, a_0 \otimes b)_{A^*}$, then $n < m$ and $n, m \in X \to z'_m < z'_n$: $z_m = A^*(t_m \otimes \mathbf{E}_b)(z'_m) < A^*(t_n \otimes \mathbf{E}_b)(z'_n) = z_n$, hence $A^*(t_n \otimes \mathbf{E}_b)(z'_m) \leq A^*(t_m \otimes \mathbf{E}_b)(z'_m) < A^*(t_n \otimes \mathbf{E}_b)(z'_n)$, and so $z'_m < z'_n$. We have therefore constructed a s.d.s. in $A^*(a_0 \otimes b) = A(a_0)(b)$. Now observe that $A(a_0)$ is finite dimensional (12.3.12 below!), hence the preptyx $A(a_0)$ is a ptyx, so $A(a_0)(b)$ is well-founded.

- Assume that there is an infinite $X \subset I\!N$ such that $n \leq m$, $n, m \in X \to t_n \not\leq t_m$. Assume that $\mathsf{Tr}(a_0) = \{(x_1, c_1), ..., (x_p, c_p)\}$; if $t \in I(a_0, a)$ we define a sequence $s(t)$ by: $s(t) = (y_1, ..., y_p)$, where y_i is s.t. $t(c_i)(x_i) = y_i$. Now, one easily checks that: if $s(t) = (y_1, ..., y_p)$, $s(t') = (y'_1, ..., y'_p)$, then $t \leq t' \leftrightarrow y_1 \leq y'_1 \wedge ... \wedge y_p \leq y'_p$. We define a partition of $C = \{(n, m)\,;\, n < m \wedge n, m \in X\}$ by $C = C_1 \cup ... \cup C_p$, with $(n, m) \in C_i$ iff $y_1^n \leq y_1^m, ..., y_{i-1}^n \leq y_{i-1}^m$, $y_i^n > y_i^m$, where $s(t_n) = (y_1^n, ..., y_p^n)$, $s(t_m) = (y_1^m, ..., y_p^m)$. Ramsey's theorem applied to this partition yields $Y \subset X$, Y infinite together with $i_0 \leq p$ s.t. $(n, m) \in C_{i_0}$ for all $n, m \in Y$, $n < m$. Then the sequence

$(y_{i_0}^n)_{n \in Y}$ is a s.d.s. in $0n$, contradiction. $\square$

12.3.12. Proposition (12.1.33).
If A and a are finite dimensional, so is $A(a)$.

Proof. This is an analogue of 12.1.33; we easily get $\mathsf{dim}\Big(A(a)\Big) \leq \mathsf{dim}(A) \cdot 2^{\mathsf{dim}(a)}$. $\square$

12.3.13. Theorem (12.1.34).
Assume that X is a finite set of finite dimensional ptykes of type $\boldsymbol{\sigma}$; define another set X' by:

$$a \in X' \leftrightarrow \exists a_1, a_2, a_3 \in X \; \exists t_1, t_2, t_3, t_1 \in I(a_i, a) \;, \qquad (i = 1, 2, 3)$$

such that $\mathsf{Tr}(a) = \mathsf{rg}\Big(\mathsf{Tr}(t_1)\Big) \cup \mathsf{rg}\Big(\mathsf{Tr}(t_2)\Big) \cup \mathsf{rg}\Big(\mathsf{Tr}(t_3)\Big)$.

a_1

t_1

t_2

a_2 a

T_3

a_3

Finally, define X'' by: $a \in X'' \leftrightarrow \exists a' \in X' \; I(a, a') \neq \emptyset$, and define the category $\mathcal{C}_{X''}$ by:

objects: elements of X''.
morphisms: $I(a, a')$.

Now, assume that A is a functor from $\mathcal{C}_{X''}$ to $\boldsymbol{\tau}$, with the following features:

(i) If $a \in X''$ and $z \in \mathsf{Tr}\Big(A(a)\Big)$, then one can express z as $\mathsf{Tr}\Big(A(t)\Big)(z_0)$ for some $a_0 \in X$, $t \in I(a_0, a)$, $z_0 \in \mathsf{Tr}\Big(A(a_0)\Big)$. Furthermore, the condition "$\mathsf{rg}\Big(\mathsf{Tr}(t)\Big)$ minimal for C" renders t uniquely determined.

(ii) $A(a)$ is finite dimensional for all $a \in X$.

(iii) If $a, a' \in X''$, if $t, u \in I(a, a')$, then

$$t \leq u \to A(t) \leq A(u) \ .$$

Then there is one and only one ptyx F of types $\boldsymbol{\sigma} \to \boldsymbol{\tau}$ such that:

1. $F \upharpoonright \mathcal{C}_{X''} = A$.

2. $\mathsf{Tr}(F)$ is a set of pair $(z\,;\,a_0)$ with $a_0 \in X$.

<u>Proof</u>. We define the functor F^* in a way very close to 12.1.34; the details are left to the reader. □

12.3.14. <u>Remarks</u> (12.1.35).

(i) 12.3.13 extends to natural transformations: see 12.1.35 (i).

(ii) The question of the characterization of finite dimensional ptykes of type $\boldsymbol{\sigma} \to \boldsymbol{\tau}$ by means of a finite amount of information *is not solved* by 12.3.13: even if we know the similar characterization for the type $\boldsymbol{\tau}$, there is still the problem of the finiteness of $\mathcal{C}_{X''}$. Since this category is a category of finite dimensional objects, this question reduces to the finiteness of X'', which in turn can be reduced to the finiteness of X'. Now we see that X' will be finite, provided the following condition is satisfied by $\boldsymbol{\sigma}_{fd}$: if $A, B \in |\boldsymbol{\sigma}_{fd}|$, then $A \cdot B \in |\boldsymbol{\sigma}_{fd}|$. There is no evidence in general that $A \cdot B$ is even an element of $\mathcal{C}$ such that $\boldsymbol{\sigma} = \mathcal{C} \to \boldsymbol{O}$. See Exercise 12.3.15 below. However, in practice 12.3.13 can be used to give finite characterizations, especially in the case of the finite types, generated from $\boldsymbol{O}$ by means of $\to$ and X... .

12.3.15. Exercise.

(i) Assume that $\boldsymbol{\sigma} = \mathcal{C} \to \boldsymbol{O}$. Assume that $\mathcal{C}$ has the following property: given $a, b \in |\mathcal{C}_{fd}|$, $c \in |\mathcal{C}|$, together with $a \xrightarrow{t} c$, $b \xrightarrow{u} c$, there exists $d \in |\mathcal{C}_{fd}|$ together with morphisms $a \xrightarrow{t'} d$, $b \xrightarrow{u'} d$, $d \xrightarrow{v} c$, rendering the diagram

$$\begin{tikzcd}
a \arrow[dr, "t'"] \arrow[drr, "t"] & & \\
& d \arrow[r, "v"] & c \\
b \arrow[ur, "u'"] \arrow[urr, "u"'] & &
\end{tikzcd}$$

commutative; moreover, if (d_1, t'_1, u'_1, v_1) is another solution, there is a unique w: $d \xrightarrow{w} d_1$ such that: $t'_1 = wt'$, $u'_1 = wu'$, $v = v_1 w$.

Prove that, if A and $B \in |\boldsymbol{\sigma}|$, so does their product $A \cdot B$.

(ii) With the notation of (i), the 3-uple (t', u', d) is called a minimal cover of (a, b). We assume not that it has the property of (i) together with: For all $a, b \in |\mathcal{C}_{fd}|$ there are only finitely many minimal covers of (a, b).

Prove that, if A and $B \in |\boldsymbol{\sigma}_{fd}|$, so does their product $A \cdot B$.

(iii) Assume that $\boldsymbol{\sigma}$ has the property that: $A, B \in |\boldsymbol{\sigma}_{fd}| \to A \cdot B \in |\boldsymbol{\sigma}_{fd}|$; prove that given $A, B \in |\boldsymbol{\sigma}_{fd}|$, the set of $C \in |\boldsymbol{\sigma}_{fd}|$ such that for some $T \in I(A, C)$, $U \in I(B, C)$, we have $\mathsf{Tr}(C) = \mathsf{rg}\big(\mathsf{Tr}(T)\big) \cup \mathsf{rg}\big(\mathsf{Tr}(U)\big)$.

From this deduce that the category $\boldsymbol{\sigma}$ enjoys the properties of $\mathcal{C}$ described in (i) and (ii).

(*Hint. (i) and (ii) follow from rather immediate considerations as to the normal form w.r.t.* $A \cdot B$. *(iii) is obtained by considering the finite set* $\{a \in |\mathcal{C}|\,;\, \exists z\, (z, a) \in \mathsf{Tr}(A \cdot B)\}$, *and by showing that the minimal covers of* (A, B) *are determined by their restriction to the finite category generated from this set*$\ldots$)

12.3.16. Discussion (12.1.37).

In general, we cannot define weakly finite ptykes of type $\boldsymbol{\sigma} \to \boldsymbol{\tau}$; however, in practice, we have often a concept of weakly finite ptyx in $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$; then it is possible to define weakly finite ptykes of type $\boldsymbol{\sigma} \to \boldsymbol{\tau}$ to be those A's which send weakly finite ptykes of type $\boldsymbol{\sigma}$ on weakly finite ptykes of type $\boldsymbol{\tau}$. This will be the case with finite types.

Similarly, it will be possible in practice to define various notions of recursive tykes of type $\boldsymbol{\sigma} \to \boldsymbol{\tau}$; this is left to the reader. In particular, this definition is not difficult to give in the case of ptykes of finite types... .

The non-algebraic theory is not very satisfactory, except if we assume that $\boldsymbol{\tau} = \boldsymbol{O}$; this is not so bad because from that we can handle the general case $\boldsymbol{\sigma} \to \boldsymbol{\tau}$, when $\boldsymbol{\tau}$ is of the form $\boldsymbol{\tau}' \to \boldsymbol{O}$, using the isomorphism between $\boldsymbol{\sigma} \to \boldsymbol{\tau}$ and $\boldsymbol{\sigma} \times \boldsymbol{\tau}' \to \boldsymbol{O}$.

12.2.17. Theorem (12.1.40).
If $a \in |\boldsymbol{\sigma}|$, let us denote by $\underline{a} + \mathsf{Id}^{\boldsymbol{\sigma}}$ the element of $|\boldsymbol{\sigma} \to \boldsymbol{\sigma}|$ defined by:

$$(\underline{a} + \mathsf{Id}^{\boldsymbol{\sigma}})(b) = a + b \qquad (a + \mathsf{Id}^{\boldsymbol{\sigma}})(t) = \mathbf{E}_a + t \ .$$

Similarly, if $t \in I(a, a')$, let us denote by $t + \mathsf{Id}^{\boldsymbol{\sigma}}$ the element of $I(\underline{a} + \mathsf{Id}^{\boldsymbol{\sigma}}, \underline{a}' + \mathsf{Id}^{\boldsymbol{\sigma}})$ defined by: $(t + \mathsf{Id}^{\boldsymbol{\sigma}})(a) = t + \mathbf{E}_a$.

(i) If A is of kind $\boldsymbol{\sigma} \to \boldsymbol{O}$, then A and $A \circ (\underline{a} + \mathsf{Id}^{\boldsymbol{\sigma}})$ are of the same kind.

(ii) If $T \in I(A, B)$, then T and $T \circ (\underline{t} + \mathsf{Id}^{\boldsymbol{\sigma}})$ are of the same kind.

Proof. See 12.1.40. □

12.3.18. Outline of the construction (12.1.42).
Separation of variables and related topics can be easily carried out on the model of what we did for $\boldsymbol{\sigma} = \mathbf{DIL}$, in the general context of $\boldsymbol{\sigma} \to \boldsymbol{O}$. There is, however, an important point: we used the unicity of the sum decomposition in $\mathbf{DIL}$; we cannot therefore transfer the argument to the case where $\boldsymbol{\sigma}$ is not connected. However, it will be possible to render the decomposition unique, by a representation $\boldsymbol{\sigma} = \prod_i \boldsymbol{\sigma}_i$; if the indices i are themselves well-ordered, it will be possible to render unique the sum

decomposition (see 12.4.11 for more details). Hence, we obtain a separation of variables, which depends on a well-ordering of the components of $\boldsymbol{\sigma}$.

12.3.19. Definition.
Define $\Xi^{cl}_{\boldsymbol{\sigma}}$ to be the sum $\sum_{i<0n} B_i$ of ptykes of type $\boldsymbol{\sigma}$ (if $\boldsymbol{\sigma} = \mathcal{C} \to \boldsymbol{O}$ is a proper class: this is always true, except for $\mathcal{C} = \emptyset$), listed in a certain order, which we do not specify. When $A \in |\boldsymbol{\sigma} \to \boldsymbol{O}|$, we define

$$\|A\| = A(\Xi^{cl}_{\boldsymbol{\sigma}}) \ .$$

12.3.20. Remarks.

(i) The meaning of the expression $A(\Xi^{cl}_{\boldsymbol{\sigma}})$ is the following: let a_x be the sum of th first x points in our given enumeration of $|\boldsymbol{\sigma}|$; then

$$\|A\| = \varinjlim{}^{*} \Big(A(a_x), A(\mathbf{E}_{a_x a_y})\Big) \ .$$

This direct limit is in general a proper class. But this is a well-ordered one.

(Proof. Assume that (z_n) is a s.d.s. in $\|A\|$, then it is easy to produce another s.d.s. in some $A(a_x)$... . □

This ordinal class corresponds to the ordinal classes $F(0n)$ that we used for dilators.

(ii) $\|A\|$ is of course independent of the order of summation in $\Xi^{cl}_{\boldsymbol{\sigma}}$: if $\Xi'^{cl}_{\boldsymbol{\sigma}}$ is another sum, then we can easily build a natural transformation $T \in I(\Xi^{cl}_{\boldsymbol{\sigma}}, \Xi^{cl}_{\boldsymbol{\sigma}})$, and so $A(T) \in I(\|A\|, \|A\|')$, hence $\|A\| \leq \|A'\|$. By symmetry $\|A\| = \|A'\|$.

(iii) If we define the predecessors of A as we say in 12.3.18, then it is likely that the class of predecessors of A, ordered by the predecessor relation, is a well-founded order of height $\|A\|$.

(iv) If we define $\|T\| = T(\Xi^{cl}_{\boldsymbol{\sigma}})$, we obtain $\| \cdot \|$ as a functor, but this functor depends on the enumeration of $|\boldsymbol{\sigma}|$... . Let us call such a functor a **norm** for $\boldsymbol{\sigma} \to \boldsymbol{O}$. Then "the" norm has the following property: (A_i, T_{ij}) has a direct limit in $\boldsymbol{\sigma} \to \boldsymbol{O}$ iff $(\|A_i\|, \|T_{ij}\|)$ has a direct limit (of course among ordinal classes).

12.4. Products

12.4.1. Definition.

Assume that $(\boldsymbol{\sigma}_i)_{i<x}$ are types; then we define a new type $\boldsymbol{\sigma} = \prod_{i<x} \boldsymbol{\sigma}_i$ as follows: if $\boldsymbol{\sigma}_i = \mathcal{C}_i \to \boldsymbol{O}$, let $\mathcal{C}$ be the disjoint union of the categories $\mathcal{C}_i$; then $\boldsymbol{\sigma} = \mathcal{C} \to \boldsymbol{O}$.

A particular case is the case $x = 2$; then we use the notation $\boldsymbol{\sigma} \times \boldsymbol{\tau}$.

12.4.2. Notations.

(i) A functor A from $\mathcal{C} \to \boldsymbol{O}$ can be identified with the family $(A_i)_{i<x}$, with $A_i = A \upharpoonright \mathcal{C}_i$. In the same way $T \in I(A, B)$ can be identified with the family $(T_i)_{i<x}$, with $T_i = T \upharpoonright \mathcal{C}_i$.

(ii) This induces the notation $A = \bigotimes_{i<x} A_i$, $T = \bigotimes_{i<x} T_i$. An equivalent of $\prod_{i<x} \boldsymbol{\sigma}_i$ is:
objects: families $(A_i)_{i<x}$, with $A_i \in |\boldsymbol{\sigma}_i|$.
morphisms: families $(T_i)_{i<x}$, with $T_i \in I(A_i, B_i)$... .

(iii) In the case $x = 2$, we use $A \otimes B$, $T \otimes U$.

(iv) Typical functors are the ptykes π^i of type $(\boldsymbol{\sigma} \to \boldsymbol{\sigma}_i)$:

$$\pi^i\Big(\bigotimes_{j<x} A_j\Big) = A_i \qquad \pi^i\Big(\bigotimes_{j<x} T_j\Big) = T_i \ .$$

In the case $i = 2$, our notations are slightly inconsistent:

$$\pi^1(A \otimes B) = A\,, \ \ \text{etc...} \qquad \pi^2(A \otimes B) = B\,, \ \ \text{etc...}\ .$$

12.4.3. Definition.

(i) $\mathsf{Tr}\Big(\bigotimes_{i<x} A_i\Big) = \{(z, i)\,;\, z \in \mathsf{Tr}(A_i)\}\,.$

(ii) $\mathsf{Tr}\Big(\bigotimes_{i<x} T_i\Big)\Big((z, i)\Big) = (\mathsf{Tr}(T_j)(z), i)\,.$

12.4.4. Remark.

This is nothing but the application of the general definition 12.2.3. In particular we can apply the general results concerning the trace.

12.4.5. Theorem.

The following are equivalent:

(i) $\Big(\bigotimes\limits_{l<x} A^l, \bigotimes\limits_{l<x} T_i^l\Big) = \varinjlim \Big(\bigotimes\limits_{l<x} A_i^l, \bigotimes\limits_{l<x} T_{ij}^l\Big).$

(ii) $(A^l, T_i^l) = \varinjlim (A_i^l, T_{ij}^l)$ for all $l < x$.

Proof. Left to the reader. $\square$

12.4.6. Theorem.

The following are equivalent:

(i) $\bigotimes\limits_{l<x} T_1^l \wedge \bigotimes\limits_{l<x} T_2^l = \bigotimes\limits_{l<x} T_3^l.$

(ii) $T_1^l \wedge T_2^l = T_3^l$ for all $l < x$.

Proof. Left to the reader. $\square$

12.4.7. Theorem.

$\mathsf{dim}\Big(\bigotimes\limits_{i<x} A_i\Big) = \sum\limits_{i<x} \mathsf{dim}(A_i)$ (cardinal sum).

In particular, $\bigotimes\limits_{i<x} A_i$ is finite dimensional iff all A_i's are finite dimensional, and all A_i's, but a finite number, are $\underline{0}$.

Proof. Left to the reader. $\square$

12.4.8. Remark.

In particular, if we know the finite dimensional ptykes of alltypes $\boldsymbol{\sigma}_i$, then we have an effective way of generating the finite dimensional ptykes of type $\prod\limits_{i<x} \boldsymbol{\sigma}_i$.

12.4.9. Proposition.
The preptykes of type $\prod_{i<x} \boldsymbol{\sigma}_i$ are exactly the $\bigotimes_{i<x} A_i$, with A_i preptyx of type $\boldsymbol{\sigma}_i$ for all $i < x$.

Proof. Left to the reader. $\square$

12.4.10. Discussion.
We only discuss the notion of recursive and weakly finite ptykes of type X:

(i) If the notions of recursive ptyx of type $\boldsymbol{\sigma}$ and of type $\boldsymbol{\tau}$ make sense, then we say that a ptyx A of type $\boldsymbol{\sigma} \times \boldsymbol{\tau}$ is recursive iff $\pi^1 A$ and $\pi^2 A$ are recursive.

(ii) The same thing for weak finiteness: if we have given a meaning to "weakly finite" in $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$, then A is weakly finite iff $\pi^1 A$ and $\pi^2 A$ are weakly finite.

12.4.11. Discussion.
The principal problem, if we want to carry out the non algebraic theory for products, is that, due to the lack of connectedness of the products $\prod_{i<x} \boldsymbol{\sigma}_i$, we lose the unicity of the sum decomposition. However, it is possible to obtain unique decomposition in the following situation:

(i) Assume that $\boldsymbol{\sigma} = \prod_{i<x} \boldsymbol{\sigma}_i$, and that the $\boldsymbol{\sigma}_i$'s are connected; then, if $A \in |\boldsymbol{\sigma}|$, write $\pi^i A = \sum_{j<x_i} A^i_j$ (the unique decomposition of $\pi^i A$ in $\boldsymbol{\sigma}_i$). Let $y = \sum_{j<x} x_i$, and define $(B_j)_{j<y}$ by: $B_{\sum_{j<i} x_j + k}$ = the ptyx C of type $\boldsymbol{\sigma}$ such that

- ${}^{k'}C = \underline{0}$ when $k' \neq k$.
- ${}^{k}C = A^i_k$.

(Of course we assume $k < x_i$.)

Then $A = \sum_{j<y} B_j$, and the coefficients B_j, as well as the ordinal

y, have been defined in a unique way. In the same way, we should get unique decompositions for natural transformations. These decompositions are made unique by the idea that we first list objects (morphisms) of $\mathcal{C}_0$, then of $\mathcal{C}_1, \dots$.

(ii) This idea can be used to yield unicity of the decomposition in general types:

- we represent $\boldsymbol{\sigma}$ as a product of types $\boldsymbol{\sigma}_i$ $(i \in I)$.
- then we well-order I: we are therefore in the situation of (i) (recall that $\boldsymbol{\sigma} \sim \prod\limits_{i<x} \boldsymbol{\sigma}_i$).

If the components $\boldsymbol{\sigma}_i$ of $\boldsymbol{\sigma}$ are enumerated in a well-ordered way, $(\boldsymbol{\sigma}_i)_{i<x}$, and we want to define a norm $\|\cdot\|$ on $\boldsymbol{\sigma}$, then the natural solution seems to be: $\|A\| = \sum\limits_{i<x} \|\pi^i A\|$, provided norms are defined on the $\boldsymbol{\sigma}_i$'s.

This norm has the property that (A_i, T_{ij}) has a direct limit in $\boldsymbol{\sigma}$ iff $(\|A_i\|, \|T_{ij}\|)$ has a direct limit in $\boldsymbol{\sigma}$.

Finally, can we say something as to the predecessor relation in $\boldsymbol{\sigma}$? Assuming that the components $\boldsymbol{\sigma}_i$ of $\boldsymbol{\sigma}$ are well-ordered, and that the predecessor relations are defined in the types $\boldsymbol{\sigma}_i$, then the predecessors of $\bigotimes\limits_{i<x} A_i$ are the $\bigotimes\limits_{i<x} A'_i$, where: for some $j < x$,

$$A'_i = A_i \text{ for all } i < j$$

$$A'_j \text{ is a predecessor of } A_j$$

$$A'_i = \underline{0} \text{ for all } i > j\,.$$

This definition is obviously connected to the norm $\|\cdot\|$: the well-founded class $\{B\,;\, B \;\mathsf{pred}\; A\}$ is of height $\|A\|$.

There is an obvious analogy between the sum (of ordinals, dilators, ptykes...) and the product of types... .

12.5. Ptykes of finite types

The finite types are defined by:

(i) $\boldsymbol{O}$ is a finite type.

(ii) If $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are finite types, so is $\boldsymbol{\sigma} \to \boldsymbol{\tau}$.

(iii) If $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are finite types, so is $\boldsymbol{\sigma} \times \boldsymbol{\tau}$.

(iv) The only finite types are those given by (i)–(iii).

Most of the important facts concerning finite types, more precisely, ptykes of finite types, can be extracted from the results of 12.2–12.4; there are, however, a few additional remarks that may be of some interest:

1. It is easily shown, for all finite types $\boldsymbol{\sigma}$, that:

$$A, B \in |\boldsymbol{\sigma}_{fd}| \to A \cdot B \in |\boldsymbol{\sigma}_{fd}| \ .$$

   Hence, using 12.3.14–12.3.15, it will be possible to apply Theorem 12.3.13, and to obtain an effective enumeration of all ptykes of type $\boldsymbol{\sigma} \to \boldsymbol{\tau}$ which are finite dimensional.

2. From 1, it is now easy to obtain various concepts of recursive ptykes of type $\boldsymbol{\sigma}$.

3. In a similar way, weakly finite ptykes are defined by:

   - weakly finite ptykes of type $\boldsymbol{O}$ are integers.
   - weakly finite ptykes of type $\boldsymbol{\sigma} \to \boldsymbol{\tau}$ are ptykes A such that $A(a)$ is weakly finite, for all $a \in |\boldsymbol{\sigma}|$, a weakly finite.
   - weakly finite ptykes of type $\boldsymbol{\sigma} \times \boldsymbol{\tau}$ are ptykes A such that $\pi^1 A$ and $\pi^2 A$ are weakly finite.

4. We now turn our attention towards connectedness of

   - finite types; each type $\boldsymbol{\sigma}$ can be represented as a finite product $\boldsymbol{\sigma}_0 \times ... \times \boldsymbol{\sigma}_n$ of connected finite types.

- $\boldsymbol{O}$ is connected (i.e. $n = 0$).
- if $\boldsymbol{\tau} \sim \boldsymbol{\tau}_0 \times ... \times \boldsymbol{\tau}_n$, then

$$\boldsymbol{\sigma} \rightarrow \boldsymbol{\tau} \sim (\boldsymbol{\sigma} \rightarrow \boldsymbol{\tau}_0) \times ... \times (\boldsymbol{\sigma} \rightarrow \boldsymbol{\tau}_n) \ .$$

- if $\boldsymbol{\sigma} \sim \boldsymbol{\sigma}_0 \times ... \times \boldsymbol{\sigma}_n$, $\boldsymbol{\tau} \sim \boldsymbol{\tau}_0 \times ... \times \boldsymbol{\tau}_m$, then

$$\boldsymbol{\sigma} \times \boldsymbol{\tau} \sim \boldsymbol{\sigma}_0 \times ... \times \boldsymbol{\sigma}_n \times \boldsymbol{\tau}_0 \times ... \times \boldsymbol{\tau}_m \ .$$

Let us observe that this decomposition lists the components $\boldsymbol{\sigma}_0, ..., \boldsymbol{\sigma}_n$ in a certain order, hence the observations made at the end of Section 12.4 make it possible to define predecessors and norms for all ptykes of finite types.

5. If we define $h(\boldsymbol{\sigma}) = 0$, $h(\boldsymbol{\sigma} \rightarrow \boldsymbol{\tau}) = \mathsf{sup}\Big(h(\boldsymbol{\sigma}) + 1, h(\boldsymbol{\tau})\Big)$, $h(\boldsymbol{\sigma} \times \boldsymbol{\tau}) = \mathsf{sup}\Big(h(\boldsymbol{\sigma}), h(\boldsymbol{\tau})\Big)$, then it is easily proved that

$$\{e\,;\, e \text{ is the index of a recursive ptyx of type } \boldsymbol{\sigma}\}$$

is $\Pi^1_{h(\boldsymbol{\sigma})+1}$ complete. (When $h(\boldsymbol{\sigma}) = 0$, the concept of "recursive" we use must allow infinite objects... .) The proof would use a generalization of the $\boldsymbol{\beta}$-completeness theorem to finite types: see Exercise 10.B.4.

12.A. Gödel's T and ptykes

The purpose of this section is to give a model of Gödel's T by means of ptykes of finite type. There are, however, two essential difficulties in this task:

(i) The types of T are built from the atomic type $\boldsymbol{\iota}$ instead of $\boldsymbol{O}$; this difference can be formally abolished by identifying $\boldsymbol{\iota}$ with $\boldsymbol{O}$. But this means that the objects of T are now viewed as hereditary operations (functors) acting on ordinals!

(ii) Of course, one wants this interpretation of T to bear some relation with the original interpretation; but then a new difficulty arises: consider for instance an object of type $\boldsymbol{\iota} \to \boldsymbol{\iota}$ in T, representing the function

$$\begin{cases} f(2n) = n \\ f(2n+1) = 0 \end{cases} .$$

It is not possible to find any ptyx A of type $\boldsymbol{O} \to \boldsymbol{O}$ coinciding with f on integers: it is well known that dilators are increasing functions ($n \leq m \to I(n,m) \neq \emptyset$; $I(n,m) \neq \emptyset \to I\Big(A(n), A(m)\Big) \neq \emptyset$...). We overcome this difficulty by considering a system T', which is a variant of T:
T' contains one more atomic symbol: the term $+_{\boldsymbol{\iota}}$ of type $\boldsymbol{\iota} \to (\boldsymbol{\iota} \to \boldsymbol{\iota})$; the rules for $_+\boldsymbol{\iota}$ are:

$$t + \bar{0} \Rightarrow t \qquad t + S(u) \Rightarrow S(t+u) \ .$$

All the other rules of T are unchanged, except the rule for R:

$$R^{\boldsymbol{\sigma}}(t, u, \bar{0}) \Rightarrow t$$

$$R^{\boldsymbol{\sigma}}\Big(t, u, S(v)\Big) \Rightarrow R^{\boldsymbol{\sigma}}(t,u,v) \ +_{\boldsymbol{\sigma}} \ u(R^{\boldsymbol{\sigma}}(t,u,v), v)$$

with

$$+_{\boldsymbol{\sigma} \to \boldsymbol{\tau}} = \lambda x \ \lambda y \ \lambda z \ +_{\boldsymbol{\tau}} \ \Big(x(z), y(z)\Big)$$

$$+_{\boldsymbol{\sigma} \times \boldsymbol{\tau}} = \lambda x \ \lambda y \ +_{\boldsymbol{\sigma}} \ (\pi^1 x, \pi^1 y) \otimes \ +_{\boldsymbol{\tau}} \ (\pi^2 x, \pi^2 y) \ .$$

In fact this system T' was implicitly used in the proof of Howard's majoration theorem (7.A.25). Given a term of T', the majorizing term constructed in 7.A.25 can be written in an obvious way in the system T'. Observe by the way that T' can be viewed as a subsystem of T. T' is in some sense an "unwinding" of T. The mathematical structure of T' seems to be far more interesting than the mathematical structure of T... .

To each term of T', we shall associate a ptyx of the same type; more precisely, assume that $t(x_1, ..., x_n)$ is a term of type $\boldsymbol{\tau}$, and that all free variables of t are among $x_1, ..., x_n$ (of respective types $\boldsymbol{\sigma}_1, ..., \boldsymbol{\sigma}_n$); then we define, given ptykes $A_1, ..., A_n$ of respective types $\boldsymbol{\sigma}_1, ..., \boldsymbol{\sigma}_n$, a ptyx $t^*(A_1, ..., A_n)$ of type $\boldsymbol{\tau}$; we also define, when $T_1 \in I(A_1, B_1), ..., T_n \in I(A_n, B_n)$, a morphism $t^*(T_1, ..., T_n) \in I\Big(t^*(A_1, ..., A_n), t^*(B_1, ..., B_n)\Big)$, in such a way that the following hold:

1. $t^*(\mathbf{E}_{A_1}, ..., \mathbf{E}_{A_n}) = \mathbf{E}_{t^*(A_1,...,A_n)}$.

2. $t^*(T_1, ..., T_n)\ t^*(U_1, ..., U_n) = t^*(T_1U_1, ..., T_nU_n)$.

3. $t^*(T_1, ..., T_n) \wedge t^*(U_1, ..., U_n) = t^*(T_1 \wedge U_1, ..., T_n \wedge U_n)$.

4. If $(A_1, T_1^i) = \varinjlim (A_1^i, T_1^{ij}), ..., (A_n, T_n^i) = \varinjlim (A_n^i, T_n^{ij})$, then $\Big(t^*(A_1, ..., A_n), t^*(T_1^i, ..., T_n^i)\Big) = \varinjlim \Big(t^*(A_1^i, ..., A_n^i), t^*(T_1^{ij}, ..., T_n^{ij})\Big)$. (In other terms, t^* is a ptyx of type $\boldsymbol{\sigma}_1 \rightarrow (... \rightarrow (\boldsymbol{\sigma}_n \rightarrow \boldsymbol{\tau})...)$.)

12.A.1. <u>Case of a variable</u>.
If $t(x_1, ..., x_n) = x_i$, define $t^*(A_1, ..., A)n) = A_i$, $t^*(T_1, ..., T_n) = T_i$. Conditions 1–4 are trivially satisfied.

12.A.2. <u>Case of $\bar{0}$, s, $+_{\boldsymbol{l}}$</u>.

$\bar{0}^*$ is 0 (the ordinal 0)

s^* is the dilator $\mathsf{Id} + \underline{1}$

$_+\boldsymbol{l}^*$ is the functor (bilator) sum .

(In fact, we should have written, for instance: $\bar{0}^*(A_1, ..., A_n) = 0$, $\bar{0}^*(T_1, ..., T_n) = \mathbf{E}_0, \dots$) Conditions 1–4 are immediately fulfilled.

12.A.3. Case of AP.
We consider $t(u)(x_1, ..., x_n)$,that we can rewrite $t(x_1, ..., x_n)\ u(x_1, ..., x_n)$; we define

$$t(u)^*(A_1, ..., A_n) = t^*(A_1, ..., A_n)\ u^*(A_1, ..., A_n)$$

$$t(u)^*(T_1, ..., T_n) = t^*(T_1, ..., T_n)\ u^*(T_1, ..., T_n)\ .$$

Then $t(u)^*$ enjoy 1–4.

Proof. 1 and 2 come from the similar properties of t^* and u^*; 3 comes from 12.3.9, 4 from 12.3.8. □

12.A.4. Case of λ.
If t^* has been defined, and $\lambda x t(x_1, ..., x_n)$ is $\lambda x\Big(x_1, ..., x_n, x)\Big)$, we define

$$\Big((\lambda xt)^*(A_1, ..., A_n)\Big)(A) = t^*(A_1, ..., A_n, A)$$

$$\Big((\lambda xt)^*(T_1, ..., T_n)\Big)(T) = t^*(T_1, ..., T_n, T)$$

$$\Big((\lambda xt)^*(T_1, ..., T_n)A = t^*(T_1, ..., T_n, \mathbf{E}_A)\ .$$

By 1 and 2 it is immediate that $(\lambda xt)^*(A_1, ..., A_n)$ is a functor of the appropriate type, and $(\lambda xt)^*(T_1, ..., T_n)$ is a natural transformation. Using 3 and 4, it is immediate that $(\lambda xt)^*(A_1, ..., A_n)$ is a ptyx of the appropriate type. Furthermore λxt^* enjoys 1–4.

Proof. 1 and 2 are immediate; 3 and 4 follow from 12.3.9 and 12.3.8. □

12.A.5. Case of π^1, π^2.
Assume that t^* has been defined; then

$$(\pi^1 t)^*(A_1, ..., A_n) = \pi^1\Big(t^*(A_1, ..., A_n)\Big)$$

$$(\pi^1 t)^*(T_1, ..., T_n) = \pi^1\Big(t^*(T_1, ..., T_n)\Big)$$

$$(\pi^2 t)^*(A_1, ..., A_n) = \pi^2\Big(t^*(A_1, ..., A_n)\Big)$$

$$(\pi^2 t)^*(T_1, ..., T_n) = \pi^2\Big(t^*(T_1, ..., T_n)\Big)$$

where π^1 and π^2 are the ptykes of 12.4.2 (iv).

1–4 are satisfied.

Proof. Because π^1 and π^2 are ptykes... . □

12.A.6. Case of $\otimes$.

Consider $(t \otimes u)(x_1, ..., x_n)$, that we can rewrite as $t(x_1, ..., x_n) \otimes u(x_1, ..., x_n)$; define

$$(t \otimes u)^*(A_1, ..., A_n) = t^*(A_1, ..., A_n) \otimes u^*(A_1, ..., A_n)$$

$$(t \otimes u)^*(T_1, ..., T_n) = t^*(T_1, ..., T_n) \otimes u^*(T_1, ..., T_n) \ .$$

Conditions 1–4 are fulfilled.

Proof. Immediate. □

12.A.7. Case of $R^{\boldsymbol{\sigma}}$.

We define

(1) $R^*(A, B, 0) = A\,.$
(1)′ $R^*(A, B, \mathbf{E}_0) = \mathbf{E}_A\,.$

(2) $R^*(A, B, x+1) = R^*(A, B, x) + B(R^*(A, B, x), x)\,.$
(2)′ $R^*(T, U, f + \mathbf{E}_1) = R^*(T, U, f) + U(R^*(T, U, f), f)\,.$
(2)″ $R^*(T, U, f + \mathbf{E}_{01}) = R^*(T, U, f) + \mathbf{E}_{\underline{0}B'(R^*(A', B', x'), x')}$ when $T \in I(A, A')$, $U \in I(B, B')$, $f \in I(x, x')$.

(3) $R^*(A, B, x) = \sup\limits_{y<x} R^*(A, B, y)$ when x is a limit.
(3)′ $R^*\Big(A, B, \bigcup\limits_i f_i\Big) = \bigcup\limits_i R^*(A, B, f_i)$.

The first thing is to show that these definitions make sense; we show, by induction on x, the existence of a functor F_x from $\mathbf{ON} < x$ to $\boldsymbol{\tau}$ such that $F_x(y) = R^*(A, B, y)$, $F_x(f) = R^*(\mathbf{E}_A, \mathbf{E}_B, f)$ and which is a "flower", i.e. $F_x(\mathbf{E}_{yy'}) = \mathbf{E}_{F_x(y)F_x(y')}$. The details offer no difficulty and are left to the reader... . Then it remains to prove that the functor B^* preserved direct limits and pull-backs. We establish the existence of a normal form w.r.t. R (a three-variable normal form): assume that $z \in \mathsf{Tr}\Big(R^*(A, B, x)\Big)$, then we define the normal form of z by induction on x:

(i) If $x = 0$; then $z \in \mathsf{Tr}(A)$; consider $a_0, b_0, x_0, t_0 \in I(a_0, A)$, $u_0 \in I(b_0, B)$, $f_0 \in I(x_0, x)$: $\mathsf{rg}\Big(\mathsf{Tr}(t_0)\Big) = \{z\}$, $\mathsf{rg}\Big(\mathsf{Tr}(u_0)\Big) = \emptyset$, $\mathsf{rg}\Big(\mathsf{Tr}(f_0)\Big) = \emptyset$ (hence $x_0 = 0$, $b_0 = \underline{0}$); then $z \in \mathsf{rg}\Big(\mathsf{Tr}\Big(R^*(t_0, u_0, f_0)\Big)\Big)$. The fact that this solution is minimum is left as an exercise to the reader.

(ii) If $x = y + 1$, $z \in \mathsf{Tr}\Big(R^*(A, B, y) + B\Big(R^*(A, B, y), y\Big)\Big)$; we establish a lemma:

12.A.8. <u>Lemma</u>.
If A and B are of type $\boldsymbol{\sigma}$, then $\mathsf{Tr}(A + B) = \mathsf{Tr}(A) \cup \mathsf{rg}\Big(\mathsf{Tr}(\mathbf{E}_{\underline{0}A} + \mathbf{E}_B)\Big)$.

<u>Proof</u>. Easy induction on $\boldsymbol{\sigma}$.... . □

Now two cases may occur:

1. $z \in \mathsf{Tr}\Big(R^*(A, B, y)\Big)$; then the induction hypothesis yields a normal form for z, by means of t_0, u_0, and $f_0 \in I(x_0, y)$; from this we easily obtain a normal form by replacing f_0 by $f_0 + \mathbf{E}_{01}$.

2. If $z \in \mathsf{rg}\Big(\mathsf{Tr}(\mathbf{E}_{\underline{0}A'} + \mathbf{E}_{B'})\Big)$ with $A' = R^*(A, B, y)$, $B' = B(R^*(A, B, y), y)$: assume that $z = \mathsf{Tr}(\mathbf{E}_{\underline{0}A'} + \mathbf{E}_{B'})(z')$ with $z' \in \mathsf{Tr}(B)$, and write $z = \mathsf{Tr}(B)(T, g)(v)$, with $T \in I\Big(C, R^*(A, B, y)\Big)$, $g \in I(y', y)$; T, g, v are unique if we require $\mathsf{rg}\Big(\mathsf{Tr}(T)\Big)$, $\mathsf{rg}(g)$ minimal for inclusion.... If $\mathsf{rg}(T) = \{u_1, ..., u_n\}$, write, for $i = 1, ..., n$ $u_i \in \mathsf{Tr}\Big(R^*(A, B, y)\Big)$ under their normal forms, which exist, by the induction hypothesis: $u_i = \mathsf{Tr}\Big(R^*(t_i, u_i, f_i)\Big)(z_i)$; consider the sets:

$$X = \mathsf{rg}\Big(\mathsf{Tr}(t_1)\Big) \cup ... \cup \mathsf{rg}\Big(\mathsf{Tr}(t_n)\Big)$$

$$Y = \mathsf{rg}\Big(\mathsf{Tr}(u_1)\Big) \cup ... \cup \mathsf{rg}\Big(\mathsf{Tr}(u_n)\Big) \cup \{(v, A), y')\}$$

$$Z = \mathsf{rg}\Big(\mathsf{Tr}(f_1)\Big) \cup ... \cup \mathsf{rg}\Big(\mathsf{Tr}(f_n)\Big) \cup \mathsf{rg}\Big(\mathsf{Tr}(g)\Big) \cup \{y\} .$$

   If t_0, u_0, f_0 are such that $\mathsf{rg}\Big(\mathsf{Tr}(t_0)\Big) = X$, $\mathsf{rg}\Big(\mathsf{Tr}(u_0)\Big) = Y$, $\mathsf{rg}\Big(\mathsf{Tr}(f_0)\Big) = Z$, with $t_0 \in I(a_0, A)$, $u_0 \in I(b_0, B)$, $f_0 \in I(x_0, x)$; then z has a normal form by means of t_0, u_0, f_0: $z \in \mathsf{rg}\Big(\mathsf{Tr}\Big(R^*(t_0, u_0.f_0)\Big)\Big)$. The fact that this 3-uple is minimum for inclusion is left to the reader.

(iii) If x is limit, then one easily sees that

$$\mathsf{Tr}\Big(R^*(A,B,x)\Big) = \bigcup_{y<x} \mathsf{Tr}\Big((A,B,y)\Big) \ ;$$

if $z \in \mathsf{Tr}\Big(R^*(A,B,x)\Big)$, we have $z \in \mathsf{Tr}\Big(R^*(A,B,y)\Big)$ for some $y < x$, and the induction hypothesis yields a normal form for z by means of t_0, u_0, and $f_0 \in I(x_0, y)$; it suffices to replace f_0 by the function $f_1 \in I(x_0, x)$ s.t. $f_1(z) = z$ for all $z < x_0$... . The unicity is completely trivial.

We have succeeded in definining R^*. Of course, if R is viewed as $R(x_1, ...,$
$x_n)$, then we proceed as in 12.A.2... .

12.A.9. Theorem.
The interpretation $(\ )^*$ is a model of T': more precisely, if $t(x_1, ..., x_n) \Rightarrow t'(x_1, ..., x_n)$, we have $t^* = t'^*$.

Proof. The theorem means that, from

$$t(x_1, ..., x_n) \Rightarrow t'(x_1, ..., x_n) \ ,$$

we can infer that

$$t^*(A_1, ..., A_n) = t'^*(A_1, ..., A_n)$$

$$t^*(T_1, ..., T_n) = t^*(T_1, ..., T_n)$$

for all $A_1, ..., A_n$, $T_1, ..., T_n$ of appropriate types. But:

12.A.10. Proposition.

$$t^*\Big(A_1, ..., A_n, u^*(A_1, ..., A_n)\Big) = v^*(A_1, ..., A_n)$$

$$t^*\Big(T_1, ..., T_n, u^*(T_1, ..., T_n)\Big) = v^*(T_1, ..., T_n)$$

with

$$v(x_1, ..., x_n) = t\Big(x_1, ..., x_n, u(x_1, ..., x_n)\Big) \ .$$

Proof. Straightforward induction on t.... $\square$

Now we observe that * is compatible with the conversion rules:

(i) $\lambda xt(u) \Rightarrow t(u)$: we must show that $(\lambda xt)^*(A_1, ..., A_n)\Big(u^*(A_1, ..., A_n)\Big) = v^*(a_1, ..., a_n)$, with $v\Big(x_1, ..., x_n) = t\Big(x_1, ..., x_n, u(A_1, ..., A_n)\Big)$. But $(\lambda xt)^*(A_1, ..., A_n)(B) = t^*(A_1, ..., A_n, B)$, hence the property follows from 12.A.10; the case of morphisms is similar.

(ii) $\pi^1(t \otimes u) \Rightarrow t$: obviously $\pi^1(t \otimes u)^*(A_1, ..., A_n) = \pi^1\Big(t^*(A_1, ..., A_n) \otimes u^*(A_1, ..., A_n)\Big) = t^*(A_1, ..., A_n)$. The case of morphisms is similar.

(iii) $\pi^2(t \otimes u) \Rightarrow u$: symmetric to (ii).

(iv) $R(t, u, \bar{0}) \Rightarrow t$: we must show that $R(t, u, \bar{0})^*(A_1, ..., A_n) = t^*(A_1, ..., A_n)$; but $R(t, u, \bar{0})^*(A_1, ..., A_n) = R^*\Big(t^*(A_1, ..., A_n), u^*(A_1, ..., A_n), \bar{0}^*\Big) = t^*(A_1, ..., A_n)$ since $\bar{0}^* = 0$. For similar reasons $R(t, u, \bar{0})^*(T_1, ..., T_n) = t^*(T_1, ..., T_n)$.

(v) $R\Big(t, u, S(v)\Big)^* \Rightarrow v$, with $v = R(t, u, v) + u(R(t, u, v), v)$. We must show that $R\Big(t, u, S(v)\Big)^*(A_1, ..., A_n) = v^*(A_1, ..., A_n)$. But $R\Big(t, u, S(v)\Big)^*(A_1, ..., A_n) = R^*\Big(t^*(A_1, ..., A_n), u^*(A_1, ..., A_n), v^*(A_1, ..., A_n) + 1\Big) = R^*\Big(t^*(A_1, ..., A_n), u^*(A_1, ..., A_n), v^*(A_1, ..., A_n)\Big) + u^*(A_1, ..., A_n)\Big(R^*\Big(t^*(A_1, ..., A_n), u^*(A_1, ..., A_n), v^*(A_1, ..., A_n)\Big), v^*(A_1, ..., A_n)\Big) = w^*(A_1, ..., A_n)$. The same method proves the result for morphisms (we are implicitly using the fact that $+^*_{\boldsymbol{\sigma}}$ is the functor: $A +^* B = A + B$, $T +^* U = T + U$).

(vi) $t + \bar{0} \Rightarrow t$: trivial.

(vii) $t + S(u) \Rightarrow S(t + u)$: trivial.

It is not easy to establish, by induction on a reduction $t \Rightarrow t'$, that $t^* = t'^*$:

– This holds for the conversion rules.

– This is preserved by transitivity

$$\frac{t \Rightarrow t' \qquad t' \Rightarrow t''}{t \Rightarrow t''}$$

– This is preserved by the rules

$$\frac{t \Rightarrow t'}{\lambda xt \Rightarrow \lambda xt'} \qquad \frac{t \Rightarrow t' \qquad u \Rightarrow u'}{t(u) \Rightarrow t'(u)}$$

$$\frac{t \Rightarrow t' \qquad u \Rightarrow u'}{t \otimes u \Rightarrow t' \otimes u'} \qquad \frac{t \Rightarrow t'}{\pi^1 t \Rightarrow \pi^1 t'} \qquad \frac{t \Rightarrow t'}{\pi^2 t \Rightarrow \pi^2 t'}$$

hence the theorem follows... . □

12.A.11. Comments.

(i) The objects of T' can be viewed as ptykes; concretely, this means that, when t is a closed term of type $\boldsymbol{l} \to \boldsymbol{l}$, then the function $\tilde{t}$ from $\mathbb{N}$ to $\mathbb{N}$ associated with t by $t(\bar{n}) \Rightarrow \overline{\tilde{t}(n)}$ (see 7.A.23), that one usually identifies with t, is "naturally" extended into a dilator (ptyx of type $\boldsymbol{O} \to \boldsymbol{O}$); in particular the expression $\tilde{t}(x)$ makes sense for any ordinal x. It is remarkable that the data contained in t are enough to make such an extension in a perfectly natural way: all that we do in fact is to write the definition of the associated ptykes t^*:

1. By using the usual reduction rules: for instance from $R(a, b, 0) \Rightarrow a$, we define $R^*(A, B, 0) = A$.
2. By transferring 1 to morphisms: typically $R^*(T, U, \mathbf{E}_0) = T$.
3. By extending "by direct limits": typically, in the clauses (3) and $(3)'$ of the definition of R^*.

The only definition which does not follow "mechanically" from the intended interpretation of T' is $(2)''$ in the clauses for R^*; but this clause follows from the fact that R^* is a "flower".

Hence the construction of t^* from t is a very natural and intrinsic process. It is not an exaggeration to say that the conversion rules are sufficiently powerful to define the terms of T' on arbitrary ordinal arguments... .

(ii) An old question is the following: to find an intrinsic "ordinal assignment" to terms of T, i.e., associate ordinals to all terms of T in a natural way. Here this becomes possible:

1. To each term t of T, we associate a term t_1 of T', which is formally the same term. (But the reduction rules are not the same....)
2. Then we consider the ptyx t_1^*. Let us now give some particular cases, depending on the type $\boldsymbol{\sigma}$ of t. The reader will easily find out the precise general definition:

   - if $\boldsymbol{\sigma} = \boldsymbol{l}$, then t_1^* is a finite ordinal, i.e. an integer; this is the normal form of t_1 in T'. Let $\|t\| = t_1^*$.
   - if $\boldsymbol{\sigma} = \boldsymbol{l} \to \boldsymbol{l}$, then t_1^* is a dilator; we define $\|t\| = t_1^*(\omega)$.
   - if $\boldsymbol{\sigma} = (\boldsymbol{l} \to \boldsymbol{l}) \to \boldsymbol{l}$, then t_1^* is a ptyx of type $\mathbf{DIL} \to \mathbf{ON}$; we define $\|t_1^*\| = t_1^*(\Xi_1)$, where Ξ_1 is the sum of all recursive and weakly finite dilators.
   - if $\boldsymbol{\sigma} = \Big((\boldsymbol{l} \to \boldsymbol{l}) \to \boldsymbol{l}\Big) \to \boldsymbol{l}$, we consider the ptykes A of type 2, which send weakly finite dilators on integers: we call them weakly finite. We define Ξ_2 to be the sum of all recursive weakly finite ptykes of type 2, and we let $\|t\| = t_1^*(\Xi_2)$.

   Of course we must explain these typical examples:

   - In the case of type $\boldsymbol{l}$, there is nothing special to say, except that the ordinal $\|t\|$ is $< \omega$.
   - The case of tupe $\boldsymbol{l} \to \boldsymbol{l}$ is obtained as follows: we want that $\|t(u)\| \leq \|t\|$ for all u of type $\boldsymbol{l}$; since we have observed that $\|u\| < \omega$, remark that $\|t(u)\| = t_1^*(u_1^*) \leq t_1^*(\omega) = \|t\|$.
   - In the case of type $(\boldsymbol{l} \to \boldsymbol{l}) \to \boldsymbol{l}$, we still want $\|t(u)\| \leq \|t\|$ for all u of type $\boldsymbol{l} \to \boldsymbol{l}$; however, here we must take an additional requirement: we want the ordinal assignment to be preserved by further extensions; in a reasonable extension U of T,t he ordinal $\|t\|$ must be the same. What reasonable condition can we ask for U? A reasonable hypothesis will be that all the dilators (ptykes of type $\boldsymbol{O} \to \boldsymbol{O}$) corresponding to objects of type $\boldsymbol{l} \to \boldsymbol{l}$ of U, will still be weakly finite and recursive. Hence we are naturally

led to $t_1^*(D) \leq \|t\|$ for all D, weakly finite and recursive dilator, and from that, the value $\|t\| = t_1^*(\Xi_1)$ naturally follows.

- In the case of type $\Big((\boldsymbol{l} \to \boldsymbol{l}) \to \boldsymbol{l}\Big) \to \boldsymbol{l}$, we argue just as above: what is a reasonable general property of all ptykes appearing as u_1^*, for some u in some extension U of T: typically, to send weakly finite dilators on integers, as elements u_1^*, $u \in I$ do. This gives the explanation for the values of $\|t\|$ in that case.

We have so far proposed no ordinal for the types involving the product. The question is delicate, however, one can take for a reasonable hypothesis that

$$\|t\| \geq \|\pi^1 t\|, \|\pi^2 t\|$$

hence

$$\|t\| = \|\pi^1 t\| + \|\pi^2 t\|$$

could be a reasonable solution... .

(iii) When $\boldsymbol{\sigma}$ is finite type, the natural question to ask is to determine the sets:

$$X_{\boldsymbol{\sigma}} = \{|t|\,;\, t \text{ closed term of type } \boldsymbol{\sigma} \text{ of } T\}\ .$$

- It is plain that $X_{\boldsymbol{l}} = \omega$.
- $X_{\boldsymbol{l} \to \boldsymbol{l}}$ is the Howard ordinal η_0: Since **PA** (or **HA**) and T are "equivalent" systems, this can be viewed as a new evidence that η_0 is "the" ordinal of arithmetic; for indications as to the proof of this fact, see (iv) below. A direct proof of this equality has been given by Päppinghaus [120] (1982).
- $X_{(\boldsymbol{l} \to \boldsymbol{l}) \to \boldsymbol{l}}$ is a subset of the first stable ordinal σ_0, by 11.C.9, 11/C/12/ This set is not an initial segment of σ_0: for instance, it contains an initial segment equal to η_0, and the next point is ω_1^{CK} ($\omega_1^{CK} = \Big\|\lambda x^{\boldsymbol{l} \to \boldsymbol{l}} x\Big(x(\bar{0})\Big)\Big\|$)... . The question of the determination of the ordinal

$$\mathsf{sup}\,\{z\,;\, z \in X_{(\boldsymbol{l} \to \boldsymbol{l}) \to \boldsymbol{l}}\}$$

is an open question. Maybe this ordinal is strictly less than the first recursively inaccessible... .

- $X_{\left((\boldsymbol{l}\to\boldsymbol{l})\to\boldsymbol{l}\right)\to\boldsymbol{l}}$ is a subset of π^1_3; of course, such a set will mainly consist of stable ordinals, and its structure is not yet very clear... .

The interest of these ordinals lies in the fact that, besides the now rather familiar association of the recursive ordinal η_0 to arithmetic, we now exhibit "non recursive ordinals of **PA**", typically the ordinals $x_{\boldsymbol{\sigma}} = \mathsf{sup}\,\{z\,;\, z \in X_{\boldsymbol{\sigma}}\}$ (of course the order type of $X_{\boldsymbol{\sigma}}$ is always recursive...). The natural conjecture is the following:

Find generalized (elementary) principles of induction, involving the ordinals $x_{\boldsymbol{\sigma}}$, in such a way that formulas of a certain logical complexity (say K) which are theorems of **PA***, are provable by this kind of induction up t $x_{\boldsymbol{\sigma}}$, $\boldsymbol{\sigma}$ depending only on K.*

(iv) An interesting neighbor is Feferman's system [77] of hereditarily replete function over the ordinals: Feferman's paper was the first serious attempt to analyze the idea of ordinal denotation by means of families of functions. The precise details are of no interest here; for us, we can consider Feferman's system as being T' + constants $\bar{x}$ for all $x \in 0n$, together with equations:

1. $S(\bar{x}) = \overline{x+1}$
2. $\bar{x} + \bar{y} = \overline{x+y}$
3. $\lambda x t(x)(u) = t(u)$
4. $\pi^1(t \otimes u) = t$
   $\pi^2(t \otimes u) = u$
5. $R(t, u, \bar{0}) = t$
   $R(t, u, \overline{x+1}) = R(t, u, \bar{x}) + u(R(t, u, x), x)$
   $R(t, u, \bar{x}) = \mathop{\mathsf{sup}}\limits_{y<x} R(t, u, \bar{y})$

when x is limit (with an ad hoc notion of sup), together with principles expressing that $=$ is an equality... .

This system can obviously be interpreted in the same way as T', if we set $\bar{x}^* = x \ldots$. Of course, since every closed term of type $\boldsymbol{O}$ in this system has a unique normal form $\bar{x}$, the question raised in [77] by Feferman, can be translated as follows in our terminology: *which ordinals x are such that $= \bar{x}$, for some closed t of type $\boldsymbol{O}$, involving no parameters $\bar{y}$, with $y > \omega$?* Feferman's conjecture (established by Weyhrauch [121]) was that these ordinals are exactly the ordinals $< \eta_0$. The conjecture proves that $X_{\boldsymbol{l} \to \boldsymbol{l}} = \eta_0$.

(Proof. If $x \in X_{\boldsymbol{l} \to \boldsymbol{l}}$, write $x = t^*(\omega)$ for some $t \in T'$, t closed; then $t(\bar{\omega})$ is a term of Feferman's system, and $t(\bar{\omega}) = \bar{x}$ in this system, and by Weyhrauch's result $x < \eta_0$. Conversely, if $x < \eta_0$, then x can be expressed by means of a term t in Feferman's system, using only $\bar{\omega}$ as parameter: $t(\bar{\omega}) = \bar{x}$; but then $t^*(\omega) = x$, and $x = \|\lambda g t(g)\|$.$\square$)

(v) The relation of T' (and therefore of T) to ptykes is a rather new and unexpected phenomenon. I hope that this could produce a revival of the interest for T. Observe that the ptyx interpretation answers (at least partly) the question: what is the meaning of closed normal forms $\neq \boldsymbol{l}$? (A closed normal form of type $\boldsymbol{l}$ is an integer; but closed normal forms of higher types are more delicate to interpret.) Typically consider the normal forms $\lambda x(\bar{1} + x)$ and $\lambda x(x + \bar{1})$ in T: they both represent the successor function: $n \rightsquigarrow n + 1$. But the associated dilators are $\underline{1} + \mathsf{Id}$ and $\mathsf{Id} + \underline{1}$, which are different. Hence the interpretation by means of ptykes helps us to understand why these two normal forms are different: the associated dilators are not the same. (But this says nothing as to the difference between S and $\lambda x(x + \bar{1})$!)

12.A.12. Exercise.

Show that, when t is a closed term of $\boldsymbol{\sigma}$, t^* is a weakly finite ptyx, and is recursive.

12.A.13. Question.
What is the relation between the ordinal $|t|$ and the structure of the computation of the normal form of t? It seems reasonable that $|t|$, viewed as a direct limit, can be shown to exist by elementary ways; of course these elementary methods would only enable us to construct the linear order $|t|$, and nothing more; presumably a (weak) well-foundedness assumption as to $|t|$ would then entail normalization for t... .