Siena, 18-19 Mai 2007

TRUTH

MODALITY

INTERSUBJECTIVITY

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FROM : LE POINT AVEUGLE

T2 : VERS L'IMPERFECTION

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I — The subject

1- THE BLIND SPOT

- Inability of logic to say anything as to truth, complexity, etc.
- Tarski's arrogant essentialism : A true when... A is true :
  La Palice : Un 1/4 d'heure avant sa mort, il était encore en vie
- Meta and Byzantium : Turtles all the way down
- Non-ideological analysis retains only three layers :
  -1 : Truth, provability, consistency : You have got mail
  -2 : Functions, morphisms, categories : The letter says...
  -3 : Procedurality : This letter was brought by...
- Example, the old Barbara :
  -1 : Transitivity of inclusion : Łukasiewicz
  -2 : Composition of morphisms : Curry-Howard
  -3 : Geometry of interaction : feedback equation

2- THE SUBJECT

- **No satisfying status in usual logic :**
  Frege : Sense vs. denotation ; the object already there
  Epistemic logic : He who says nothing has something to hide
- **Refusal of the subject leads to subjectivism :**
  Ptolemy : Absoluteness of Earth leads to epicycles
  Jurassic Park : Absolute truth leads to metasystems
- **Categories handle the subject through morphisms :**
  Essentialism : Morphism derives from « form »
  Layer -3 : Because -2 cannot be primitive
- **Subjectivism at work in set-theory :**
  Quantum : Objective wave, observed through measurement
  Observation : Points (bips) as eigenvectors of measurement
  Intersubjectivity : System of commuting measurements

3- NON COMM. : FINITE DIMENSION

- **Diagonalisation** w.r.t. orthonormal basis holds for :
  Hermitians : $u = u^*$
  Unitaries : $uu^* = u^*u = I$
  Normal : $uu^* = u^*u$ (synchretic notion)
  Proof : Characteristic polynomial $\det(u - \lambda I) = 0$ admits
    a complex root, hence an eigenvector e_1
    Normality implies $u(\mathrm{e}_1^\perp) \subset \mathrm{e}_1^\perp$; redo with $u \upharpoonright \mathrm{e}_1^\perp$
- **Diagonal** matrices form a commutative algebra
- Infinite « diagonalisation » through operator algebras :
  C* : Continous functions on compact space $\mathcal{C}(X)$
  v Neumann : Bounded measurable functions $\mathcal{L}^\infty(X, \mu)$
- N.C. geometry (Connes) : operator algebra as sort of :
  Function space over a « non-set »
- Subject creates sets by choosing a base

II — Quantum coherent spaces

4- COHERENT SPACES

- **The origin of linear logic :**
  Categories : Instead of pseudo-topology : $\bigcup \rightsquigarrow \varinjlim$
  Stability : Pull-backs, indeed algebraic differences
- **Two equivalent versions :**
  Essentialist : $(|X|, \frown_X)$ $\quad a \in X \;\Leftrightarrow\; a$ clique
  Existentialist : $a \underset{\sim}{\downarrow} b \quad \Leftrightarrow \quad \sharp(a \cap b) \leq 1$
  Adjunction : $\sharp(F[a] \cap b) = \sharp(F \cap (a \times b)) \quad (a \in X, b \in \sim Y)$
- **Generalisation through Banach C. S. and Quantum C. S. :**
  BCS : Essentialist $\frown_X \rightsquigarrow \|\cdot\|_X \quad a \in X \;\Leftrightarrow\; \|a\|_X \leq 1$
  QCS : Existentialist $a \underset{\sim}{\downarrow} b \quad \Leftrightarrow \quad 0 \leq \mathrm{tr}(a \cdot b) \leq 1$
  Adjunction : $\mathrm{tr}(F[a] \cdot b) = \mathrm{tr}(F \cdot (a \otimes b)) \quad (a \in X, b \in \sim Y)$
  Dictionary : $\cap, \times, \sharp \quad \rightsquigarrow \quad \cdot, \otimes, \mathrm{tr}$

5- HERMITIAN GEOMETRY

- Space $\mathbb{C}^n$ equipped with sesquilinear form :
  $\langle x \mid y \rangle := \sum x_n \bar{y}_n$
- Banach space $\|x\| := \langle x \mid x \rangle^{1/2}$ Self-dual :
  anti-isomorphism $x \mapsto \langle \cdot \mid x \rangle$
- Endormorphisms as square matrices : $[m_{ij}] \in \mathcal{M}_n(\mathbb{C})$
  Adjunction, seen as transconjugation : $[m_{ij}]^* := [\bar{m}_{ji}]$
- The trace $\mathrm{tr}[m_{ij}] := \sum m_{ii}$ is cyclic :
  $$\mathrm{tr}(uv) = \mathrm{tr}(vu)$$
- $a \subset \{1, \ldots, n\}$ yields projection E_a onto subspace $\mathbb{C}^a$
- Fits with the dictionary :
  Intersection : $E_a \cdot E_b = E_{a \cap b}$
  Cartesian product : $E_a \otimes E_b = E_{a \times b}$
  Cardinality : $\mathrm{tr}(E_a) = \dim(\mathbb{C}^a) = \sharp(a)$

6- BEWARE OF THE SUBJECT !

- **Subjectivity : choice of an orthonormal basis** $\langle \mathrm{e}_i \mid \mathrm{e}_j \rangle = \delta_{ij}$
  Example : $\mathbb{C}^n$ **: vector space with « canonical » base**
- **Matrices as subjective endormorphisms :**
  Change of base : Matrix $[\mathrm{e}_1, \ldots, \mathrm{e}_n]$ $M' := u^* M u$
  Unitaries : A.k.a. changes of base, enjoy $uu^* = u^* u = I$
  Isometricity : Characterised by $\langle u(x) \mid u(y) \rangle = \langle x \mid y \rangle$
- **Objectivity of all operations :**
  Form : Ensured by isometricity
  Adjoint : Definable from $\langle a^*(x) \mid y \rangle := \langle x \mid a(y) \rangle$
  Trace : Ensured by cyclicity : $\mathrm{tr}(u^* a u) = \mathrm{tr}(u u^* a) = \mathrm{tr}(a)$
- **Through diagonalisation, objective analogues for :**
  Functions $\{1, \ldots, n\} \rightsquigarrow \mathbb{R}$ **: Hermitians** $u^* = u$
  Positive functions : Positive hermitians $\langle u(x) \mid x \rangle \geq 0$
  Subsets : Projections : $u^* = u, u^2 = u$

7- FROM BOOLEANS TO SPINS

- **Usual booleans :** $\{1\}, \{2\} \subset \{1, 2\}$
- **Objective, base-free, version :** $u = u^*, \mathrm{tr}(u) = 1, \det(u) = 0$
- **Hermitians parametrised by Pauli matrices :**

$$u = 1/2 \begin{bmatrix} t + z & x - iy \\ x + iy & t - z \end{bmatrix}$$

Quantum booleans : $t = 1, x^2 + y^2 + z^2 = 1$

Spin : Relative to axis of $\mathbb{R}^3$ (up : true, down : false)

Booleans : Relative to the axis $\vec{z}$

- **Measurement through probabilistic preselection w.r.t. base :**

$$1/2 \begin{bmatrix} 1 + z & x - iy \\ x + iy & 1 - z \end{bmatrix} \mapsto 1/2 \begin{bmatrix} 1 + z & 0 \\ 0 & 1 - z \end{bmatrix}$$

$= \lambda \mathtt{t} + (1 - \lambda)\mathtt{f}$ **with** $\lambda := 1/2(1 + z)$

8- ETA AND SUBJECTIVITY

- **Dimension 2 : identity handled by twist** $\sigma(x \otimes y) := y \otimes x$

$$\sigma := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \neq \tau := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- $\mathrm{tr}(\sigma \cdot a \otimes b) = \mathrm{tr}(a \cdot b)$ **; hence** $\sigma[a] = a$
- **But** $\tau[\begin{bmatrix} x & y \\ z & t \end{bmatrix}] = \begin{bmatrix} x & 0 \\ 0 & t \end{bmatrix}$ **handles measurement w.r.t. axis** $\vec{z}$
- τ **is the incarnation of** σ **(=** η**-spansion)**
- **Non-sets (general spins) needed to witness the difference**

III — C^*-ALGEBRAS

9- HILBERT SPACE

- Complex vector space $\mathbb{H}$ equipped with form :
  Sesquilinear : $\langle \lambda x \mid \mu y \rangle = \lambda \bar{\mu} \langle x \mid y \rangle$
  Definite positive : $\langle x \mid x \rangle > 0 \; (x \neq 0)$ ($\Rightarrow \langle y \mid x \rangle = \overline{\langle x \mid y \rangle}$)
- Complete (Banach space) w.r.t. norm $\|x\| \;:=\; \langle x \mid x \rangle^{1/2}$
  Cauchy-Schwarz : $|\langle x \mid y \rangle| \leq \|x\| \cdot \|y\|$ (equal iff colinear) ;
  Median : $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$
- ℓ^2 : square summable sequences $\langle (x_n) \mid (y_n) \rangle := \sum x_n \bar{y}_n$
- Median identity induces projections on closed convex sets :
  Closed subspace : Yields a linear map $\pi^2 = \pi$
  Kernel $\varphi^{-1}(0)$: Every linear form written as $\langle \cdot \mid x \rangle$
- Dual space $\mathbb{H}^\sharp$ « isomorphic » to $\mathbb{H}$
  Adjunction : $\langle u^*(x) \mid y \rangle \;=\; \langle x \mid u(y) \rangle$
  C*-identity : (after Cauchy-Schwarz) $\|u\|^2 = \|uu^*\|$

10- C*-ALGEBRAS

- **Complex Banach algebra with unit and operation $(\cdot)^*$:**
  Involutive : $u^{**} = u \quad (\lambda u)^* = \bar{\lambda} u^* \quad (uv)^* = v^* u^*$
  Stellar norm : $\|u\|^2 = \|uu^*\|$
- **Examples (generic ; generic commutative) :**
  Bounded operators : $\mathcal{B}(\mathbb{H})$, **more generally closed self-adjoint subalgebra**
  Continuous fonctions : $\mathcal{C}(X)$ **(X compact), with pointwise sum, product, conjugation**

11- THE SPECTRUM

- **Sort of eigenvalues :**

  $$\mathrm{sp}(u) \;=\; \{\lambda \in \mathbb{C}; u - \lambda I \quad \mathrm{non-invertible}\}$$

  Gel'fand : $\mathrm{sp}(u)$ **compact and non-empty**

  Proof : $u - zI$ **always invertible yields bounded entire map**
  Impossible by Liouville's theorem

  Spectral radius : $r(u) \;:=\; \sup\{|\lambda|; \lambda \in \mathrm{sp}(u)\}$

  $$r(u) \;=\; \lim \|u^n\|^{1/n}$$

  Polynomials : $\mathrm{sp}(P(u)) = P(\mathrm{sp}(u)) \quad \mathrm{sp}(u^*) = \overline{\mathrm{sp}(u)}$

- **Assume u hermitian (generalises to normal, e.g., unitary) :**

  Stone-Weierstraß : **Yields** $f(u)$ **for** $f \in \mathcal{C}(\mathrm{sp}(u))$
  C^***-algebra generated by** u **isomorphic with** $\mathcal{C}(\mathrm{sp}(u))$

  General case : **Commutative** $\mathcal{A}$ **isomorphic with** $\mathcal{C}(\chi_{\mathcal{A}})$
  Compact space $\chi_{\mathcal{A}}$ **of characters**

12- SPECTRAL CALCULUS

- **Assuming commutativity, operators behave like functions**
- **Commutativity yields subjective set-theoretical illusions**
- **Spectrum analysis, valid for a single normal operator :**

  Unitary : $\mathbf{sp}(u) \subset \mathbb{T}$
  Hermitian : $\mathbf{sp}(u) \subset \mathbb{R}$
  Positive : $\mathbf{sp}(u) \subset \mathbb{R}^+$
  Projection : $\mathbf{sp}(u) \subset \{0, 1\}$
  Symmetry : $\mathbf{sp}(u) \subset \{-1, +1\}$
  Partial symmetry : $\mathbf{sp}(u) \subset \{-1, 0, +1\}$ **(i.e., $u^3 = u$)**
- **Square root of a positive hermitian : $u^{1/2}$**
- $\mathbf{sp}(uv) \cup \{0\} = \mathbf{sp}(vu) \cup \{0\}$

  Partial isometries : Such that uu^* projection
  Closed under $(\cdot)^*$: $\mathbf{sp}(u^*u) \subset \{0, 1\}$ and u^*u hermitian

13- The GNS construction

- **Yields a representation of $\mathcal{C}^*$-algebra $\mathcal{A}$ in $\mathcal{B}(\mathbb{H}_\rho)$**
- **Representation depends on a state ρ**
- **Summing all such representations is faithful : $\mathcal{A} \hookrightarrow \mathcal{B}(\bigoplus_\rho \mathbb{H}_\rho)$**
- **Gel'fand-Neumark-Segal construction :**

  State : Linear form on $\mathcal{A}$ s.t. $\rho(I) = 1$, $u \geq 0 \Rightarrow \rho(u) \geq 0$

  Pre-Hilbert space : $\langle u \mid v \rangle = \rho(v^* u)$ positive on $\mathcal{A}$

  Hilbert space : $\mathbb{H}_\rho$ obtained by separation/completion

  Left regular representation : $\varphi_u(v) := u \cdot v$

  Norm-decreasing :

$$\|\varphi_u(v)\|^2 = \rho(v^* u^* u v) \leq \rho(v^*(\|u\|^2_{\mathcal{A}} I) v) = \|u\|^2_{\mathcal{A}} \|v\|^2$$

14- FAITHFULNESS

- **The norm is « algebraic » :**

  Positive : $\|h\| = \sup\{\lambda; \lambda \in \mathrm{sp}(h)\}$

  General case : $\|u\|^2 = \|uu^*\|$
- **Non-faithful $*$-iso shrinks spectrum of some positive h :**

  $$\mathrm{sp}(\varphi(h)) \subsetneq \mathrm{sp}(h)$$

  Only if φ non-injective
- **Simple $*$-algebra admits only one norm**
- **Typical example : $\mathcal{M}_n(\mathbb{C})$**
- **Direct system** $\hookrightarrow \ldots \mathcal{M}_{2^n}(\mathbb{C}) \hookrightarrow \mathcal{M}_{2^{n+1}}(\mathbb{C}) \hookrightarrow \ldots$
- **Whose algebraic direct limit inherits :**

  Stellar norm : Complete to get the CAR algebra

  Trace : Cyclic state (from normalised traces $2^{-n}\mathrm{tr}(\cdot)$)
- **GNS applied to CAR w.r.t. trace yields the hyperfinite factor**

15- Canon. Anticomm. Relations

- **Relate creators $\kappa(a)$ and their adjoints $\zeta(a)$ (annihilators) :**

$$\kappa(a)\zeta(b) + \kappa(b)\zeta(a) = \delta_{ab} \cdot I$$

$$\kappa(a)\kappa(b) + \kappa(b)\kappa(a) = 0$$

- **a, b in set A (or Hilbert space $\mathbb{H}$, in case $\delta_{ab} \rightsquigarrow \langle a \mid b \rangle$)**
- **$\kappa(1), \ldots, \kappa(n), \zeta(1), \ldots, \zeta(n)$ generate algebra $\mathrm{CAR}(n)$**

  Of dimension $2^{2n} = 2^n \times 2^n$

  Isomorphic with $\mathcal{M}_{2^n}(\mathbb{C})$
- **Fock space $\Lambda\mathbb{H} \ := \ \bigoplus \Lambda_n \mathbb{H}$**

$$\langle x_1 \wedge \ldots \wedge x_n \mid y_1 \wedge \ldots \wedge y_n \rangle \ := \ \det[\langle x_i \mid y_j \rangle]$$

- **Fock representation (irreducible, leads to type I_∞ factor) :**

$$\kappa(a)(z) \ := \ a \wedge z$$

$$\zeta(a)(x_1 \wedge \ldots \wedge x_n) \ := \ \sum (-1)^i \langle x_i \mid a \rangle x_1 \wedge \ldots \wedge \widehat{x_i} \wedge \ldots \wedge x_n$$

  $\kappa(a)$ partial isometry between « half-space without a » onto « half-space with a »

IV — von Neumann algebras

16- BACKGROUND

- Completion of a C^*-algebra w.r.t. directed sups of hermitians
- Passage $\mathcal{C}(X) \rightsquigarrow \mathcal{L}^\infty(X,\mu)$, two problems :
  Not canonical : Depends on μ (null sets), e.g., $\sup_n x^{1/n}$
  Not separable : weird C^*-algebras
  However, $\mathcal{L}^\infty$ is the dual of the separable $\mathcal{L}^1$
- Definitions of a vN algebra :
  Intrinsic : Dual C^*-algebra ;
  Predual made of normal (ultraweakly continuous) forms
  Implemented : Self-adjoint subalgebra of $\mathcal{B}(\mathbb{H})$ equal to its :
  Weak closure : $u_i \longrightarrow 0 \quad :\Leftrightarrow \quad \forall x \in \mathbb{H}\ \langle u_i(x) \mid x\rangle \longrightarrow 0$
  Strong closure : $u_i \longrightarrow 0 \quad :\Leftrightarrow \quad \forall x \in \mathbb{H}\ \|u_i(x)\| \longrightarrow 0$
  Bicommutant : $\mathcal{A}^c := \{u \in \mathcal{B}(\mathbb{H}); \forall v \in \mathcal{A} \quad uv = vu\}$
- Product in $\mathcal{B}(\mathbb{H})$ strongly continuous on balls
- « Dini's theorem » A weak increasing limit is a strong limit

17- COMMUTATIVE CASE

- A commutative vN algebra is a C^*-algebra, hence a $\mathcal{C}(X)$
  X extremely disconnected (closure of open sets still open !)
  Bad topology : too discrete (Scott domains are too coarse)
  No example (but abstract nonsense : Stone-Čech $\beta\mathbb{N}$)
  Replacing βX with X yields $\mathcal{L}^\infty(X, \mu)$ (non intrinsic)
- The center $\mathcal{Z}$ of $\mathcal{A}$ is a vN algebra ($= \mathcal{A} \cap \mathcal{A}^c$)

$$\mathcal{Z} = \mathcal{L}^\infty(X, \mu) \qquad \mathcal{A} = \int A_x d\mu(x)$$

- Factor : « connected » vN algebra (i.e., trivial center $\mathcal{Z} = \mathbb{C}I$)
- Every vN algebra is a sum of factors

18- CLASSIFICATION OF FACTORS

- Define preorder $\preccurlyeq$ between the projections of $\mathcal{A}$, combining :
  Inclusion : $\pi \subset \pi'$ when $\pi\pi' = \pi$
  Equivalence : $\pi \simeq \pi'$ when $\exists u\ u^*u = \pi$ and $uu^* = \pi'$
  Infinite projection : $\pi \simeq \pi' \subsetneq \pi$
- $\mathcal{A}$ factor iff $\preccurlyeq$ total ; if separable predual, look at $\preccurlyeq / \simeq$:
  $\{0, \ldots, n\}$: Type I_n
  $\mathbb{N} \cup \{\infty\}$: Type I_∞
  $[0, 1]$: Type II_1
  $[0, \infty]$: Type II_∞
  $\{0, \infty\}$: Type III
- Type I corresponds to trivial algebras $\mathcal{B}(\mathbb{H})$, of type $I_{\dim(\mathbb{H})}$
- Type III corresponds to completely infinite algebras
- Types I_∞, II_∞ semi-finite : admit non-trivial tracial weights :

$$\varpi(h) \in [0, \infty] \quad (h \geq 0) \qquad \varpi(uu^*) = \varpi(u^*u)$$

19- FACTORS OF TYPE II_1

- **Fractal dimension :** $\dim \pi := 1/2$ **when** $\pi \simeq I - \pi$**, etc.**
- **Extends into a trace, i.e., a normal cyclic state (unique)**
  Normal : Ultraweaky (i.e., weakly on the unit ball) continuous
- **The bicommutant of GNS** $CAR(\mathbb{N})$ **in** $\mathbb{H}_{\mathrm{tr}(\cdot)}$ **is a** II_1 **factor**
- **If** G **discrete group, take Hilbert space** $\ell^2(G)$ **:**
  Convolution : $\ell^2(G) \times \ell^2(G) \mapsto \ell^\infty(G)$
  Left algebra : $\mathcal{A}[G] := \{u \in \ell^2(G); u * \ell^2(G) \subset \ell^2(G)\}$
  Right algebra : Idem ; its commutant is the left algebra
- G **is i.c.c. when all its conjugacy classes are infinite**
  If G **is i.c.c., then** $\mathcal{A}[G]$ **is a factor**
  Of type II_1 **; its unique trace is given by** $\mathrm{tr}((x_g)) := x_e$

20- HYPERFINITENESS

- $\mathcal{A}$ hyperfinite iff bicommutant of rep. of some CAR algebra
- Type I factors are hyperfinite (use irreducible Fock rep.)
- Up to $*$-iso, only one hyperfinite II_1 factor

  Murray-von Neumann factor $\mathcal{R}_0$, a.k.a. hyperfinite factor
- « The » hyperfinite factor can be obtained in various ways :

  CAR : GNS w.r.t. trace

  Local finiteness : $\mathcal{A}[G] = \mathcal{R}_0$ if $G = \bigcup_n G_n$, G_n finite

  Amenability : $\mathcal{A}[G] = \mathcal{R}_0$ iff G admits an invariant mean

  Crossed product : $\mathcal{R}_0 \rtimes G \sim \mathcal{R}_0$ if G amenable group of external automorphisms (needs no be i.c.c.)
- Locally finite $\Rightarrow$ amenable ; many amenable discrete groups

  Free groups not amenable ; $\mathcal{A}[G]$ not hyperfinite for G free
- Connes : $\mathcal{A}$ hyperfinite iff exists projection of norm 1 :

$$E : \mathcal{B}(\mathbb{H}) \mapsto \mathcal{A} \quad E(x) = x \ (x \in \mathcal{A}) \quad \|E(x)\| \leq \|x\|$$

V — GEOMETRY OF INTERACTION

21- Limitations of QCS

- Problems in infinite dimension with the trace :
  Type I_∞ : The twist σ is not trace-class
  Type II_1 : Trace defined, but $\mathrm{tr}(\sigma \cdot (a \otimes b)) = 0$
- Layer -2 (categories, morphisms) badly fitted for :
  Truth : There should be « bad » morphisms ; no way !
  Complexity : No natural way to avoid having « all » functions
- Layer -2 rests upon the hypothesis :

  The potential is the set of all possibilities
- A multiplicative proof-net can be seen as :
  Layer -3 : A permutation (unitary operator) σ_0 of the literals
  Layer -2 : All travels $\sigma_0\tau$ induced by σ_0
- The travels can be obtained by means of the functor « Fock » :

$$u \rightsquigarrow \Lambda i u$$

22- The invariant of GoI

- **Fock functor :** $\mathrm{tr}(\Lambda i u \cdot \Lambda i v) = \mathrm{tr}(\Lambda - uv) = \det(I - uv)$
- $\det(I - uv)$ **as the layer** -3 **(GoI) invariant**
- $\det \begin{bmatrix} I-a & -b \\ -c & I-d \end{bmatrix} = \det \begin{bmatrix} I-a & -b+(I-a)(I-a)^{-1}b \\ -c & I-(d+c(I-a)^{-1}b) \end{bmatrix}$
- $\det(I - F(a \oplus b)) = \det(I - Fa) \cdot \det(I - F[a]b)$

  Compare with : $\mathrm{tr}(F(a \otimes b)) = \mathrm{tr}(F[a] \cdot b)$
- « **Heating coefficient** » $\det(I - Fa)$ **:**

  Nilpotency of Fa **: Logical rules ensure** $\det(I - Fa) = 1$

  Introspective : Killed by layer -2

  Truth : Measures soundness (not quite a « truth value »)

  Project : (α, u) **(**$\alpha \in \mathbb{C}$ **wager,** $u \in \mathcal{R}_0, \|u\| \leq 1$ **design)**

  Duality : $(\alpha, u) \downarrow\!\sim (\beta, v) \quad \Leftrightarrow \quad \alpha\beta \det(I - uv) \neq 1$

  $\alpha\beta \det(I - uv) \neq 1$ **supposes that** $r(uv) < 1$

23- SUBJECTIVE TRUTH

- As in ludics, derive truth from winning, provided :
  Consistency : $(\alpha, u) \downarrow\!\sim (\beta, v)$ cannot both win
- Finite dim. : $\alpha = \beta = 1, u, v$ part. perm. $\Rightarrow (\alpha, u) \not\downarrow\!\sim (\beta, v)$
  Either $r(uv) = 1$ or uv nilpotent
- Subjective notion of truth :
  Viewpoint (« base ») : Max. comm. subalg. $\mathcal{P} = \mathcal{P}^c \subset \mathcal{R}_0$
  Success w.r.t. $\mathcal{P}$: $\alpha = 1$ and $u\mathcal{P}u^* \subset \mathcal{P}$
  u in the normaliser of $\mathcal{P}$, sort of « partial permutation »
- Unlike ludics, no « professional losers »
- Behaviour B may be true, false ($\sim \mathrm{B}$ true) depending on $\mathcal{P}$
- A theorem may even be false w.r.t. « inadequate » viewpoint
  Intersubjectivity : Sharing of $\mathcal{P}$ in Mod. Ponens $A, A \Rightarrow B / B$
  Supports of A, B must belong to $\mathcal{P}$

24- MODALITY AND COMPLEXITY

- Functoriality of « bang » requires truth constraints
- $!A$ means « A is true »; subjective
- Necessity, infinity, change of polarity :
  Sort of measurement, reduction to base, to a « set-theory »
- Subjective truth : heating $\det(I - Fa) = 1$ (Fa nilpotent)
  Termination : Subjectivity induces computability
  Modified viewp. : Comm. C^*-algebra $\mathcal{P}$ with $\mathcal{P}^c$ commutative
  Complexity : Normaliser of $\mathcal{P}$ of « bounded complexity »
  - Locally finite G s.t. normaliser made of finite combinations
    $u \in \mathcal{A}[G_n]$; nilpotency bounded by $\sharp(G_n)$