# Transcendental syntax II: non-deterministic case

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*Donner un sens plus pur aux mots de la tribu :*
*à P.-L., ami et compagnon de route de 30 ans.*

**Abstract**

This second installment expands our analytics of determinism [5] to the non-deterministic case, in close relationship with additive proof-nets.

# 1 Introduction

## 1.1 Non-determinism

The characteristic of the analytic layer is to be self-contained, i.e., beyond interpretation, thus both indisputable and meaningless. Not to be confused with semantics, which is a structured, therefore biased — for better or for worse — approach. Analytics splits into two sublayers, *constative* and *performative*.

A reasonable approximation to analyticity is given by pure $\lambda$-calculus, normal terms being constative — explicit —, the others being performative — implicit. The knitting constat/performance relies on strong normalisation and the Church-Rosser property. The $\pi$-calculus — or rather calculi, since the notion is rather disposable — would like to be the non-deterministic $\lambda$-calculus, a sort of untyped linear logic. Whatever standpoint — analytic or synthetic — we adopt, $\pi$-calculi are either too cooked or too raw: this half-baked approach is by no means an approximation to a non-deterministic analytics.

By the way, $\lambda$-calculus is not quite analytic, i.e., not self-contained: this is due to the external character of the rewriting in charge of performance. Moreover, the approach involves a functional bias which can be seen as an external polarisation: the distinction between arguments and values. Our first installment [5] fixes this want of analyticity by means of *stars* and *constellations*: constats are nothing but uncoloured constellations and performance — a.k.a. colour-elimination — is obtained by *matching* rays of complementary colours, e.g., $t$ and $u$, in `green` and `magenta`: the physical plugging between $t$ and $u$ is handled by means of the m.g.u. (most general unifier) $\theta$ s.t. $t\theta = u\theta$.

Stars and constellations are offsprings of the *proof-nets* of linear logic. They can also be viewed as cousins — since both ultimately come from Herbrand and Gentzen — of the *resolution* at work in the ill-fated logic programming: a clause $\Gamma \vdash A$ can be seen as a star, with a `magenta` tail $\Gamma$ and a `green` (resp. `uncoloured`) head $A$ (resp. $A$).

However, resolution does not provide a good analytics. First because the variables occurring in the head and tail may not be quite the same; second, because of the mess produced by multiple matchings. By requiring all rays of a star to use exactly the same variables, [5] fixes once for all the first drawback. By requiring any two rays of the constellation to be disjoint i.e., not matchable — thus coping with multiple matchings —, [5] produced a nice *deterministic* analytics. This limitation is however too drastic, since *additive* operations of logic are basically non-deterministic: $A \,\&\, B$ denotes a non deterministic choice between $A$ and $B$.

In order to cope with non-determinism, it is therefore necessary to allow multiple matchings: in this way, we may encounter an alternative between $t\theta = u'\theta$ and $t\theta = u''\theta$. This liberalisation is however too brutal: since the various alternatives are not correlated, we obtain a sort of Alzheimer non-determinism — the kind at work in LOGSPACE computation. In particular, the seminal NP *satisfiablity* problem — which involves a consistent evaluation `true/false` of each propositional atom — cannot be handled in this way. The necessary coordination between the various non-deterministic choices prompts us to equip our constellations with a structure of coherent space: two stars $\mathcal{S}, \mathcal{T}$ can be plugged, i.e., may be part of the same *dendrite*[1], only when $\mathcal{S} \frown \mathcal{T}$.

This solution agrees with the analytic part of additive nets: given constellations $\mathcal{C}, \mathcal{D}$ corresponding to the premises $\vdash \Gamma, A$ and $\vdash \Gamma, B$ of a &-rule with conclusion $\vdash \Gamma, A \,\&\, B$, we can form the constellation $\mathcal{C} \breve{+} \mathcal{D}$, in which any two stars $\mathcal{S} \in \mathcal{C}$ and $\mathcal{T} \in \mathcal{D}$ are incoherent: $\mathcal{S} \smile \mathcal{T}$. Incidentally, observe that $\mathcal{C}$ and $\mathcal{D}$ may have some stars in common; they should be considered as distinct, hence the use of « $\breve{+}$ » instead of « $\breve{\cup}$ ».

When handling exponentials, rays $p(t)$ are replaced (see 4.1 below) with $p(t)_y := p(t \cdot y)$, where $y$ is a fresh variable, to the effect that a star $\mathcal{S}$ becomes $\mathcal{S}_y$: depending upon the values $i, j, \ldots$ assumed by $y$, we shall thus get copies $\mathcal{S}_i, \mathcal{S}_j, \ldots$ of the original $\mathcal{S}$. Now, *quid* of the coherence in terms of copies? Let us assume, for simplicity, that $y$ assumes closed values $y\theta = i, y\pi = j$. Inside the same copy, the coherence reduces to the original one: $\mathcal{S}_i \frown \mathcal{T}_i$ iff $\mathcal{S} \frown \mathcal{T}$. But there is no incoherence between distinct copies: $\mathcal{S}_i \frown \mathcal{T}_j$ when $i \neq j$.

Our coherence takes the form of a (finitely generated) set $\mathcal{S} \dagger \mathcal{T}$ of *forbidden* substitutions[2]. Typically, in the additive example of two incoherent $\mathcal{S}$ and $\mathcal{T}$, $\mathcal{S}_y \dagger \mathcal{T}_z$ is defined as the set of those $\theta$ s.t. $y\theta = z\theta$.

[1] Diagram in [5].

[2] Self-incoherence is the fact that a star $\mathcal{S}$ may only occur for certain instanciations $\mathcal{S}\theta$.

## 1.2 The stuff as types are made on

Coherent constellations are enough to cope with the analytics — the untyped computational structure — of additive proof-nets. But inadequate from the synthetic standpoint: the correctness criterion is not so easily handled. This synthetic inadequation prompts us to revisit our approach.

The obvious idea is to handle the logical link with premises $A, B$ and conclusion $A \& B$ by a L/R switch: either $\&_L := [\![ \frac{q_A(x)}{\boxed{q_{A\&B}(x)}} ]\!]$ or $\&_R := [\![ \frac{q_B(x)}{\boxed{q_{A\&B}(x)}} ]\!]$. Each switching of the & links thus induces a *slice*, a sort of parallel universe. But this poses a sort of problem of « free will » : when I select the position L, how do I know that I was not already in the left slice, i.e., that I could have freely selected R? Like in science fiction[3], we may have the illusion to be external to a universe (the slice) we are indeed part of. This technically translates as the failure of additive normalisation: a cut between $A \& B$ and $\sim A \oplus \sim B$ is replaced with one between $A$ and $\sim A$ (resp. $B$ and $\sim B$) depending upon the premise $\sim A$ (resp. $\sim B$) of $\sim A \oplus \sim B$. But this require this premise to be independent of the choice $A/B$: if the premise $\sim B$ (resp. $\sim A$) is chosen in slice $A$ (resp. $B$), the procedure fails.

The early approach to additives involved boxes, an *ad hoc* solution to the question. The proof-nets introduced in [4] used boolean weights accounting for the various choices L/R, i.e., for the slicing. Normalisation involved a *global* rewriting corresponding to the evaluation of those weights, i.e., an external viewpoint which makes it not quite analytic. Moreover, the question of logical correctness was not handled in a satisfactory way, i.e., respecting an Object/Subject opposition: the Object (the upper part of the proof-net) should be opposed to the Subject (the lower part of the proof-net) in a series of predefined tests. In this respect the criterion given in [7], although ensuring exact sequentialisation, was even less satisfactory: it cannot be reduced to a series of tests. Slightly less general, my own criterion [4] relied on tests making use of *jumps* depending upon the Object, hence not available prior to the Object tested.

It took a long time to realise that sequentialisation — the possibility of writing a proof-net by means of sequent calculus rules —, albeit useful, was not that central: what actually matters is the possibility of eliminating cuts. Indeed, it is possible to construct non sequential multiplicative connectives (see annex A below)— thus admitting no sequent calculus — enjoying normalisation. The criterion for a good proof-net is thus not sequentialisation, but cut-elimination.

Now remember that the upper part of a proof-net is a `blue` constellation and the lower part is a constellation in `yellow`/`uncolored`. This lower part is basically obtained as the normal form of another constellation using the additional colours `magenta` and `green`. This constellation is built by combining the bridges

[3] Simulacron 3, The Matrix, etc.

$[\![\,\frac{p_A(x)}{q_A(x)}\,]\!]$ (for atoms) and $[\![\,\frac{q_A(x)}{p_A(x)}\,]\!]$ for conclusions, and various LEGO bricks, e.g., $[\![\,\frac{q_A(x), q_B(x)}{q_{A\otimes B}(x)}\,]\!]$ and $[\![\,\frac{q_A(x)}{q_{A\parr B}(x)}\,]\!] + [\![\,\frac{q_B(x)}{}\,]\!]$ and $[\![\,\frac{q_B(x)}{q_{A\parr B}(x)}\,]\!] + [\![\,\frac{q_A(x)}{}\,]\!]$ for multiplicatives.

I propose to use the extra colours `cyan` and `red` as variants of `magenta` and `green` to form a third position of the & switch: besides $\&_L$ and $\&_R$, $\&_M := ([\![\,\frac{q_A(x)}{q_{A\&B}(x)}\,]\!] \breve{+} [\![\,\frac{q_B(x)}{q_{A\&B}(x)}\,]\!]) + [\![\,\frac{q_{A\&B}(x)}{q_{A\&B}(x)}\,]\!]$. $\&_M$ obviously normalises into $[\![\,\frac{q_A(x)}{q_{A\&B}(x)}\,]\!] \breve{+} [\![\,\frac{q_B(x)}{q_{A\&B}(x)}\,]\!] = \&_L \breve{+} \&_R$, the point being that the normalisation `cyan`/`red` can be postponed.

In presence of « mosaic » switchings like $\&_M$, the normal form becomes the sum of several copies $[\![\,p_\Gamma(x)\,]\!]_1, \ldots, [\![\,p_\Gamma(x)\,]\!]_k$ of the same star. Using Church-Rosser, this normal form can be obtained in two steps, first eliminate the colours `yellow`/`blue` and `magenta`/`green`, yielding a normal form $\mathcal{C}$, then the colours `cyan`/`red`. Let $\mathcal{C}_i$ $(i = 1, \ldots, k)$ be the subconstellation of $\mathcal{C}$ contributing to $[\![\,p_\Gamma(x)\,]\!]_i$ (in general $\mathcal{C} \supsetneq \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_k$). The additional *independence* condition, indeed an offspring of my first criterion for quantifiers [3], writes:

$$\mathcal{C}_1 \cap \ldots \cap \mathcal{C}_k \neq \emptyset$$

Let us now see how the criterion ensures « free will ». Consider a proof-net with a single conclusion $(A\&B)\otimes C$, and switch it, with the downmost « & » on `M`. We get two $\mathcal{C}_i$, say $\mathcal{C}_L$ and $\mathcal{C}_R$ corresponding to $\&_L$ and $\&_R$; each of them is a sum $[\![\frac{}{q_{A\&B}(x)}]\!]_i + [\![\frac{q_{A\&B}(x)}{p_{(A\&B)\otimes C}}]\!]_i$. Now, two stars with the same rays are considered equal only if they have been built in the same way, from the same dendrite. Therefore, the two $[\![\frac{}{q_{A\&B}(x)}]\!]_i$ are distinct. The independence condition thus requires the two $[\![\frac{q_{A\&B}(x)}{p_{(A\&B)\otimes C}}]\!]_i$ to be equal; in other terms, what happens in the proof-net above $C$ should not depend upon the choice `L`/`R` between $A$ and $B$.

# 2 Analytics proper

## 2.1 Background: substitutions

Since analytics basically means « self-contained », it implies « finite »: the *etcætera* at work in infinite structures is the obvious reference to something not quite present. Hence the use of functional terms and Herbrand's unification to avoid the use of actual infinity. A functional language involving function letters and variables is therefore fixed once for all.

A *substitution* $\theta$ assigns to each variable $x, y, z, \ldots$ a term[4] $x\theta, y\theta, z\theta$. In order to stay finite, we assume that the substitution is idle ($w\theta = w$) on almost all variables $w$; it can therefore be presented through its action on a finite set of variables. A substitution can be extended to all terms by means of:

$$t\theta := t[x\theta/x, y\theta/y, z\theta/z, \ldots]$$

Hence the possibility of composing substitutions: $x(\pi\theta) := (x\pi)\theta$. Composition induces a preorder:

$$\theta \preceq \upsilon \quad \Leftrightarrow \quad \exists\theta' \; \upsilon = \theta\theta'$$

Hence the identity $t\iota := t$ is the smallest substitution.

An equation $t = u$ between terms may be solved by a *unifier* $\theta$:

$$t\theta = u\theta$$

A celebrated result of Herbrand states the existence of a most general (or smallest) unifier (m.g.u.) — if any —, i.e., of a $\theta$ such that any unifier $\upsilon$ can uniquely be written $\upsilon = \theta\theta'$.

*Matching* is unification with bound variables: in order to match $t$ and $u$, we first modify their respective variables to make them distinct, then we unify. Matching induces a structure of conditional lattice on substitutions: assume that $\theta, \upsilon$ are idle for variables other than $x, y, z$ and let $f$ be ternary; then $\theta$ and $\upsilon$ have a supremum iff $f(x\theta, y\theta, z\theta)$ and $f(x\upsilon, y\upsilon, z\upsilon)$ are not *disjoint*, i.e., do match; the l.u.b. $\pi = \theta \vee \upsilon$ is obtained by writing the result of the matching as $f(x\pi, y\pi, z\pi)$.

The terms $t$ stands for the potentiality of all its substitutes. However, our analytics makes use of *exceptions*: typically, $f(x, y)$, unless $x = y$. The general form an exception is that of an *ideal*; by this I mean an upward-closed set of substitutions with — up to equivalence — finitely many minimal elements $\theta_1, \ldots, \theta_k$. For instance « unless $x = y$ » is the ideal generated by the substitition $x\theta = y\theta := x$. Two ideals are of special interest: the empty one $\emptyset$ and the full (or trivial) one $\Omega$ generated by the identity substitution $\iota$.

*Quid* of the behaviour of an ideal $\mathcal{I}$ under a substitution $\theta$? Indeed, given $\upsilon \in \mathcal{I}$, we may form (if possible) the l.u.b. $\upsilon \vee \theta$ which can be written as a composition $\theta(\theta^{-1}\upsilon)$; then the ideal $\theta^{-1}\mathcal{I}$ is defined as the set $\{\theta^{-1}\upsilon; \upsilon \in \mathcal{I}\}$. Its generators are the images of the generators of $\mathcal{I}$ under the map $\theta^{-1}\cdot$; note that, since this map is partial, some generators may be missing. Typically, coming back to the ideal « unless $x = y$ », $\theta^{-1}\mathcal{I} = \emptyset$ when $x\theta, y\theta$ are disjoint. But $\theta^{-1}\mathcal{I}$ is the trivial ideal $\Omega$ when $x\theta = y\theta$.

Ideals are used to describe dynamic situations of the sort « $\mathcal{S}$ and $\mathcal{T}$ are coherent except for certain values ». These forbidden values are the substitutions of some ideal $\mathcal{S} \dagger \mathcal{T}$, so that « except for » keeps a finitary meaning. When associating ideals to expressions depending upon variables $x, y, z$, only those variables do matter: if $x\theta = x\pi, y\theta = y\pi, z\theta = z\pi$ and $\theta \in \mathcal{I}$, then $\pi \in \mathcal{I}$. In particular, if we deal with a closed expression, the ideal is either $\emptyset$ or $\Omega$.

[4]The notation $x\theta$ is friendler than $\theta x$.

## 2.2 Stars and constellations

A *star* $\mathcal{S}$ is set of *rays*, $t_1, \ldots, t_{n+1}$, i.e., terms with *exactly* the same variables.

The variables in the star are bound, i.e., local to the star: we should indeed use the notation $\nu X.[\![\, t_1, \ldots, t_{n+1} \,]\!]$, where $X = x_1, \ldots, x_k$ are the variables occuring in the rays $t_i$. Since this notation is a pain in the ass, we shall prefer a sort of $\alpha$-conversion, the replacement $X = x_1, \ldots, x_k$ with $Y = y_1, \ldots, y_k$ in $[\![\, t_1, \ldots, t_{n+1} \,]\!]$ yielding $[\![\, t_i[Y/X], \ldots, t_j[Y/X] \,]\!]$. The renaming of variables corresponds to the substitution $x_i\theta := y_i$. More generally, substitutions act on stars: if $\theta$ is any substitution, then $\mathcal{S}\theta := [\![\, t_1\theta, \ldots, t_{n+1}\theta \,]\!]$.

A *constellation* $\mathcal{C}$ is a finite set of stars[5] together with a sort of coherence: the set of the $\mathcal{S}\theta$ ($\mathcal{S} \in \mathcal{C}$, $\theta$ substitution) is equipped with a binary symmetric relation $\mathcal{S}\theta \smile \mathcal{T}\pi$ s.t. $\mathcal{S}\theta \smile \mathcal{S}\theta$ implies $\mathcal{S}\theta \smile \mathcal{T}\pi$. This relation, restricted to the set of *self-coherent* $\mathcal{S}\theta$ ($\mathcal{S}\theta \asymp \mathcal{S}\theta$) is therefore a coherent space.

This coherent space should be *finitary*, i.e., presented by means of ideals $\mathcal{S} \dagger \mathcal{T}$. Or rather $\mathcal{S} \dagger \mathcal{T}'$: we must first rename the variables of $\mathcal{T}$ to make them distinct from those of $\mathcal{S}$. Then $\mathcal{S}\theta$ and $\mathcal{T}\pi$ can be simultaneously written as $\mathcal{S}\upsilon, \mathcal{T}'\upsilon$. The set $\mathcal{S} \dagger \mathcal{T}' := \{\upsilon; \mathcal{S}\upsilon \smile \mathcal{T}'\upsilon\}$ is required to be an ideal. In other terms, coherence is assumed unless... it falls into a finitely generated set of exceptions. Typically, if $\mathcal{S}, \mathcal{T}$ respectively depend upon $x, y$, they may be coherent unless $x = y$. Applying a substitution restricts the coherence: $\mathcal{S} \dagger \mathcal{T}$ is replaced with $\theta^{-1}(\mathcal{S} \dagger \mathcal{T})$ which may be trivial, i.e., equal to $\Omega$.

Define $\mathcal{S}^\dagger$ as the ideal $\{\theta; \mathcal{S}\theta \smile \mathcal{S}\theta\}$; if $\theta \in \mathcal{S}^\dagger$, then $\mathcal{S}\theta$ is self-incoherent, in other terms, unfit. We can therefore see $\mathcal{S}^\dagger$ as a set of *forbidden* substitutions. An extreme case is when $\mathcal{S}^\dagger = \Omega$, in case all substitutions are forbidden: in particular $\mathcal{S}$ is self-incoherent. Such stars are not quite excluded, but they can be neglected, counted as nil. This sheds a new light as to the expression « forbidden »: a forbidden substitution kills the star, hence the notation $\mathcal{S}^\dagger$ for the ideal of star-slayers.

The coherent space structure is subject to the following restriction:

*If $s \in \mathcal{S}$ and $t \in \mathcal{T}$ and $s\theta = t\pi$, then either $\mathcal{S}\theta \smile \mathcal{T}\pi$ or $\mathcal{S} = \mathcal{T}$ and $s = t$.*

The deterministic case corresponds to a flat coherence $\mathcal{S} \dagger \mathcal{T} = \emptyset$. The condition above reduces to the non-matchability required in [5].

## 2.3 Colours and explicitation

*Colours* are specific unary function symbols, used only in prefix. Terms of distinct colours do not match.

Colours come by pairs: `green`/`magenta`, `blue`/`yellow`, `red`/`cyan`; instead of `green`$(t)$, `magenta`$(t)$ (resp. `blue`$(t)$, `yellow`$(t)$, or `red`$(t)$, `cyan`$(t)$), we write $t$, $t$ (resp. $t$, $t$ or $t$, $t$). The colours `green,blue` and `red` are *additive*, `magenta,yellow` and `cyan` being *substractive*. In logical contexts, we try to use additive colours for conclusions, negative ones for premises.

[5]Possibly with repetitions; due to the coherence, we cannot however speak of a multiset.

The process of explicitation consists in eliminating colours in a constellation $\mathcal{C}$; w.l.o.g. we can assume that only one pair of colours, say green/magenta has been used: in presence of a second pair blue/yellow, replace $t$, $t$ with $f(t)$, $f(t)$, where $f$ is a new function letter.

A *dendrite* (diagram in [5]) is a connected and acyclic graph whose vertices are stars of $\mathcal{C}$; if a star is used several times, its variables should be renamed so as to make them distinct. Edges are formal equations $t = u$ between rays of complementary colours.

The *actualisation* $\mathcal{D}$ of a dendrite $\Delta$ consists in matching the uncoloured terms underlying each edge: $t = u$ becomes the actual matching $t\theta = u\theta'$. $\mathcal{D}$ can thus be seen as a set of substituted stars $\mathcal{S}\theta$; we require $\mathcal{D}$ to be a clique in the coherent space $\mathcal{C}$. The requirement encompasses self-coherence: if $\mathcal{S}\theta \in \mathcal{D}$, then $\theta \notin \mathcal{S}^{\dagger}$.

*Strong normalisation* is defined as expected:

1. There are only finitely many correct dendrites. In other terms, for an appropriate $N$, all dendrites of size $N + 1$ fail. Since a dendrite of size $N+2$ contains dendrites of size $N+1$, there is no point in forming dendrites of bigger sizes.

2. A ray of the dendrite is *free* when not involved in an edge; we require that there is no *closed* — without free ray — correct dendrite.

A dendrite with no coloured free ray is styled *uncoloured*; the residual star $\mathcal{D}_r$ of an uncoloured correct dendrite consists in the free rays of the actualised dendrite.

The star $\mathcal{D}_r$ is equipped with the ideal consisting of those $\pi$ s.t. $\mathcal{D}\pi$ fails to be a clique when substituted by $\pi$. And $\mathcal{D}_r\pi \smile \mathcal{E}_r\pi'$ when $\mathcal{D}\pi \cup \mathcal{E}\pi'$ fails to be a clique.

By the way, observe that the same star may occur several times in the normal form, typically when obtained from different dendrites. These « occurrences » are necessarily incoherent.

### 2.4 Church-Rosser

An uncoloured term is a term not beginning with a colour function; since colours are up to my choice, I may forget certain pairs, to the effect that certain coloured terms may be styled uncoloured. As to colour-elimination, in presence of two pairs green/magenta and blue/yellow, I may either eliminate them in a single step or forget that blue/yellow are colours and eliminate the sole green/magenta; then remember that blue/yellow are colours and eliminate them from the normal form. This induces three possible ways of eliminating two pairs of colours.

These three ways are basically equivalent, except for a small discrepancy: the non-normalisable three-coloured star $[\![ x, x, x ]\!]$ ($x$ in green, magenta, blue) can be normalised in two steps (`blue`/`yellow`, then `green`/`magenta`). I proposed in [5] a way to fix this minor issue, which although technically correct, is a sort of minor unknitting constat/performance.

In the sequel of the paper, I will act as if Church-Rosser were completely unproblematic. This is slightly incorrect, but the real point is not there anyway.

# 3 Additives

## 3.1 The analytics of additives

$p_A$ and $p_B$ are defined from $p_{A\&B}$ by: $p_A(x) := p_{A\&B}(\mathtt{l}\cdot x), p_B(x) := p_{A\&B}(\mathtt{r}\cdot x)$. In the same way, $p_A$ and $p_B$ would be defined from $p_{A\oplus B}$ by:
$p_A(x) := p_{A\oplus B}(\mathtt{l}\cdot x), p_B(x) := p_{A\oplus B}(\mathtt{r}\cdot x)$.

**&-rule:** if the proof $\pi$ of $\vdash \Gamma, \underline{\Delta}, \underline{\Delta}', A \,\&\, B$ comes from proofs $\nu$ of $\vdash \Gamma, \underline{\Delta}, A$, and $\mu$ of $\vdash \Gamma, \underline{\Delta}', B$, then $\pi^\bullet := \nu^\bullet \overset{\smile}{+} \mu^\bullet$, i.e., the disjoint union of $\nu^\bullet$ and $\mu^\bullet$ in which any two stars $\mathcal{S} \in \nu^\bullet$ and $\mathcal{T} \in \mu^\bullet$ are incoherent: $\mathcal{S} \dagger \mathcal{T} = \Omega$.

**$\oplus_L$-rule (resp. $\oplus_R$-rule):** if the proof $\pi$ of $\vdash \Gamma, \underline{\Delta}, A \oplus B$ comes from a proof $\nu$ of $\vdash \Gamma, \underline{\Delta}, A$ (resp. $\vdash \Gamma, \underline{\Delta}, B$), then $\pi^\bullet := \nu^\bullet$.

Real logic is to be found in the combination vehicle + ordeals. Sequent calculus can be seen as a collection of recipes producing vehicles, a sort of shorthand, by no means the real thing. We must therefore give up the idea of a sequentialisation in the sense of the reduction to a preset system of sequent calculus rules. For instance, there are other additive constructions, typically a 4-ary analogue, the « multibox » [2] — which combines proofs of $\vdash \Gamma, \underline{\Delta}, A, C$, $\vdash \Gamma, \underline{\Delta}, A, D$, $\vdash \Gamma, \underline{\Delta}, B, C$, $\vdash \Gamma, \underline{\Delta}, B, D$, yielding $\vdash \Gamma, \underline{\Delta}, A \,\&\, B, C \,\&\, D$. This direct construction cannot be reduced to the binary case: the translation would keep track of the order in which the two &-rules have been performed.

Sequentialisation being out, we concentrate upon an additive criterion ensuring normalisation. The general idea is that an additive cut between — say — a &-rule and a $\oplus_L$-rule, i.e., the physical plugging between $A \& B$ and $\sim\!A \oplus \sim\!B$ amounts at plugging the left premise $A$ of the &-rule with the premise $\sim\!A$ of the $\oplus_L$-rule: as observed in [5], this operation which does not affect the vehicles, is basically a change of syntheticity. The right premise of the &-rule, i.e., $\mu^\bullet$, since incoherent with $\nu^\bullet$, will not contribute to the eventual normal form, as if erased by this change of syntheticity.

The delicate point comes from the fact that nothing forbids intricate situations, typically that of an incoherent sum $\oplus_L + \oplus_R$, the two rules being respectively incoherent with the premises $A$ and $B$ of the &-rule: the physical plugging of $A \& B$ and $\sim\!A \oplus \sim\!B$ amounts at two incoherent pluggings $A/\sim\!A$, $B/\sim\!B$: an unwelcome possibility that the correctness criterion is supposed to avoid.

## 3.2 The synthetics of additives

We shall make use of *mosaics*, i.e., ordeals built with the extra colours `red` and `cyan`. Depending upon a switching, i.e., a choice `L`/`R`/`M` (left/right/mosaic) for the $A \,\&\, B$, the following LEGO bricks are introduced:

$\&_L$: $[\![ \frac{q_A(x)}{q_{A\&B}(x)} ]\!]$.

$\&_R$: $[\![ \frac{q_B(x)}{q_{A\&B}(x)} ]\!]$.

$\&_M$: $([\![ \frac{q_A(x)}{q_{A\&B}(x)} ]\!] \overset{\smile}{+} [\![ \frac{q_B(x)}{q_{A\&B}(x)} ]\!]) + [\![ \frac{q_{A\&B}(x)}{q_{A\&B}(x)} ]\!]$.

$\oplus$: $[\![ \frac{q_A(x)}{q_{A\oplus B}(x)} ]\!] \overset{\smile}{+} [\![ \frac{q_B(x)}{q_{A\oplus B}(x)} ]\!]$.

The notations $+$ and $\overset{\smile}{+}$ are respectively for coherent and incoherent sums. $\&_M$ can be seen as the sum $\&_L + \overset{\smile}{+} \&_R$ (into which it normalises) together with a possible pruning at the level of the edge $q_{A\&B}(x) = q_{A\&B}(x)$.

As to the correctness criterion, once a switching has been chosen, we first require the normal form to be $[\![ p_\Gamma(x) ]\!]_1 \overset{\smile}{+} \ldots \overset{\smile}{+} [\![ p_\Gamma(x) ]\!]_k$ ($k \neq 0$). Now, $[\![ p_\Gamma(x) ]\!]_i$ comes from a dendrite with internal edges using the three pairs of colours; if we decide to connect the `green`/`magenta` and `blue`/`yellow` edges, we get a dendrite in `red`/`cyan`/`uncoloured` which can be seen as a constellation $\mathcal{C}_i$ — the mosaic normal form. We require that $\mathcal{C}_1 \cap \ldots \cap \mathcal{C}_k \neq \emptyset$ (*independence*).

Vehicles obtained from sequent calculus proofs are easily shown to enjoy our correctness condition. Let us concentrate on independence: the basic case is that of a terminal & link switched on `M`: the star containing the conclusion $p_{A\&B}(x)$, i.e., $[\![ \frac{q_{A\&B}(x)}{q_{A\&B}(x)} ]\!]$ is in $\mathcal{C}_1 \cap \ldots \cap \mathcal{C}_k$. The condition is preserved by tensor product: if the proof $\pi$ of $\vdash \Gamma, \Delta, A \otimes B$ comes from proofs $\nu$ of $\vdash \Gamma, A$ and $\mu$ of $\vdash \Delta, B$, then $\pi^\bullet := \nu^\bullet + \mu^\bullet$. If the vehicle $\nu^\bullet$ (resp. $\mu^\bullet$) enjoys independence, there is an independent star $\mathcal{S} \in \mathcal{C}_1 \cap \ldots \cap \mathcal{C}_k$ (resp. $\mathcal{T} \in \mathcal{D}_1 \cap \ldots \cap \mathcal{D}_l$). If the conclusion $p_A(x)$ (resp. $p_B(x)$) is not in $\mathcal{S}$ (resp. $\mathcal{T}$), then it remains independent in $\pi^\bullet$. If $\mathcal{S} = [\![ t_1, \ldots, t_m, p_A(x) ]\!]$ and $\mathcal{T} = [\![ p_B(x), u_1, \ldots, u_n ]\!]$, then $[\![ t_1, \ldots, t_m, p_{A\otimes B}(x), u_1, \ldots, u_n ]\!]$ is independent.

## 3.3 Normalisation

The main knitting of logic, the adequation usine/usage, rests upon cut-elimination, which is an analytic operation — a performance. However, in order to show its convergence, we must introduce a counterpart corresponding to the replacement of a cut with simpler ones. In this *synthetic cut-elimination*, which affects the ordeals but not the vehicle, what matters is the preservation of correctness. Since our condition now involves mosaics, we must first consider the case of a cut $[C] := [(A \otimes B) \otimes (\sim A \parr \sim B)]$: its replacement with two cuts $[A \otimes \sim A]$ and $[B \otimes \sim B]$, should preserve independence. Let us set the $\parr$ switch of $\sim A \parr \sim B$ on `L`, so that the normal form involves dendrites $\mathcal{C}_i$ which become, after replacing our cut with two cuts, dendrites $\mathcal{C}'_i$ and let $\mathcal{S} \in \mathcal{C}_1 \cap \ldots \cap \mathcal{C}_k$ be an « independent star ». If $p_C(x) \notin \mathcal{S}$, then $\mathcal{S} \in \mathcal{C}'_1 \cap \ldots \cap \mathcal{C}'_k$; if $p_C(x) \in \mathcal{S}$, then the star $\mathcal{S}'$ containing the rays $p_A(x)$ and $p_{\sim A}(x)$ is the same in all $\mathcal{C}'_i$, hence $\mathcal{S}' \in \mathcal{C}'_1 \cap \ldots \cap \mathcal{C}'_k$.

Now let us consider an additive cut $[C] := [(A \,\&\, B) \otimes (\sim A \oplus \sim B)]$; the synthetic cut-elimination replaces it with the two cuts $[A \otimes \sim A]$ and $[B \otimes \sim B]$, the difference with the multiplicative case being that, provided on selects a switching, only one of those two cuts is actually used.

A (non mosaic) ordeal for $\Gamma, [C]$ is a sum $\mathcal{S} + \mathcal{T} + \mathcal{T}'_1$ or $\mathcal{S} + \mathcal{T} + \mathcal{T}'_2$, with $\mathcal{T} := ([\![ \frac{q_{\sim A}(x)}{\boxed{q_{\sim A \oplus \sim B}(x)}} ]\!] \breve{+} [\![ \frac{q_{\sim A}(x)}{\boxed{q_{\sim A \oplus \sim B}(x)}} ]\!]) + [\![ \frac{q_{A\&B}, q_{\sim A \oplus \sim B}}{\boxed{q_C(x)}} ]\!] + [\![ \frac{q_C(x)}{p_C(x)} ]\!]$, and $\mathcal{T}'_1 := [\![ \frac{q_A(x)}{\boxed{q_{A\&B}(x)}} ]\!]$, $\mathcal{T}'_2 := [\![ \frac{q_B(x)}{\boxed{q_{A\&B}(x)}} ]\!]$. $\mathcal{T} + \mathcal{T}'_1$ and $\mathcal{T} + \mathcal{T}'_2$ admit $\mathcal{U}_1 := [\![ \frac{q_A(x), q_{\sim A}(x)}{p_C(x)} ]\!] \breve{+} [\![ \frac{q_A(x), q_{\sim B}(x)}{p_C(x)} ]\!]$ and $\mathcal{U}_2 := [\![ \frac{q_B(x), q_{\sim A}(x)}{p_C(x)} ]\!] \breve{+} [\![ \frac{q_B(x), q_{\sim B}(x)}{p_C(x)} ]\!]$ as normal forms.

Since $\boxed{\mathcal{V}} + \mathcal{S} + \mathcal{T} + \mathcal{T}'_i$ strongly normalises into $[\![ p_\Gamma(x), p_C(x) ]\!]$, $\boxed{\mathcal{V}} + \mathcal{S}$ strongly normalises into some $\mathcal{W}$ s.t. $\mathcal{W} + \mathcal{U}_i$ normalises into $[\![ p_\Gamma(x), p_C(x) ]\!]$. The part of $\mathcal{W}$ contributing to the normal form is made of at most four stars containing the various rays of $p_\Gamma(x)$ as well as $\boxed{q_A(x)}, \boxed{q_B(x)}, \boxed{q_{\sim A}(x)}, \boxed{q_{\sim B}(x)}$: stars $\sharp 1, \sharp 2, \sharp 3, \sharp 4$ respectively contain the rays $\boxed{q_A(x)}, \boxed{q_B(x)}, \boxed{q_{\sim A}(x)}, \boxed{q_{\sim B}(x)}$. Our claim is that only three stars are present, i.e., that one among $\sharp 3, \sharp 4$ is absent. To prove the claim, we introduce a mosaic, namely the position $\&_M$ of the switch. Then $\mathcal{T}'_1, \mathcal{T}'_2$ become $\mathcal{T}'_3 := ([\![ \frac{q_A(x)}{\boxed{q_{A\&B}(x)}} ]\!] \breve{+} [\![ \frac{q_B(x)}{\boxed{q_{A\&B}(x)}} ]\!]) + [\![ \frac{q_{A\&B}(x)}{\boxed{q_{A\&B}(x)}} ]\!]$ and the normal form of $\mathcal{S} + \mathcal{T} + \mathcal{T}'_3$ becomes the mosaic

$$\mathcal{U}_3 := ([\![ \frac{q_A(x)}{\boxed{q_{A\&B}(x)}} ]\!] \breve{+} [\![ \frac{q_B(x)}{\boxed{q_{A\&B}(x)}} ]\!]) + ([\![ \frac{q_{A\&B}(x), q_{\sim A}(x)}{p_C(x)} ]\!] \breve{+} [\![ \frac{q_{A\&B}(x), q_{\sim B}(x)}{p_C(x)} ]\!]).$$

The mosaic normal form of $\boxed{\mathcal{V}} + \mathcal{S} + \mathcal{T} + \mathcal{T}'_3$ contains subconstellations $\mathcal{C}_1$ and $\mathcal{C}_2$ corresponding to $\mathcal{T}'_1$ and $\mathcal{T}'_2$. Since $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$, the common star must be the one containing $p_C(x)$; this means that, in both cases the same star (among $[\![ \frac{q_{A\&B}(x), q_{\sim A}(x)}{p_C(x)} ]\!]$ and $[\![ \frac{q_{A\&B}(x), q_{\sim B}(x)}{p_C(x)} ]\!]$) has been used; this is possible only if one among $\sharp 3, \sharp 4$ is absent. What is excluded by the independence condition is a possible correlation, e.g., $\sharp 1 \smile \sharp 3$, $\sharp 2 \smile \sharp 4$.

One should now consider a general mosaic switching and prove that independence is preserved; this is straightforward.

# 4 Exponentials once more

## 4.1 Fixing a bug

First I must acknowledege a mistake in [5]: the printed version systematically used $p_A(t) \cdot x$ instead of $p_A(t \cdot x)$. Since the two are isomorphic, this mistake has little consequences; however, since $p_A(t) \cdot x$ is not an instance of $p_{A \olessthan B}(x)$, it is not compatible with $\eta$-expansion. The analytics of exponentials should be:

**Dereliction:** if the proof $\pi$ of $\vdash \Gamma, \underline{\Delta}, \underline{A}$ comes from a proof $\nu$ of $\vdash \Gamma, \underline{\Delta}, A$, then $\pi^\bullet := \nu^\bullet$ in which the terms $p_A(t)$ have been replaced with $p_A(t \cdot \mathtt{d})$, where $\mathtt{d}$ is a constant.

**Weakening:** if the proof $\pi$ of $\vdash \Gamma, \underline{\Delta}, \underline{A}$ comes from a proof $\nu$ of $\vdash \Gamma, \underline{\Delta}$, then $\pi^\bullet := \nu^\bullet$.

**Contraction:** if the proof $\pi$ of $\vdash \Gamma, \underline{\Delta}, \underline{A}$ comes from a proof $\nu$ of $\vdash \Gamma, \underline{\Delta}, \underline{A}', \underline{A}"$, then $\pi^\bullet := \nu^\bullet$ in which the terms $p_{A'}(t \cdot u)$ (resp. $p_{A"}(t \cdot u)$) have been replaced with $p_A(t \cdot (\mathtt{l} \cdot u))$ (resp. $p_A(t \cdot (\mathtt{r} \cdot u))$).

**$\ltimes$-rule:** if the proof $\pi$ of $\vdash \Gamma, \underline{\Delta}, A \ltimes B$ comes from a proof $\nu$ of $\vdash \Gamma, \underline{\Delta}, \underline{A}, B$, then $\pi^\bullet := \nu^\bullet$, with $p_A, p_B$ *defined* by $p_A(x) := p_{A \ltimes B}(\mathtt{l} \cdot x)$, $p_B(x) := p_{A \ltimes B}(\mathtt{r} \cdot x)$.

**$\olessthan$-rule:** if the proof $\pi$ of $\vdash \Gamma', \underline{\Delta}, \underline{\Delta}', A \olessthan B$ comes from proofs $\nu$ of $\vdash \underline{\Delta}, A$ and $\mu$ of $\vdash \Gamma', \underline{\Delta}', B$, we define $p_A(x) := p_{A \ltimes B}(\mathtt{l} \cdot x), p_B(x) := p_{A \olessthan B}(\mathtt{r} \cdot x)$. We modify $\nu^\bullet$ into $\nu_1^\bullet$ by replacing all $p_A(t)$ with $p_A(t \cdot y)$, with $y$ a fresh variable. Due to this variable, $\nu_1^\bullet$ is no longer a constellation; we homegeneise $\nu_1^\bullet$ into $\nu_2^\bullet$ by replacing all terms $p_C(t \cdot u)$ with $C \in \Delta$ with $p_C(t \cdot (u \cdot y))$. We define $\pi^\bullet := \nu_2^\bullet + \mu^\bullet$.

Its synthetics should be:

$\olessthan_\delta$: $$[\![ \frac{q_A(x \cdot x), q_B(x)}{\boxed{q_{A \olessthan B}(x)}} ]\!].$$

$\olessthan_\mathtt{1}$: $$[\![ \frac{q_A(x \cdot \mathtt{1}), q_B(x)}{\boxed{q_{A \olessthan B}(x)}} ]\!].$$

$\ltimes_R$: $$[\![ \frac{q_B(x)}{\boxed{q_{A \ltimes B}(x)}} ]\!] + [\![ \frac{q_A(x \cdot y)}{} ]\!] + [\![ \frac{q_A(x' \cdot y')}{} ]\!], \textit{ except if } x = x'.$$

$\ltimes_{L'}$: $$[\![ \frac{q_A(x \cdot y)}{\boxed{q_{A \ltimes B}(x \cdot y)}} ]\!].$$

Since $\ltimes_{L'}$ is *cancelling*, we may as well consider the incoherent sum $\ltimes_R \breve{+} \ltimes_{L'}$, hence the sole $\olessthan$ is switched.

## 4.2 Duplication of switches

The perfect case (multiplicatives and additives) makes use of an *external* switching: we select, for any $\parr$ and & links one of the positions `L/R` or `L/R/M` of the switch. When passing to exponentials, we meet a problem with hidden conclusions: switches are duplicated, typically $p_A(\ldots(x \cdot t_m)\ldots \cdot t_1)$ with unspecified $t_1, \ldots, t_m$. If we want to respect the opposition Object/Subject, these switches should be set without knowing the actual value of the $t_i$ (which depend upon the vehicle). Moreover, these switches should be independent: $p_{A\parr B}(x \cdot t)$ and $p_{A\parr B}(x \cdot u)$ should simultaneously assume the four choices `L/L`, `L/R`, `R/L`, `R/R`; a generic switching `L/R` of $p_{A\parr B}(x \cdot y)$ would only yield the choices `L/L` and `R/R`.

I thus propose to replace the switches `L/R` (or `L/R/M`, $\delta/1$) with non deterministic sums. In which case the normal form should be the sum of several copies of $[\![\, p_\Gamma(x)\,]\!]$, one for each switching of the full net. But this poses various problems, the most immediate of them being the number of these copies: there should be enough of them, one for each switching.

## 4.3 Probabilistic constellations

The idea is to replace constellations with a probabilistic analogue: stars now receive positive weights and a constellation takes the form of a sum $\Sigma\lambda_{\mathcal{S}} \cdot \mathcal{S}$ with the $\lambda_{\mathcal{S}} \neq 0$. When combined to form a dendrite, the various weights do multiply, yielding a weight $\prod(\lambda_{\mathcal{S}})^{m_{\mathcal{S}}}$ for the dendrite and its residual star, where $m_{\mathcal{S}}$ is the multiplicity of $\mathcal{S}$ in the dendrite.

We now replace our switches by non deterministic sums; the coefficients — rational to ensure exact computatibility — are chosen in such a way that the normal form should be of the form $\sum \lambda_i \cdot [\![\, p_\Gamma(x)\,]\!]_i$ with $\sum \lambda_i = 1$.

$\parr$ **switch:** $\parr_L := [\![\, \frac{q_A(x)}{\boxed{q_{A\parr B}(x)}} \,]\!] + [\![\, q_B(x) \,]\!]$ and $\parr_R := [\![\, \frac{q_B(x)}{\boxed{q_{A\parr B}(x)}} \,]\!] + [\![\, q_A(x) \,]\!]$

are made of two stars. Now, if $\parr_L$ is replaced with $a \cdot \parr_L$, the expected normal form $[\![\, p_\Gamma(x)\,]\!]$ becomes $a^2 \cdot [\![\, p_\Gamma(x)\,]\!]$. This is why the $\parr$ switch is replaced with the weighted sum $\frac{3}{5} \cdot \parr_L \,\breve{+}\, \frac{4}{5} \cdot \parr_R$.

& **switch:** replaced with the constellation $\frac{1}{4} \cdot [\![\, \frac{q_A(x)}{\boxed{q_{A\&B}(x)}} \,]\!] \,\breve{+}\, \frac{1}{4} \cdot [\![\, \frac{q_B(x)}{\boxed{q_{A\&B}(x)}} \,]\!] \,\breve{+}$ $((\frac{1}{4} \cdot [\![\, \frac{q_A(x)}{\boxed{q_{A\&B}(x)}} \,]\!] \,\breve{+}\, \frac{1}{4} \cdot [\![\, \frac{q_B(x)}{\boxed{q_{A\&B}(x)}} \,]\!]) + \frac{1}{2} \cdot [\![\, \frac{q_{A\&B}(x)}{\boxed{q_{A\&B}(x)}} \,]\!])$.

! **switch:** replaced with the non deterministic choice $\frac{1}{2} \cdot !_\delta \,\breve{+}\, \frac{1}{2} \cdot !_1$.

As to correctness, we require that the normal form should be sum of weighted copies $\sum_{i \in I} \lambda_i [\![\, p_\Gamma(x)\,]\!]_i$ with total weight $\sum_{i \in I} \lambda_i = 1$. Moreover, we require independence for any non empty $J \subset I$ s.t. the $[\![\, p_\Gamma(x)\,]\!]_j$ are consistently switched: if $[\![\, p_\Gamma(x)\,]\!]_j$ and $[\![\, p_\Gamma(x)\,]\!]_k$ both involve a choice `L/R`, `L/R/M`, $\delta/1$ for the same occurrence of a switch, the choices are the same.

# 5 Open questions

## 5.1 Analytics

It seems that we reached a stable solution for a non deterministic analytics. However, since the resut is new, minor improvement can be expected.

**Church-Rosser** remains slightly awkward. Is there a neighbouring formulation of colour-elimination which makes it completely satisfactory?

**Hypercoherences:** the coherence relation is basically unary and binary. One may need a more general notion allowing pairwise coherent stars to be globally incoherent; in case one should modify the definition and allow multiple incoherences. One should of course pay attention to keep everything finite.

## 5.2 Synthetics

We were basically concerned with the right opposition Object/Subject enabling to to accommodate additives. This shoud serve as a basis for the construction and study of (box-free) proof-nets for full propositional calculus ($\otimes, \parr, \&, \oplus, \otimes, \ltimes$).

When writing this paper, I had the hope of defining general, non sequential, perfect (i.e., multiplicative/additive) connectives. I didn't succeed, presumably because additive correctness is not yet familiar. I enclose in annex the solution in the multiplicative case, which can serve as a model for the general perfect case and is anyway needed to define « first order individuals » [6].

# A Multiplicatives

## A.1 The calculus of partitions

Multiplicatives can be approached through a *calculus of partitions*, whose lineaments are due to Danos & Regnier [1]. Let $n > 0$; a $n$-ary (multiplicative) connective is a set $\mathcal{C}$ of partitions of $\{1, \ldots, n\}$ subject to certain constraints.

If $E, F$ are two partitions of $\{1, \ldots, n\}$, consider the following bipartite graph: its vertices are the elements of the disjoint sum $E + F$ and its edges $i = 1, \ldots, n$ link $\mathbf{e} \in E$ and $\mathbf{f} \in F$ exactly when $i \in \mathbf{e} \cap \mathbf{f}$. $E, F$ are *orthogonal*, $E \perp F$, when the bipartite graph is a topological tree, i.e., is connected and acyclic. If $p, q$ are the respective cardinalities of $E, F$, the Euler-Poincaré invariant is $p + q - n = 1$.

A n-ary *connective* is a set $\mathcal{C}$ of partitions of $\{1, \ldots, n\}$ equal to its biorthogonal; it must be non-trivial, i.e., neither empty nor full. In which case its orthogonal (or negation) $\sim\mathcal{C}$ is in turn a n-ary connective. A n-ary connective receives a *weight*, namely the common cardinality of all its partitions; if $\mathcal{C}$ has weight $w$, then $\sim\mathcal{C}$ has weight $n + 1 - w$. The typical example is the binary connective $\parr$ consisting of the sole partition $\{\{1\}, \{2\}\}$ whose orthogonal, $\otimes$ consists of the sole partition $\{\{1, 2\}\}$.

All connectives cannot be constructed from those two, typically ¶ (4-ary) which consists of $\{\{1,2\},\{3,4\}\}$ and $\{\{2,3\},\{4,1\}\}$; its negation $\sim$¶ consists of $\{\{1,3\},\{2\},\{4\}\}$ and $\{\{2,4\},\{1\},\{3\}\}$. The connectives ¶, $\sim$¶, for which proof-nets can be constructed, cannot be expressed in sequent calculus: they are not sequential, i.e., desperately non-classical in some sense. However, since they enjoy cut-elimination, they are as good as the others,... provided one finds a proper use for them.

$n$-ary connectives can be compared: $\mathcal{C} \subset \mathcal{D}$ means that there is a bijection $\varphi$ of $\{1,\ldots,n\}$ s.t. $\{\{1,\varphi(1)\},\ldots,\{n,\varphi(n)\}\} \perp \mathcal{C} \parr \sim\mathcal{D}$. In logical terms, $\mathcal{C} \subset \mathcal{D}$ corresponds to the implication $\mathcal{D}[x_{\varphi(1)},\ldots,x_{\varphi(n)}] \multimap \mathcal{C}[x_1,\ldots,x_n]$; the implication is on the « wrong » sense because our approach is not in terms of proofs, but in terms of switchings. If $\mathcal{C} \subset \mathcal{D}$, their weights $w, w'$ are such that $n + w + n + 1 - w' - 2n = 1$, hence $w = w'$.

Indeed, if $E \in \mathcal{C}$, then $\varphi(E) \in \mathcal{D}$: another numerical invariant, the number of partitions, a.k.a. *size*, of $\mathcal{C}$ is of interest: if $\mathcal{C} \subset \mathcal{D}$, then $\mathcal{D}$ has a greater size. A typical example of an inclusion is that (with $\varphi(i) := i$) between $(1 \otimes 2) \parr 3$ and $1 \otimes (2 \parr 3)$; the sizes are respectively 1 ($\{\{1,2\},\{3\}\}$) and 2 ($\{\{1,2\},\{3\}\}$ and $\{\{1,3\},\{2\}\}$).

If $\mathcal{C} \subset \mathcal{D}$ and $\mathcal{D} \subset \mathcal{C}$, then $\mathcal{C}$ and $\mathcal{D}$ have the same size. Then the bijection $\varphi$ which « proves » the inclusion $\mathcal{C} \subset \mathcal{D}$ is indeed an isomorphism.

## A.2 Consistency

We have not yet justified the expression « connective ». A connective must combine propositions, which poses the problem of atoms. We could of course use propositional variables, but they belong with the theory of quantification, hence are not quite propositional, even if we often ignore this detail in practice. Moreover, the usual multiplicative or additive constants are not quite first order, hence we apparently have no solid ground on which to apply our connectives.

Unless we consider the connectives themselves as logical constants: the proofs of $\mathcal{C}$ are exactly the elements of $\sim\mathcal{C}$. Typically, an inclusion $\mathcal{C} \subset \mathcal{D}$ corresponds to a certain proof $\varphi$ of $\mathcal{C} \parr \sim\mathcal{D}$, i.e., of $\mathcal{D} \multimap \mathcal{C}$.

This introduction of new constants poses a logical problem: since $\sim\mathcal{C}$ is never empty, the constant $\mathcal{C}$ admits a proof, hence the apparent failure of consistency. This logical issue is a minor one compared to — say — normalisation; but it should anyway be addressed. The idea is that not all partitions should be considered as proofs.

We therefore introduce the following requirement: a *proof* of $\mathcal{C}$ of consists in a partition $\{\mathbf{a}_1,\ldots,\mathbf{a}_k\} \perp \mathcal{C}$ with the $\mathbf{a}_i$ of cardinality 2. Therefore, if $\mathcal{C}$ is of odd arity, it cannot be provable.

We get indeed a litte more than consistency, namely the fact that $\mathcal{C}$ and $\sim\mathcal{C}$ cannot be both provable. This is due to the fact that two proofs $\{\mathbf{a}_1,\ldots,\mathbf{a}_k\}$ and $\{\mathbf{b}_1,\ldots,\mathbf{b}_l\}$ cannot be orthogonal: we would get $k = l = \frac{n}{2}$ in which case the Euler-Poincaré invariant is $k + k - n = 0$, hence $\neq 1$.

## A.3 Composition

Connectives are intended to be applied to propositions, which basically amounts at composing them. The general idea is to define $\mathcal{C}[\mathcal{D}_1, \ldots, \mathcal{D}_n]$, with a predictable mess at the level of the indexing. The simplest solution consists in a change of indexing set: it is no longer $\{1, \ldots, n\}$, but a non empty set $I$ enjoying the following property:

$$i, j \in I \Rightarrow i \cap j = \emptyset$$

We say that $\mathcal{C}$ is of arity $I$; we define $|I| := \bigcup I$.

In order to perform composition, we must introduce rooted partitions. A partition $\{\mathbf{a}_1, \ldots, \mathbf{a}_k\}$ of $I$ can be *rooted* by selecting one its elements — say $\mathbf{a}_k$ for notational convenience — and replace it with $\{\mathbf{a}_1, \ldots, \mathbf{a}_k \cup \{|I|\}\}$. It thus becomes a partition of $I \cup \{|I|\}$ in which the element $\{|I|\}$ does not stand alone. There are $k$ different ways to root $\{\mathbf{a}_1, \ldots, \mathbf{a}_k\}$. Incidentally, we understand what is wrong with the 0-ary case: it cannot be rooted.

Partitions (rooted or not) can be seen as sort of constellations. Stars are made of rays $i \in I$, $|I|$ which are coloured in two different ways, typically the $i \in I$ in `yellow` and $|I|$ `uncoloured`. In that case, the rooted partition could be written $[\![ \frac{\mathbf{a}_1}{} ]\!] + \ldots + [\![ \frac{\mathbf{a}_k}{|I|} ]\!]$.

Now, if the connectives $\mathcal{D}_i$ ($i \in I$) are of arities $J_i$ with $|J_i| = i$, then $\mathcal{C}[\ldots \mathcal{D}_i \ldots]$ will be a connective of arity $J := \bigcup_{i \in I} J_i$; observe that $|J| = |I|$. Take rooted partitions $E, F_i$ in $\mathcal{C}$ and the $\mathcal{D}_i$. Paint $E$ in `magenta` and `uncoloured`, the $F_i$ in `yellow` and `green` and eliminate the colours `green`/`magenta` from $E + \sum F_i$. The result is a constellation in `yellow` and `uncoloured` which can be seen as a rooted partition of $J$. $\mathcal{C}[\ldots \mathcal{D}_i \ldots]$ is defined, not quite as the set of the rooted partitions obtained in this way, but as its biorthogonal.

Orthogonality between partitions $E, F$ of the same arity, one of them rooted, is defined by using complementary colours, say `green` for $i \in I$ in $E$, `magenta` for for $i \in I$ in $E$ and non complementary colours for the root $|I|$ — if any —, say `uncoloured` vs. `blue`. Orthogonality requires $E + F$ to normalise into the star made of the respective roots; this is indeed equivalent to the orthogonality of the underlying unrooted partitions.

## A.4 A fundamental result

The main result concerning composition is that connectives are closed under composition. In other terms[6], the equation

$$\mathcal{C}[\ldots \mathcal{D}_i \ldots] = \mathcal{C}^{\perp}[\ldots, \mathcal{D}_i^{\perp} \ldots]^{\perp\perp} \tag{1}$$

Indeed, connectives are hardly accessed through their complete set of partitions, but only through a sort of dense subset, some $\mathcal{C}_0$ such that $\mathcal{C} = \mathcal{C}_0^{\perp\perp}$. A typical example is given by the switchings of a proof-net which do not quite

[6] For questions of legibility, I use the alternative notation $\mathcal{C}^{\perp}$ for negation.

yield all tests corresponding to connective $\mathcal{C}$, but only a dense subset $\mathcal{C}_0$. We can indeed see equation (1) as the abstract form of sequentialisation, namely that passing the tests is enough for logical correctness. Forgetting sequentialisation, too tied to sequential connectives, equation (1) still means logical correctness, indeed the adequation of l'usine ($\mathcal{C}_0$) w.r.t. l'usage ($\mathcal{C}$).

In order to prove (1), we shall use proof-nets. More precisely, the atoms $j \in J$ will be duplicated in either $j'$ or $j''$ depending they occur in the $\mathcal{D}_i$ or in the $\mathcal{D}_1^{\perp}$; ditto for the roots, $|J'_i|$ and $|J''_i|$, $|J'| = |I'|$, $|J''| = |I''|$.

**$\eta$-expansion** We want to prove the orthogonality relation

$$\mathcal{C}[\ldots \mathcal{D}_i \ldots] \perp \mathcal{C}^{\perp}[\ldots \mathcal{D}_i^{\perp} \ldots] \tag{2}$$

from the orthogonalities $\mathcal{C} \perp \mathcal{C}^{\perp}$ and $\mathcal{D}_i \perp \mathcal{D}_i^{\perp}$. For this, we draw a proof-net in five colours, the $\eta$-expanded identity link between $\mathcal{C}[\ldots \mathcal{D}_i \ldots]$ and $\mathcal{C}^{\perp}[\ldots \mathcal{D}_i^{\perp} \ldots]$:

`blue`: the $[\![ \boxed{j'}, \boxed{j''} ]\!]$ $(j \in J)$.

`yellow`/`green`: the $\dfrac{\ldots j' \ldots}{\boxed{i'}}$ $(j \in i)$ and $\dfrac{\ldots j'' \ldots}{\boxed{i''}}$ $(j \in i)$ for $i \in I$.

`magenta`/`uncoloured`: $\dfrac{\ldots i' \ldots}{|I'|}$ $(i \in I)$ and $\dfrac{\ldots i'' \ldots}{|I''|}$ $(i \in I)$.

The proof-net can be switched by selecting rooted partitions in $\mathcal{D}_i$ or $\mathcal{C}$ for each link written. Now, if we forget the lower part in `magenta`/`uncoloured`, we have in fact several proof-nets with two conclusions $\boxed{i'}$ and $\boxed{i''}$. These proof-nets are correct, i.e., normalise into $[\![ \boxed{i'}, \boxed{i''} ]\!]$ whatever switchings we choose for $\dfrac{\ldots j' \ldots}{\boxed{i'}}$ and $\dfrac{\ldots j'' \ldots}{\boxed{i''}}$. Now, the $[\![ \boxed{i'}, \boxed{i''} ]\!]$ stashed above the $\dfrac{\ldots i' \ldots}{|I'|}$ and $\dfrac{\ldots i'' \ldots}{|I''|}$ yields a correct proof-net, hence the result.

**Cut-elimination** We want to prove the orthogonality relation

$$\mathcal{C}[\mathcal{D}_1, \ldots, \mathcal{D}_n]^{\perp} \perp \mathcal{C}^{\perp}[\mathcal{D}_1^{\perp}, \ldots, \mathcal{D}_n^{\perp}]^{\perp} \tag{3}$$

from the orthogonalities $\mathcal{C}^{\perp} \perp \mathcal{C}^{\perp\perp}$ and $\mathcal{D}_i^{\perp} \perp \mathcal{D}_i^{\perp\perp}$. Now take partitions $E' \perp \mathcal{C}[\mathcal{D}_1, \ldots, \mathcal{D}_n]$ and $F'' \perp \mathcal{C}^{\perp}[\mathcal{D}_1^{\perp}, \ldots, \mathcal{D}_n^{\perp}]$; we want to prove that $E \perp F$.

For this, we draw again a proof-net in five colours, the cut between the proofs $E$ and $F$ of $\mathcal{C}[\ldots \mathcal{D}_i \ldots]$ and $\mathcal{C}^{\perp}[\ldots \mathcal{D}_i^{\perp} \ldots]$:

`blue`: $\boxed{E'}$ and $\boxed{F''}$.

`yellow`/`green`: the $\dfrac{\ldots j' \ldots}{\boxed{i'}}$ $(j \in i)$ and $\dfrac{\ldots j'' \ldots}{\boxed{i''}}$ $(j \in i)$ for $i \in I$.

`magenta`/`green`: $\dfrac{\ldots i' \ldots}{\boxed{|I'|}}$ $(i \in I)$ and $\dfrac{\ldots i'' \ldots}{\boxed{|I''|}}$ $(i \in I)$.

`magenta`/`uncoloured`: $\frac{|I'|,|I'|}{[|I'| \otimes |I'|]}$.

This proof-net is correct by assumption. Forget the lower part in `magenta`/`green`/`uncoloured` and switch the upper part. The normal form yields two partitions in green of $I'$ and $I''$ which respectively belong to $\mathcal{C}^{\perp\perp}$ and $\mathcal{C}^{\perp}$ and are thus orthogonal. This means that the lower part can be replaced with cuts $\frac{i',i''}{[i' \otimes i'']}$ and remain correct.

We are still not done, since we must replace these cuts with cuts on the atoms. If there were only one of them, say $\frac{i',i''}{[i' \otimes i'']}$, then we would observe that $E'$ and $F''$ are respectively orthogonal to $\mathcal{D}_i^{\perp\perp}$ and $\mathcal{D}_i^{\perp}$ and proceed as above, thus replacing our cut with cuts on its atoms while preserving correctness. In general, we can switch all links distinct from $\frac{\ldots\ j'\ \ldots}{i'}$ and $\frac{\ldots\ j''\ \ldots}{i''}$ so as to reduce to the situation with a single cut.

However, this process of concentrating on a single cut $\frac{i',i''}{[i' \otimes i'']}$ which involves a switching of links and a normalisation changes the original $E' \cup F''$ into something else. It cannot be *a priori* excluded that the normalised partition connects some $j'$ with some $k''$, in which case our argument would fail. But, since the switchings of $\mathcal{D}_i$ involve the choice of a root, we can attach it to the element of the partition containing $j$; ditto with $\mathcal{D}_i^{\perp}$ and $k$: correctness then forbids $j'$ and $k''$ to be in the same element of the partition.

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## References


[1] V. Danos and L. Regnier. **The structure of multiplicatives**. *Archive for Mathematical Logic*, 28:181 – 203, 1989.

[2] L. Tortora de Falco. **The additive multiboxes**. *Annals of Pure and Applied Logic*, 120:65 – 102, 2003.

[3] J.-Y. Girard. **Quantifiers in linear logic**. In dalla Chiara and Galavotti, editors, *Temi e prospettive della logica e della filosofia della scienza contemporanee*, pages 95 – 130, Bologna, 1989. CLUEB.

[4] J.-Y. Girard. **Proof-nets : the parallel syntax for proof-theory**. In Ursini and Agliano, editors, *Logic and Algebra*, New York, 1996. Marcel Dekker.

[5] J.-Y. Girard. **Transcendental syntax I: deterministic case**. *Mathematical Structures in Computer Science*, pages 1–23, 2015. *Computing with lambda-terms. A special issue dedicated to Corrado Böhm for his 90th birthday.*

[6] J.-Y. Girard. **Transcendental syntax III: equality**. *Logical Methods in Computer Science*, 2016. *Special issue dedicated to Pierre-Louis Curien for his 60th birthday.*

[7] D.J.D Hughes and R.D. van Glabbeek. **Proof nets for unit-free multiplicative-additive linear logic (Extended abstract).** In *18th IEEE Symposium on Logic in Computer Science (LICS 2003)*, pages 1 – 10. IEEE Computer Society, 2003.