Truth, modality and intersubjectivity

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Quantum physics together with the experimental (and slightly controversial) *quantum computing*, induces a twist in our vision of computation, thence — since computing and logic are intimately linked — in our approach to logic and foundations. In this paper, we shall discuss the most mistreated notion of logic, *truth*.

1 Introduction

1.1 Revisiting foundations

Is there something more frozen than « foundations »? A quick glance at the list « foundations of mathematics » :

`http ://www.cs.nyu.edu/mailman/listinfo/fom`

shows a paradigm close to archaic astronomy : truth is a primitive (like Earth), around which several systems and meta-systems gravitate (like the epicycles of Ptolemy). This being orchestrated by Doctors of the Law, in charge of the latest developments of Hilbert's program, i.e., of a certain form of *finitism* obsolete since Gödel's theorem (1931 !), but still in honour in this sort of *Jurassic Park*.

Let us put it bluntly : these people confuse foundations with *prejudices*. Of course, it cannot be excluded that the deep layers behave accordingly to our preconceptions ; but who thinks in that way should draw the conclusions and quit. My personal bias, the one followed in this paper, is that the real *hypostases* are very different from our familiar (mis)conceptions : I shall thence propose a *disturbing* approach to foundations. This viewpoint is by no means « non standard », it is on the contrary most standard ; but it relies on ideas developed in the last century and prompted by quantum physics, the claim being that *operator algebra is more primitive than set theory.*

1.2 Sets vs. operators

In terms of foundations, the most impressive achievement of the turn of the century is to be found outside logic — not to speak of the the aforementioned *Jurassic Park* — : in the *non commutative geometry* of Connes [1], a paradigm violently anti-set-theoretic, based upon the familiar result :

A commutative operator algebra is a function space.

Typically, a commutative C^*-algebra can be written $\mathbb{C}(X)$, the algebra of continuous functions on the compact X. Connes proposes to consider non commutative operator algebras as sorts of algebras of functions over… non existing sets, an impressive blow against set-theoretic essentialism !

Logic is *a priori* far astray from considerations internal to geometry ; but this changes our ideas of finite set, of point, of graph, etc.

- The commutative, set-theoretic, world appears as a vector space equipped with a distinguished base. All operations are organised in relation to this base, in particular they can be represented by linear functions whose matrices are diagonal in this base.
- The non-commutative world forgets the base ; there is still one, but it is *subjective*, the one where one diagonalises the hermitian operator one uses : « his » set-theory, so to speak. But, if two hermitians f and g have non commuting « set-theories », $f+g$ has a third set-theory bearing no relation to the previous.

Roughly speaking, the base is on the side of *particles* ; while an operator is *wavelike*. If the latter is objective, the former, which corresponds to set-theory, is subjective.

1.3 The three layers

In [5], I introduced three foundational layers, **-1**, **-2** and **-3**. This has nothing to do with playing with iterated metas[1] ; computationally speaking, the distinction can easily be explained on an example :

[1]A system rests on a meta-system which in turn rests on a meta-meta-system… *Turtles all the way down*, like in a famous joke ! The *Jurassic Park*, conscious of the problem, added one more turtle at the bottom, and so on. Transfinite meta-turtles, « predicative » or not, does this make convincing foundations ?

-1 : the function φ sends integers ($\mathbb{N}$) to booleans ($\mathbb{B}$), what is traditionally expressed through the implication $\mathbb{N} \Rightarrow \mathbb{B}$.

-2 : $\varphi(n) = T$ is n is prime, $\varphi(n) = F$ otherwise.

-3 : φ implements the sieve of Eratosthenes.

Level **-1** deals with inputs/outputs ; logically speaking, it corresponds to truth, logical consequence and satellites such as consistency. Level **-2** considers proofs as functions and, more generally, as *morphisms* in an appropriate category. Finally, level **-3** deals with the dynamics, i.e., with the *procedurality* of logical operations.

The central result of proof-theory, *cut-elimination*, reads as follows in the three layers :

-1 : the absurd sequent not being cut-free provable, is not provable at all, thence consistency.

-2 : the Church-Rosser property (natural deduction, proof-nets) induces the *compositionality* of proofs, i.e., the existence of an underlying category.

-3 : the cut-elimination process can be expressed as the solution of a linear equation on the Hilbert space, the *feedback equation* (19) below.

Historically speaking, layer **-1** comes from the foundational discussion of *classical* logic ; the view of proofs as functions (layer **-2**) must be ascribed to *intuitionism* ; finally, the paradigm of proofs as actions (layer **-3**) is well adapted to *linear* logic. Quantum computing admits an interpretation of level **-2** (QCS below), but its spirit is mostly of level **-3**.

1.4 A failure : quantum logic

According to Heredotus, Xerxes had the sea *beatten* for misbeahaviour ; *quantum logic* is, in its way, a punishment inflicted upon nature for making « mistakes » of logic.

According to quantum logic, everything should stay the same, but the truth values ; by the way, the idea that logic should be defined in terms of truth values, i.e., at level **-1**, is spurious : such an assumption makes the departure from classical logic difficult, nay impossible. The boolean algebra $\{T, F\}$ is therefore replaced with the structure consisting of the closed subspaces of a given Hilbert space. Unfortunately, these subspaces badly socialise : any reasonable operation requires the *commutation* of the associated orthoprojections ; typically, the intersection, which is easily defined as the product $\pi\pi'$ of the associated projections in case of commutation, has no manageable definition ortherwise. There are two ways of fixing this fundamental mismatch :

1. Either abstract everything, forget the Hilbert space : this leads to « orthomodular lattices », i.e., nowhere.
2. Or replace subspaces with their orthoprojections and close them under real linear combinations : this leads to hermitians and, eventually, at forgetting the logical nonsense about truth values. The second way was the one followed by von Neumann, who had the bad taste of introducing quantum logic, but who soon corrected his mistake by the creation of what we now call *von Neumann algebras.*

For instance, the « set-theories » of the hermitians $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ correspond to the bases $\{\vec{X}, \vec{Y}\}$ and $\{\sqrt{2}/2(\vec{X}+\vec{Y}), \sqrt{2}/2(\vec{X}-\vec{Y})\}$, but the « set-theory » of their sum $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$ does not belong in lattice theory, since it involves solving the algebraic equation $\lambda^2 - 2\lambda + 1/2 = 0$. In other terms, the order structure of subspaces does not socialise with the basic quantum operation, *superposition.* This explain the failure of approach (i) and its replacement with (ii).

This replacement supposes to relinquish the logical viewpoint ; is it therefore possible to establish a link between logic and quantum ?

1.5 Logic vs. quantum

Beyond any doubt, a relation should be established. Unfortunately, this vague question became : *find a logical explanation of quantum phenomenons...* which eventually lead to quantum « logic ». In the same way, the vague question of the relation of Earth and planets was formulated as : *find a geocentric explanation of celestial machanics* ; this program was pursed during endless centuries and led to the notorious Ptolemy's epicycles, another punishment inflicted upon nature, guilty of not following Joshua's Book.

In other terms, what is so good in logic that quantum physics should obey ? Can't we imagine that our conceptions about logic are wrong, so wrong that they are unable to cope with the quantum miracle ? Indeed, the « logical » treatment of the quantum world rests upon the prejudice that the usual operator-theoretic approach is wrong ; logicians are happy toying with their own counter-explanations of the quantum phenomenons. In particular, they seem to believe in *hidden variables*, i.e., in a thermodynamic explanation of quantum mechanics : otherwise, how to explain the attempts at exhumating the corpse of Gleason's theorem ?

Instead of teaching logic to nature, it is more reasonable to learn from her. Instead of interpreting quantum into logic, we shall interpret logic into quantum. This basically involves operator algebras, the difficult part being to find the correct way of doing so : we shall go beyond level **-1** (truth values), first to level **-2** (functions, morphisms, categories), with *quantum coherent spaces*. There we shall meet a problem with infinite dimension : what will eventually force us to move at layer **-3**.

2 Quantum coherent spaces

2.1 QCS

If $\mathbf{H}$ is a complex Hilbert space of finite dimension n, then $\mathcal{L}(\mathbf{H})$, the space of endormorphisms of $\mathbf{H}$ has (complex) dimension n^2, thence real dimension $2n^2$. Every $u \in \mathcal{L}(\mathbf{H})$ uniquely writes $u = h + ik$, with h, k *hermitian* ; it follows that the real vector space $\mathcal{H}(\mathbf{H})$ of *hermitians* has dimension n^2. This space is indeed *euclidian*, i.e., a real Hilbert space, when endowed with the bilinear form $\langle h \mid k \rangle := \mathrm{tr}(h \cdot k)$.

DEFINITION 1
Two hermitians $h, k \in \mathcal{H}(\mathbf{H})$ are polar *iff :*

$$x \overset{\downarrow}{\sim} y \quad :\Leftrightarrow \quad 0 \leq \mathrm{tr}(h \cdot k) \leq 1 \tag{1}$$

Given $\mathfrak{A} \subset \mathcal{H}(\mathbf{H})$, its polar *$\sim\mathfrak{A}$ is defined as :*

$$\sim\mathfrak{A} \quad := \quad \{k; \forall h \in \mathfrak{A} \quad 0 \leq \mathrm{tr}(h \cdot k) \leq 1\} \tag{2}$$

A quantum coherent space *(QCS) of* carrier $\mathbf{H}$ *is a subset $\mathfrak{X} \subset \mathcal{H}(\mathbf{H})$ equal to its bipolar.*

QCS yield a natural — categorical, i.e., of level **-2** — interpretation for linear logic, and also a reasonable candidate for the idea of a « typed quantum algorithm ». This has already been developed in [3, 6]. Let us just mention a point : linear logic admits a connective named « tensor » and written $\otimes$. This connective does not meet the idea of *intrication* at work in quantum physics and computing, but this does not mean that QCS (thus, linear logic) are inadapted to quantum computing. Indeed, linear logic (and QCS) have *two* tensors, the other one being called « par » and noted $\parr$: this *cotensor* actually deals with intrication. The confusion comes from the fact that, algebraically speaking, tensor and cotensor coincide in finite dimension ; therefore, this apparent mismatch is a pure question of terminology.

2.2 The adjunction

The milestone in the relation between QCS and linear logic is the interpretation of linear implication :

THEOREM 1
There is a canonical isomorphism between the set of all linear *maps from the QCS $\mathfrak{X}$ to the QCS $\mathfrak{Y}$ and a QCS $\mathfrak{X} \multimap \mathfrak{Y}$.*

If $\mathfrak{X}, \mathfrak{Y}$ are of respective carriers $\mathbf{H}, \mathbf{K}$, then $\mathfrak{X} \multimap \mathfrak{Y}$ is of carrier $\mathbf{H} \otimes \mathbf{K}$ and is defined as :

$$\mathfrak{X} \multimap \mathfrak{Y} \quad := \quad \sim\{h \otimes k; h \in \mathfrak{X}, k \in \sim\mathfrak{Y}\} \tag{3}$$

Given a linear map Φ from $\mathcal{H}(\mathbf{H})$ to $\mathcal{H}(\mathbf{K})$ sending $\mathfrak{X}$ into $\mathfrak{Y}$, one defines its *skeleton* $\mathrm{sk}(\Phi) \in \mathcal{H}(\mathbf{H} \otimes \mathbf{K})$ by :

$$\langle \mathrm{sk}(\Phi)(x \otimes y) \mid w \otimes z \rangle \quad := \quad \langle (\Phi \cdot x w^*)(y) \mid z \rangle \qquad (x, w \in \mathbf{H} \quad y, z \in \mathbf{K}) \tag{4}$$

with $(xw^*)(y) := \langle y \mid w \rangle \cdot x$. By linearity :

$$\mathrm{tr}(\mathrm{sk}(\Phi) \cdot (h \otimes yz^*)) \quad := \quad \langle (\Phi(h))(y) \mid z \rangle \qquad (h \in \mathcal{H}(\mathbf{H}) \quad y, z \in \mathbf{K}) \tag{5}$$

which shows how to recover Φ from its skeleton.

Everything rests upon the « application » $\varphi[h]$ of the skeleton $\varphi := \mathrm{sk}(\Phi)$ to $h \in \mathfrak{X}$, which is characterised by the adjunction :

$$\mathrm{tr}(\varphi[h] \cdot k) = \mathrm{tr}(\varphi \cdot (h \otimes k)) \qquad (h \in \mathcal{H}(\mathbf{H}) \quad k \in \mathcal{H}(\mathbf{K})) \tag{6}$$

For instance, the *twist* $\sigma \in \mathcal{H}(\mathbf{H} \otimes \mathbf{H})$, defined by :

$$\sigma(x \otimes y) \quad := \quad y \otimes x \tag{7}$$

is such that :

$$\mathrm{tr}(h \cdot k) = \mathrm{tr}(\sigma \cdot (h \otimes k)) \qquad (h \in \mathcal{H}(\mathbf{H}) \quad k \in \mathcal{H}(\mathbf{K})) \tag{8}$$

thence $\sigma[h] = h$, which shows that it is the skeleton of the identity map.

2.3 Coherent spaces

QCS are derived from the original[2] interpretation of linear logic, *coherent spaces* [5]. Roughly speaking, they appear as a *subjective* version of QCS, obtained by focusing on a particular *base* of the carrier.

[2]In a strong sense : linear logic is issued from coherent spaces.

A finite dimensional Hilbert space $\mathbf{H}$ can, given a base X, uniquely be written as $\mathbb{C}^X$; moreover, if we restrict our attention to subsets $a, b \ldots$ of the base, then the associated subspaces are represented by projections $\pi_a, \pi_b \ldots$ whose matrices are diagonal with entries equal to $0, 1$. Observe that :

$$\mathrm{tr}(\pi_a \pi_b) = \mathrm{tr}(\pi_{ab}) = \sharp(a \cap b) \qquad (a, b \subset X) \tag{9}$$

thence $\pi_a \underset{\sim}{\downarrow} \pi_b$ iff $\sharp(a \cap b) \leq 1$.

DEFINITION 2

A coherent space *with carrier X is a set $\mathfrak{X}$ of subsets of X equal to its bipolar, the polarity between subsets of X being defined as :*

$$a \underset{\sim}{\downarrow} b \quad \Leftrightarrow \quad \sharp(a \cap b) \leq 1 \tag{10}$$

and the adjunction (6) becomes :

$$\sharp(\varphi[a] \cap b) = \sharp(\varphi \cap (a \times b)) \qquad (a \in \mathfrak{X} \quad b \in \sim\mathfrak{Y}) \tag{11}$$

Define the binary relation $\frown\!\!\!\!\!\smile_{\mathfrak{X}}$ on the carrier X by :

$$x \frown\!\!\!\!\!\smile_{\mathfrak{X}} y \quad \Leftrightarrow \quad \{x, y\} \in \mathfrak{X} \tag{12}$$

THEOREM 2

$a \subset X$ belongs to $\mathfrak{X}$ iff a is a clique *w.r.t. $\frown\!\!\!\!\!\smile_{\mathfrak{X}}$:*

$$a \in \mathfrak{X} \quad \Leftrightarrow \quad \forall x, y \in a \; x \frown\!\!\!\!\!\smile_{\mathfrak{X}} y \tag{13}$$

which corresponds to the « official » definition of coherent spaces.

The connectives $(\otimes, \parr, \&, \oplus)$ are, since logical, subjective. This is why they naturally involve the building — more generally, the maintenance — of distinguished bases. This explains why the objective, « wavelike », artifacts (QCS) remained so long invisible : they were indeed prompted by quantum computing. This means that every QCS naturally comes with its distinguished base, and that all logical constructions lead from cliques (i.e., diagonal matrices with entries $0, 1$) to cliques. There is, however, a remarkable exception, namely the *identity axiom* $A \multimap A$; if A stands for the QCS $\mathfrak{X}$, this axiom is interpreted by the *twist* (7), which is unitary, hence by no means a projection. This interpretation therefore differs from the original one, formulated in terms of coherent spaces, for which the identity axiom is the diagonal of $X \times X$, i.e., corresponds to the projection obtained from the twist by chopping off all non-diagonal coefficients. Typically, if $\mathfrak{X}$ is of dimension 2, then the twist and the « diagonal » respectively write as :

$$\sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \delta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}$$

The distinction between σ and δ must be related to the following issues (see [3, 6]) :

- Eta-conversion : δ is an « eta-expansion » of σ.
- Quantum measurement : δ is the result of the *reduction of the wave packet* applied to σ, or rather, the deterministic process of *preselection* associated to this non-determinitic operation.

Putting together a rather obscure logical technicality, which yielded but an afflictive litterature, and one of the major discoveries of last century is unexpected. It shows that logicians completely neglected what should have been their main interest : *the relation between object and subject.*

3 Perennialisation

3.1 Perfect vs. imperfect

The major discovery of *linear logic* is the distinction between *perfect* and *imperfect*, see [5]. As in the usual language, imperfection corresponds to repetition, i.e., to *perenniality* and, eventually, to infinity. A specific connective, $!A$, the *exponential* — together with its dual $?A$ —, is in charge of perennialisation, which is expressed through various rules, typically *contraction* :

$$!A \multimap !A \otimes !A \tag{15}$$

which is the primal form of perenniality.

Technically speaking, the *perfect*, non-perennial, world is linear ; what is expressed by the *linear* implication $A \multimap B$. On the other hand, the imperfect world relies on usual (intuitionistic) implication $A \Rightarrow B$, which is not linear : it can be translated as $!A \multimap B$, which means that it allows constant, quadratic, polynomial dependencies. In *coherent Banach spaces* (CBS) — a level **-2** interpretation of linear logic [6] — formulas are interpreted by Banach spaces and $!A \Rightarrow B$ is inhabited by analytic functions from the open ball $A_{<1}$ to the closed ball $B_{\leq 1}$.

3.2 Obstacles to perennialisation

Coherent spaces admit a natural exponentiation : if $\mathfrak{X}$ is a coherent space, then the carrier of $!\mathfrak{X}$ consists of all *finite* cliques of $\mathfrak{X}$ and :

$$a \frown_{!\mathfrak{X}} b \quad \Leftrightarrow \quad a \cup b \in \mathfrak{X} \tag{16}$$

Long before coherent spaces, *Scott domains* yielded a topological interpretation of the imperfect implication $A \Rightarrow B$ by means of continuous maps from A into B.

Both interpretations have their limitations ; for instance, Scott domains are a sort of childish topology in which separately continuous are *ipso facto* continuous. Moving to the real thing, typically to *coherent Banach spaces*, poses problems : we noticed that implication corresponds to analytical maps from an *open* to a *closed* ball ; such maps do not compose. This indicates that there is a logical « mistake » as to continuity in the rules for perenniality. In such a situation, one is faced with a dilemma :

– Either change the principles of topology and restrict to a castrated version of continuity, typically to Scott domains. This is the dominant viewpoint in logic, which leads nowhere.
– Or use the real thing and try to modify the rules of perenniality in a way compatible with usual mathematics. This so far lead nowhere either, but there is something promising here : changing the rules of exponentiation alters the rate of growth of definable functions. The phenomenon was first observed for the *light exponentials* : for instance, in the system **LLL** [6], definable functions are polytime. A connection with *complexity theory* is therefore expected.

3.3 Geometry of interaction

Coming back to QCS, we quickly discover that the main obstacle to perennialisation is the limitation to finite dimension. Could we therefore admit infinite dimensional carriers ? The answer is negative, for want of a satisfactory *trace* : on the space $\mathcal{B}(\mathbf{H})$ of (bounded) endomorphisms of an infinite dimensional Hilbert space, only certain operators admit a trace : they are therefore styled « trace-class ». Unfortunately, unitaries such as the *twist* are never trace-class, hence the identity map does not belong here.

It is time to remember that von Neumann algebras are meant as generalisations of finite dimensional (matrix) algebras in which the trace is available : this is the case for vN algebras of type $\mathbf{II}_1$, among them the celebrated *hyperfinite factor*[3] $\mathcal{R}$ of Murray-von Neumann [1]. Unfortunately, moving to type $\mathbf{II}_1$ does not solve our problem : a computation, made in the spirit of (8), yields the value $\mathrm{tr}(\sigma \cdot (h \otimes k)) = 0$, which shows that the identity map does not belong there either.

A change of paradigm is therefore necessary : this is *geometry of interaction* (GoI) [2, 6]. To make the long story short, it involves the replacement of the trace with the *determinant*, what makes sense, under reasonable hypotheses, in $\mathcal{R}$; if $\varrho(u) < 1$, then define :

$$\det(I - u) \quad := \quad \exp(\mathrm{tr}(\ln(I - u))) \tag{17}$$

[3] $\mathcal{B}(\mathbf{H})$ is also a vN algebra, of trivial type $\mathbf{I}_\infty$.

$\ln(I-u)$ being definable by the usual powers series thanks to the hypothesis on the *spectral radius* $\varrho(u)$.

Everything rests upon an analogue of the adjunctions (11) and (6), typically :

$$\det(I-\varphi[u]\cdot v)=\det(I-\varphi\cdot(u\oplus v)) \tag{18}$$

which is a sort of logarithm of (6) : the trace becomes a determinant, the tensor product being replaced with a direct sum, corresponding to a decomposition $I=\pi+(I-\pi)$ of the identity into a sum of orthogonal projections.

Something close to (18) can indeed be achieved ; $\varphi[u]$ can be defined as the solution of the *feedback equation*, see [4, 6] :

$$\varphi[u]:=(I-\pi)\cdot\varphi\cdot(I-u\cdot\varphi)^{-1}\cdot(I-\pi) \tag{19}$$

Unfortunately, (18) is slightly incorrect ; it must be replaced with :

$$\det(I-\varphi[u]\cdot v)\cdot\det(I-\varphi\cdot u)=\det(I-\varphi\cdot(u\oplus v)) \tag{20}$$

The apparition of the scalar $\det(I-\varphi\cdot u)$ (a sort of truth value) radically modifies our approach to the question.

3.4 The duality of GoI

Henceforth, $\mathcal{R}$ denotes the *hyperfinite factor*.

Definition 3 (Projects)
Let $\pi\in\mathcal{R}$ be a projection ; a project of base π *is the pair $\mathfrak{p}=(\alpha,u)$ of a* wager $\alpha\in\mathbb{C}$ *and an* aim $u\in\mathcal{R}$ *such that $\|u\|<1$ and $\pi u\pi=u$.*

Definition 4 (Duality)
Two projects $\mathfrak{p}=(\alpha,u),\mathfrak{q}=(\beta,v)$ of the same base π are polar, *notation $\mathfrak{p}\underset{\sim}{\downarrow}\mathfrak{q}$ when the following holds :*[4]

1. *$\varrho(uv)<1$.*
2. *$\alpha\beta\det(I-uv)\neq 1$.*

We define, provided $\mathfrak{p}\underset{\sim}{\downarrow}\mathfrak{q}$, $\ll\mathfrak{p}\mid\mathfrak{q}\gg:=\alpha\beta\det(I-uv)$, when $\varrho(uv)<1$; the scalar $\ll\mathfrak{p}\mid\mathfrak{q}\gg$ can be viewed as the degenerated project (of base 0), $(\ll\mathfrak{p}\mid\mathfrak{q}\gg,0)$.

From this, one can define :

Definition 5 (Conducts)
Let $\pi\in\mathcal{R}$ be a projection ; a conduct of base π *is a set $\mathbf{C}$ of projects of base π equal to its bipolar.*

and reconstruct logic on this basis.

[4] $\varrho(.)$ denotes the *spectral radius*.

3.5 The adjunction of GoI

In what follows, the respective bases π, ν of the conducts $\mathbf{C}, \mathbf{D}$ are orthogonal, i.e., $\pi\nu = 0$.

Definition 6 (Linear implication)
The conduct $\mathbf{C} \otimes \sim\mathbf{D}$ *(of base* $\pi + \nu$*) is defined as :*

$$\mathbf{C} \otimes \sim\mathbf{D} \quad := \quad \sim\sim\{(\alpha\beta, u+v); (\alpha, u) \in \mathbf{C}, (\beta, v) \in \sim\mathbf{D}\} \qquad (21)$$

The conduct $\mathbf{C} \multimap \mathbf{D}$ *(of base* $\pi + \nu$*) is defined as the polar of the previous :*

$$\mathbf{C} \multimap \mathbf{D} \quad := \quad \sim\{(\alpha\beta, u+v); (\alpha, u) \in \mathbf{C}, (\beta, v) \in \sim\mathbf{D}\} \qquad (22)$$

For the next theorem, remember the definition (19) of $\varphi[u]$:

Theorem 3 (Adjunction)
$(\gamma, \varphi) \in \mathbf{C} \multimap \mathbf{D}$ *iff :*

$$\forall(\alpha, u) \in \mathbf{C}\ (\varrho(\varphi u) < 1 \text{ and } (\alpha\gamma \det(I - \varphi u), \varphi[u]) \in \mathbf{D}) \qquad (23)$$

Démonstration : The proof essentially relies on (20) ; see [6]. □

4 Subjective truth

4.1 Retrieving level -2

The three layers **-1,-2,-3** are not a sort of Trinity, like the *hypostases* of neo-platonism. The idea is that everything rests upon layer **-3** and that **-1,-2** are only *surrogates* : the problem is therefore to retrieve them from layer **-3**.

The retrieval of layer **-2** consists in defining a *category* whose objects are conducts. What is immediate is that the definition of linear implication yields a functional interpretation : « functions » from $\mathbf{C}$ to $\mathbf{D}$. If $\mathbf{C}, \mathbf{D}, \mathbf{E}$ are of respective (pairwise orthogonal) bases π, ν, μ, if $(\alpha, \varphi) \in \mathbf{C} \multimap \mathbf{D}$, $(\beta, \psi) \in \mathbf{D} \multimap \mathbf{E}$, then $(\gamma, \theta) \in \mathbf{D} \multimap \mathbf{E}$, with :

$$\theta \quad := \quad (\pi\varphi + \nu)(I - \psi\varphi)^{-1}(\pi + \psi\nu) \qquad (24)$$

$$\gamma \quad := \quad \alpha\beta \det(I - \varphi\psi) \qquad (25)$$

(24) generalises (19) ; the « composition » thus defined is associative, see [6].

Unfortunately, this is not quite a category : this is due to the constraints on bases : for instance one cannot form $\mathbf{C} \multimap \mathbf{C}$. In order to construct a category, one has to allow for a certain amount of *delocation*.

4.2 The category of projects

If $n > 0$, $\mathcal{M}_n(\mathcal{R})$ stands for the algebra of $n \times n$ matrices with coefficients in $\mathcal{R}$, i.e., $\mathcal{M}_n(\mathbb{C}) \otimes \mathcal{R}$. As a vN algebra, $\mathcal{M}_n(\mathcal{R})$ is isomorphic to $\mathcal{R}$, hence admits a unique trace ; indeed the trace is made unique by the requirement that $\mathrm{tr}(I) = 1$. We shall make a different choice, namely, $\mathrm{tr}(I) = n$.

If $\mathtt{f} \in I(m, n)$ is an injective map from $m = 1, \ldots, m$ into $n = 1, \ldots, n$, then it induces a $*$-isomorphism $\mathcal{M}_{\mathtt{f}}(\mathcal{R})$ from $\mathcal{M}_m(\mathcal{R})$ to $\mathcal{M}_n(\mathcal{R})$. This isomorphism preserves everything, but the identity ; our convention about traces makes it preserve trace as well.

The following injective maps can be defined :

$\mathtt{s}$ **:** $\mathtt{s} \in I(1,3) : \mathtt{s}(1) := 1$ (source).

$\mathtt{t}$ **:** $\mathtt{t} \in I(1,3) : \mathtt{t}(1) := 3$ (target).

$\mathtt{p}$ **:** $\mathtt{p} \in I(3,3) : \mathtt{p}(1) := 1, \mathtt{p}(2) := 3, \mathtt{p}(3) := 2$ (prefixing).

$\mathtt{q}$ **:** $\mathtt{q} \in I(3,3) : \mathtt{q}(1) := 2, \mathtt{q}(2) := 1, \mathtt{q}(3) := 3$ (postfixing).

$\mathtt{r}$ **:** $\mathtt{r} \in I(1,3) : \mathtt{r}(1) := 2$ (relay).

Definition 7 (Morphism)
If $\mathbf{C}, \mathbf{D}$ *are conducts of respective bases* π, ν*, a* morphism *from* $\mathbf{C}$ *to* $\mathbf{D}$ *is a project* $\mathfrak{f} \in \mathtt{s}(\mathbf{C}) \multimap \mathtt{t}(\mathbf{D})$ *of base* $\begin{bmatrix} \pi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \nu \end{bmatrix}$.

Typically, the identity morphism of $\mathbf{C}$ of base π is $(1, \begin{bmatrix} 0 & 0 & \pi \\ 0 & 0 & 0 \\ \pi & 0 & 0 \end{bmatrix})$.

Definition 8 (Composition)
In order to compose the morphisms $\mathfrak{f} \in \mathit{Proj}(\mathbf{C}, \mathbf{D})$ *and* $\mathfrak{g} \in \mathit{Proj}(\mathbf{D}, \mathbf{E})$*, one composes, following (24), (25),* $\mathtt{p}(\mathfrak{f}) \in \mathtt{s}(\mathbf{C}) \multimap \mathtt{r}(\mathbf{D})$ *with* $\mathtt{q}(\mathfrak{g}) \in \mathtt{r}(\mathbf{D}) \multimap \mathtt{t}(\mathbf{E})$.

These lineaments of a categorical interpretation are enough to show that level **-2** can be retrieved from level **-3**.

4.3 Retrieving level -1

The *wager* is a sort of truth value, a wager equal to 1 meaning « true » : the interaction between projects $\mathfrak{p} \in \mathbf{C}$ and $\mathfrak{q} \in \sim\mathbf{C}$ yields a « truth value » $\langle \mathfrak{p} \mid \mathfrak{q} \rangle$ necessarily distinct from « true ». The intuition is therefore that a project of the form $\mathfrak{p} = (1, u)$ is, so to speak, true ; such a *wager-free* project corresponds to the familiar notion of *proof*. And we can expect to retrieve layer **-1** by saying that a conduct is *true* when it contains a wager-free project.

Unfortunately, this fails for a simple reason : wager-free projects are not closed under « composition » : typically, $\mathbf{C}, \sim\mathbf{C}$ may both contain wager-free projects. Indeed, the problem at stake is the scalar $\det(I - \varphi\psi)$ occurring in (25), already met under the simplified form $\det(I - \varphi u)$ in (20) : this scalar is responsible for non-trivial wagers ; we are therefore seeking a way to enforce $\det(I - \varphi\psi) = 1$.

4.4 Successful projects

DEFINITION 9 (VIEWPOINTS)
A viewpoint *is a maximal commutative subalgebra $\mathcal{P} \subset \mathcal{R}$.*

This is the same as saying that $\mathcal{P}$ is a subalgebra of $\mathcal{R}$ equal to its *commutant*.

It is instructive to consider the analogue of viewpoints in finite dimension : a maximal commutative subalgebra $\mathcal{P} \subset \mathcal{M}_n(\mathbb{C})$ consists in all matrices that are diagonal w.r.t. to a specific base. In other terms, a viewpoint is the same as a base. In that case, the simplest is to take the *canonical* viewpoint, i.e., the one consisting of diagonal matrices.

DEFINITION 10 (SUCCESS)
Let $\mathcal{P}$ be a viewpoint ; a project $\mathfrak{p} = (\alpha, u)$ of base $\pi \in \mathcal{P}$ is successful *(w.r.t. $\mathcal{P}$) iff :*

1. *$\mathfrak{p}$ is wager-free : $\alpha = 1$.*
2. *u is a* partial symmetry *: $u = u^* = a^3$.*
3. *u belongs to the* normaliser *of $\mathcal{P}$: $u\mathcal{P}u \subset \mathcal{P}$.*

To say that the partial symmetry u belongs to the normaliser of $\mathcal{P}$ is the same as saying that, whenever $\nu \in \mathcal{P}$ is a projection, then $u\nu u$ is still a projection of $\mathcal{P}$.

Going back to our finite dimensional analogy, and taking for $\mathcal{P}$ the canonical viewpoint, the conditions on u read as follow : u is a symmetric matrice with entries equal to $0, 1$; moreover, there is at most one nonzero coefficient in any line (hence in any column).

DEFINITION 11 (TRUTH)
Let $\mathcal{P}$ be a viewpoint ; a conduct $\mathbf{C}$ of base $\pi \in \mathcal{P}$ is true *when it contains a successful project. It is* false *when its negation $\sim\mathbf{C}$ is true.*

THEOREM 4 (SUBJECTIVE COHERENCE)
Truth is closed under logical consequence : if $\pi, \nu, \mu \in \mathcal{P}$, if $\mathbf{C}, \mathbf{D}, \mathbf{E}$ are of pairwise orthogonal respective bases π, ν, μ, then the truths of $\mathbf{C} \multimap \mathbf{D}$ and $\mathbf{D} \multimap \mathbf{E}$ entails the truth of $\mathbf{C} \multimap \mathbf{E}$.

Démonstration : The proof may be found in [6] ; see the corollary below for a simplified version. $\square$

COROLLARY 4.1
$\mathbf{C}$ cannot be both true and false.

Démonstration : Assume that $\mathfrak{p} = (1, u) \in \mathbf{C}, \mathfrak{q} = (1, v) \in \sim \mathbf{C}$ are successful, hence $\varrho(uv) < 1$ and $\det(I - uv) \neq 1$. Since successful, both $\nu \rightsquigarrow u\nu u$ and $\nu \rightsquigarrow v\nu v$ send projections in $\mathcal{P}$ to projections of $\mathcal{P}$. In particular $(uv)^n I(vu)^n = (uv)^n (uv)^{n*}$ is a projection for all n, hence is of norm 1 or 0. Since $\|(uv)^n\|^2 = \|(uv)^n (uv)^{n*}\|$, it follows that $(uv)^n$ is of norm 0 or 1 ; but $\varrho(uv) = \lim \|(uv)^n\|^{1/n} < 1$, hence $(uv)^n$ cannot be always of norm 1. uv is therefore *nilpotent* ; as in finite dimension, the determinant $\det(I - uv)$ is thence equal to 1, a contradiction. $\square$

It is quite easy to produce a conduct which is neither true nor false, and even easier to produce a conduct which is true w.r.t. a certain viewpoint $\mathcal{P}$ and false w.r.t. another viewpoint $\mathcal{Q}$. This illustrates the *subjective* character of our notion of truth. Of course this is unexpected, nay shocking, and calls for a detailed discussion.

4.5 Modalities

If we take the best — or rather less bad — modal logic $\mathbf{S_4}$, one is stricken by the fact that the additional operations such as the *necessity* $\Box A$ bring nothing new : typically, the erasure of all symbols $\Box, \Diamond$ preserves proofs. The situation can be summarised by the question :

What is the necessity of necessity ?

The current answer is « to write useless papers ». Indeed, there has been an industry of bad modal logics, people always trying to get worse and worse systems, typically $\mathbf{S_5}$, a system in which $\Box$ commutes with $\forall$, against all principles, e.g., cut-elimination.

Linear logic yielded another answer, through the *exponentials* $!A, ?A$ which are in charge of *perenniality* ; those are indeed modalities, even if the symbols $\Box, \Diamond$ have been relinquished to avoid the infamous company of $\mathbf{S_5}$ and its likes. Indeed, linear modalities are something like « $\mathbf{S_4}$ + structural rules », i.e., $!A$ behaves differently from A, because $!A$ is the *perennialisation* of A; it is therefore impossible to erase the exponentials and preserve prooofs. Typically, the removal of modalities in *contraction* :

$$!A \multimap !A \otimes !A \tag{15}$$

yields the incorrect :

$$A \multimap A \otimes A \tag{26}$$

Interpreting exponentials in GoI amounts at finding adequate hypotheses yielding contraction (15). The most obvious one is to restrict to projects whose wager equals 1 or 0 : without entering into details, it can easily be understood that contraction on $\mathfrak{p} = (\alpha, u)$ supposes $\alpha^2 = \alpha$, i.e., $\alpha = 1$ or $\alpha = 0$. Therefore, the rule of *promotion* « from $A \multimap B$, derive $!A \multimap !B$ », should only apply when the wager in the premise equals 0 or 1. But cut-elimination supposes the *functoriality* of promotion. This requires that the class of proofs to which promotion applies should be closed under composition. Null wagers pose no problems; but wager-free projects are problematic, since they are not closed under composition : we stumble again on the problem met in section 4.3, for which we know the answer, namely the restriction to *successful* projects. This means that $!A$ is subjective, since depending on a *viewpoint* $\mathcal{P}$.

We eventually discover that the « necessity » $!A$ is exactly an *affirmation* : $!A$ means that A is true w.r.t. a certain viewpoint $\mathcal{P}$; it should therefore be noted $!_{\mathcal{P}}A$.

5 Subjectivity vs. subjectivism

5.1 Truth according to Tarski

Long ago, Tarski gave a definition of truth of the form :

$$A \wedge B \textit{ is true iff } A \textit{ is true } \texttt{and } B \textit{ is true.}$$

All the other cases being treated in the same way, to sum up :

$$A \textit{ is true iff } A \texttt{ holds}.$$

Such a definition discouraged generations of mathematicians from even thinking at logic; moreover, logicians developed a sort of esthetics[5] of the « meta »

[5] In the same way, who dares to say that anybody can, like César, put his name on a compressed car, is styled « reactionary ».

ensuring that you must be dumb if you don't understand the depth of such « definitions ».

But the king is naked and one must say it : the arrogant essentialism of the Tarskian approach hides the absence of any interesting idea as to truth. It relies on a fantasy of *objectivity* reused by logical hustlers to develop systems of their own, typically :

$$A\,\mathrm{broccoli}\,B \textit{ is true iff } A \textit{ is true } \texttt{broccoli } B \textit{ is true.}$$

the logic of *broccoli*, which is not even edible ! This is the triumph of discretionary definitions : the absence of a decent *subjective* dimension in the logical explanation eventually leads to *subjectivism*.

5.2 Object and subject in logic

One of the weirdest aspects of logic is that it has been rather shy as to the relation between object and subject. The authority on the topic, Frege, distinguished between *sense* (subjective) and *denotation* (objective), the « morning star » and the « evening star » denoting the same object. What is compatible with the standard reading of incompleteness : there are *true* properties that we cannot *know*.

This relies on the assumption that the objective space exists independently of the subject : what is slightly reactionary after the discovery of quantum phenomenons ! Instead of incorporating the quantum viewpoint, logic tried to fill the gap between object and subject, so as to reduce, as much as possible, the pregnancy of the subject.

5.3 Digression : the logic of dungeons

This is especially conspicuous in junk logics (logics for AI) ; the authors of such atrocities usually know the tune, but not the words, of correct logic. Their productions are therefore typical of logical fantasies ; they all agree on one point : the distinction between object and subject is pointless. For instance truth values have been assigned to cognitive situations, with the outcome that one can expect...

The worse paralogic ever produced is arguably the so-called *epistemic logic*. In this collection of childish riddles (the Baghdad cuckolds, etc.), one assumes that the objective world is well-constituted, that all questions have received answers, the only problem being to bridge things together : the hidden assumption here is the identification between *deduction* and *constatation*, by the way a possible definition of *totalitarism*.

Of course, this does not work, think of the *Houston* cuckolds : there is only one of them, but his intermediate initial is W., which means that, knowing the existence of one cukhold, moreover not one of his neighbours, he draws no conclusion ; hence, his colleagues, well-told in epistemic logic, slay their innocent wifes. Which once more disproves the identification « deduction = constatation », already refuted by incompleteness, indecidablity, in the narrower context of mathematical reasoning. In fact, no activity in the real world seems to match the ideal of this cristal-clear identification, except the sort of asymetric protocol in fashion in Guantanamo and similar places.

This digression enables one to introduce the expression « intersubjectivity ». In epistemic logic, this is called « common knowledge » and corresponds to the exchange of information between infallible and truthful partners : think of *Big Brother* and, more recently, the network of secret services and secret dungeons organised by the CIA.

5.4 Intersubjectivity

Intersubjectivity has definitely nothing to do with this totalitarian nightmare. If we agree that a single subject is something like the choice of a commutative algebra, intersubjectivity is the gathering of several of them, provided they commute. What I called « viewpoint » is therefore the (ideal) building of a complete intersubjectivity, this completeness being only a convenience.

Now, let us come to the paradoxical aspects of our definition of truth. The point is that truth depends on the viewpoint $\mathcal{P}$; in particular, a theorem A may become false w.r.t. the « wrong » viewpoint. This must not be taken as a sort of relativistic argument justifying the denial of various evidences. When thinking of this *subjective paradox*, one must take into account that the viewpoint is part of the *meaning* that we ascribe to A : as long as we respect this intended meaning, nothing unpleasant or really shocking can occur ; and if we depart from it, where is the paradox ?

A theorem is not a decoration that one puts on a shelf, it is a tool, which can be used as a *lemma* to produce other theorems : the *use* of A through logical consequence is the actual *meaning* of A. Now, when I relate A and $A \multimap B$ to get B, I relate them w.r.t. their intended meaning ; if distinct subjects have been in charge of A and B, then $A \multimap B$ makes sense only when these subjects « recognise each other », i.e., commute as commutative algebras. In other terms, the meaning of A is determined by its intersubjective context, since it involves the creation of a *common viewpoint*.

Indeed, the subjective paradox is not very different from the various paradoxes induced by the arising of subjectivity in modern science. For instance, after relinquishing the geocentric viewpoint, one could argue that speed, now relative to a galilean referential, no longer makes sense ; but, when studying the interaction of mechanical bodies, it is wise to choose a common referential !

To sum up :

Subjective, but not subjectivistic !

and :

Truth = Modality=Intersubjectivity.

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