Geometry of Interaction V: logic in the hyperfinite factor

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December 17, 2009

À Claire Delaleu (1991-2009)

Nous nous aimions : toi ma Petite Tortue ; moi l'oncle dont tu étais, enfant, carrément amoureuse, je crois. C'est curieux comme j'entends mieux ta voix, cette petite voix si ténue de Claire, depuis que je sais que tu ne m'appelleras plus.

Introduction

Motivations

Geometry of Interaction (GoI) reacts against the absence of any satisfactory explanation for logic. The usual one is that of a symbolic calculus of truth values, which supposes that truth values preexist and formulas as well. This was later improved into a symbolic calculus of category-theoretic diagrams, what is still unsatisfactory: since this calculus rests upon an oriented rewriting, one side is more equal, more commutative, than the other. The aim of GoI is therefore to find a space where truth, commuting diagrams, etc. are no longer primitive and where dynamical processes (proof-search, rewriting, a.k.a. normalisation) are primitive: let us call such processes *projects* (or designs).

The curious preexistence of formulas makes logic dependent over *ad hoc* syntactical choices: most convenient to write PhDs, but at the same time a severe morphological flaw. GoI should thus define formulas independently of any language: let us call such language-free formulas *behaviours* or *conducts*.

We thus bestow a central status to *deduction* from which everything (esp. syntax) should proceed, whence the syntax-free approach, projects and behaviours. This is Philosophy of Science, but not the usual sterile and incompetent comment on (or rather: against) living science we are accustomed to.

Proposing a reconstruction of logic not resting upon syntactic *a priori* is not only of utmost philosophical interest, it is is also important from various technical points, including down to earth syntactical manipulations: *light logics* ([10], ch. 16), i.e., low complexity logics, are not clearly grounded (in particular cannot be accessed through « semantics »); the present paper induces many clarifications in this area.

Architecture

This sixth GoI paper (after [4, 5, 6, 7, 8]) is the first to present a consistent and systematic reconstruction of logic. For a detailed introduction to the program, see [10], although chapters 20 and 21 are now partly obsolete.

The previous papers of the series were concerned with the representation of proofs by means of operators and the study of the *feedback equation* which accounts for normalisation. The use of operators is natural, since *proof-nets* ([10], ch. 11) dwell in matrix algebras, of which operator algebras are generalisations; among them, *von Neumann algebras* are the natural choice, since closed under directed suprema. Only lately, it occurred to me that the algebra is more basic than its inhabitants[1]. W.r.t. an appropriate choice of a vN algebra (the *hyperfinite factor*) we give a possible answer to the most basic *morphological* questioning: « what is a formula? ». Which supposes explaining what is a proof, what is truth, how formulas do socialise (connectives) etc. And which excludes the usual inductive constructions of the style « atoms are formulas, formulas are closed under… » whose limitations have already been expounded.

Any decent logical morphology rests upon implication: this was already the case at the time of Aristotle; this remains true nowadays, simply we focus upon *linear* implication[2]. We explain it through an *adjunction*, typically, in *quantum coherent spaces* (QCS, [10], ch. 17), through:

$$\mathrm{tr}(F \cdot (A \otimes B)) = \mathrm{tr}([F]A \cdot B) \tag{1}$$

which says that measuring (by means of B) the application $[F]A$ of F to A is the same as measuring F by means of $A \otimes B$. QCS have many qualities, coming from their « non-commutativity »: typically, booleans become *spins*… and two major drawbacks: that of a categorical interpretation, whence unable to explain dynamics, thus complexity; and, at a deeper level, their incompatibility with infinite dimension, the latter drawback being related to the former.

The founding adjunction of *Geometry of Interaction* (GoI) is:

$$\det(I - F \cdot (A \oplus B)) = \det(I - [F]A \cdot B) \cdot \det(I - FA) \tag{2}$$

and the problem at stake is a reorganisation of logic around (2), involving the definition of *projects* (representing proofs) and *behaviours* or *conducts* (sets of projects, representing formulas). The paper proposes the following solutions:

[1] The same is true in geometry: a curve is more primitive than its points!

[2] B.t.w., the distinction between several implications made no sense for syllogistics!

(i) Everything takes place in the[3] hyperfinite factor, for two reasons: the existence of a *trace*, enabling the use of determinants; the uniqueness of this factor, enabling the use of various isomorphisms, usually *outer*.

(ii) The basic artifact, the *project* $\mathfrak{c} = c \cdot\!+\!\cdot \gamma + C$, makes use of a *wager* c, i.e., a real number, which « homogenises » equation (2), i.e., takes care of the « extra » factor $\det(I - FA)$. For convenience determinants are replaced with their (co)logarithms, thus wagers take their values in $\mathbb{R} \cup \{\infty\}$; the default wager is 0.

(iii) The most important component of the project $\mathfrak{c}$ is its *plot* C, a hermitian of norm at most 1. C dwells, not quite in the hyperfinite factor, but in its tensorisation with an *idiom* space $\mathcal{C}$, which is a finite-dimensional vN algebra, treated *up to isomorphism*: when two projects are put in duality, their respective idioms are tensorised, i.e., remain *private*; this privacy was styled long ago [7] « communication without comprehension ». The novelty is that the idiom is non-commutative: this is crucial for the *contraction* rule; *a contrario*, « additive contraction », i.e., the superimposition at work in the additive case, is but the *commutative* form of contraction. Indeed, the idiom corresponds to the idea of *resource*: this is why idiom-free (i.e., *perennial*) projects can be duplicated.

(iv) A subtle point: the idiom is equipped with a « pseudo-trace » γ, a sort of trace that need neither be normalised nor positive. Pseudo-traces are an elegant way to cope with algebraic combinations of projects and, *in fine*, with equivalence relations, which were a bit mistreated (*bihaviours*) in my previous large scale reconstruction, *ludics* [9].

(v) Given two projects $\mathfrak{a} = a \cdot\!+\!\cdot \alpha + A, \mathfrak{b} = b \cdot\!+\!\cdot \beta + B$, idiom tensorisation yields variants $A^{\ddagger}, B^{\dagger}$ of A, B and one defines, using the cologarithmic determinant ldet (relative to the pseudo-trace $\mathrm{tr} \otimes \alpha \otimes \beta$):

$$\ll \mathfrak{a} \mid \mathfrak{b} \gg := a\beta(I_{\mathcal{B}}) + b\alpha(I_{\mathcal{A}}) + \mathrm{ldet}(I - A^{\ddagger}B^{\dagger}) \tag{3}$$

(vi) The projects $\mathfrak{a}, \mathfrak{b}$ are *polar*, notation $\mathfrak{a} \,\underset{\sim}{\downarrow}\, \mathfrak{b}$ iff $\ll \mathfrak{a} \mid \mathfrak{b} \gg \neq 0, \infty$. *Conducts* are sets of projects equal to their bipolar. The exclusion of the value ∞ corresponds to the *acyclicity* criterion of proof-nets; while the exclusion of 0 is reminiscent of *connectedness*.

(vii) With the help of conducts, one can develop multiplicatives, but hardly go beyond. In order to cope with the other logical primitives, one must consider a morphological constraint, *polarisation*, which *tames* wagers — roughly speaking, forces them to be 0.

(viii) This is not enough, for one cannot swap polarities. In order to do so, a refinement of polarisation, *lateralisation* (left = negative, right = positive), is introduced. Lateralised conducts, a.k.a. *behaviours*, are our ultimate logic artifact.

[3] Of type $\mathbf{II}_1$, the default type.

Summary of results

Let us now review the main achievements and novelties of the paper:

Polarisation: (and lateralisation) is not as expected. The tensor $\mathbf{A} \otimes \mathbf{B}$ is defined when at least one of $\mathbf{A}$ or $\mathbf{B}$ is a negative conduct. The deep reason for that difference w.r.t., say, ludics [9], is that, in the non-commutative case, the inductive analysis (look at the last rule used and act accordingly) no longer makes sense.

Exponentials: weakening on positive conducts is free of charge, the same with contraction on *perennial* conducts, i.e., idiom-free positive conducts. Exponentiation $!\mathbf{A}$ therefore depends upon an isomorphism which « kills » the idiom. W.r.t. an appropriate choice Ω, *conducts* validate the first order rules of the *iconoclast* logic **ELL** ([10], ch. 16); behaviours seem to be closer to the other *light* logic, **LLL**.

Isomorphisms: the usual isomorphisms of logic, including the most important of them all, $!\& = \otimes!$, are available as *literal* equalities, but for the distributivity $\otimes\oplus \simeq \oplus\otimes$ which is only up to isomorphism.

Witnesses: those are (positive) conducts made of projections. Witness behaviours are remarkably stable, e.g., closed under $\oplus$ and $⅋$: indeed, the two disjunctions *coincide* on witnesses! In ludics, designs were the result of a complex elaboration, involving the choice of a *first action*, thus taking care of a fruitful paradox, namely the *Gustave function* ([10], ch. 12). *Behaviours* « lateralise » certain conducts w.r.t. a *base*, i.e., a witness, thus disentangling Gustave-like situations.

Non commutativity

As to the mathematical treatment, I tried to stay, as far as possible, « non-commutative » in the sense of Connes [1]. The main novelties of this paper must be ascribed to this bias; I hope that GoI may eventually reach a state where some sophisticated results coming from operator algebra may apply.

All abuses of commuting projections which were prominent in the first items of GoI, e.g., [5], disappeared. Of course, do remain those abuses which are part of the data: as observed long ago by quantum physicists, we subjects are — individually and collectively: this is *intersubjectivity* — « commutative ». This is why connectives (which take care of *socialisation*) involve disjoint (hence commuting) carriers. The only point which remains strongly, indefectibly, « commutative » is *truth*, treated in the spirit of quantum measurement: truth is relative to the choice of a *viewpoint*, i.e. a sort of « base ».

While the logical *vulgate* defined truth as an absolute, I define (honestly, nothing to do with Tarski's plagiarism of Molière's *dormitive virtue of opium*: the « veritive value of truth ») truth *subjectively*, i.e., w.r.t. a *viewpoint*.

The paper cannot be read without some mathematical culture, e.g., some acquaintance with topology (which has little in common with the punishment known as Scott domains) and functional analysis: C^*-algebras, vN algebras. The materials are accessible in standard textbooks such as [11]; my own presentation of the topics, in the last chapters of [10], can be more accessible to a logician, since precisely written in view of logical applications.

Thanks to Esfan Haghverdi for stimulating discussions, to Thomas Seiller for signaling multiple bugs and typos and, last but not least, to Georges Skandalis for his expertise on vN algebras.

1 The adjunction

1.1 The Fuglede-Kadison determinant

In what follows, $\mathcal{A}$ is a finite factor, thus admitting a unique trace $\mathrm{tr}(\cdot)$.

THEOREM 1 (FUGLEDE & KADISON, [3])
If $u \in \mathcal{A}$ is invertible, define:

$$\det(u) := e^{\mathrm{tr}(\log(|u|))} \tag{4}$$

The determinant thus defined is multiplicative, monotonous and commutes to directed infima. The determinant can then be extended to the full $\mathcal{A}$ and is still multiplicative, monotonous and commuting to directed infima.

Contrarily to the usual determinant $\mathrm{Det}(\cdot)$, $\det(\cdot)$ takes its values in $\mathbb{R}^+$; indeed, if $\mathcal{A} = \mathcal{M}_n(\mathbb{C})$:

$$\det(u) = |\mathrm{Det}(u)|^{1/n} \tag{5}$$

If $u \geq 0$, then $|u| := \sqrt{u^*u} = u$, whence, if u is invertible, $\det(u) = e^{\mathrm{tr}(u)}$; in particular, if a is hermitian and $\|a\| < 1$, then $\det(I - a) = e^{-\mathrm{colog}(I-a)}$, with:

$$\mathrm{colog}(I - a) := a + a^2/2 + a^3/3 + \ldots \tag{6}$$

The following proposition summarises the basic properties of $\det(\cdot)$.

PROPOSITION 1

(i) $\det(u) \in \{0, 1\}$ *when u is a partial isometry;* $\det(u) = 1$ *iff u is unitary.*

(ii) $\det(u^*) = \det(u)$.

(iii) $\det(I - uv) = \det(I - vu)$.

(iv) $\det(I - u) = 1$ *when u is nilpotent.*

Proof : (iii) If v is unitary, $\det(I - uv) = \det(v(I - uv)v^*) = \det(I - uv)$. In general, write $v = \lambda_1 v_1 + \ldots + \lambda_4 v_4$, with the v_i unitary ([11], 4.1.7).

(iv) Let π be the projection of the closure of the range of u; then $\det(I - u) = \det(I - \pi u) = \det(I - u\pi)$. If $u^2 = 0$, then $u\pi = 0$ and we are done; otherwise, redo the same thing with $u\pi$, etc. □

As an extension by directed infima, the Fuglede-Kadison determinant is barely continuous.

1.2 The feedback equation

Assume that, modulo a block decomposition $I = \boldsymbol{a} + \boldsymbol{b}$ ($\boldsymbol{a}, \boldsymbol{b}$ projections)

$F := \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$ and A (with $A\boldsymbol{a} = \boldsymbol{a}A = A$) are hermitians of norm ≤ 1.

The *feedback equation*:

$$\begin{aligned} F_{11}A(x) + F_{12}(y) &= x & (7) \\ F_{21}A(x) + F_{22}(y) &= y' & (8) \end{aligned}$$

yields an operator $([F]A)(y) := y'$ such that $[F]A \cdot \boldsymbol{b} = \boldsymbol{b} \cdot [F]A = [F]A$, provided the equation (7, 8) has a solution. It turns out that:

- The solution — if it exists — is unique: a hermitian of norm ≤ 1.
- If $I - F_{11}A$ is invertible, then:

  $$[F]A = F_{22} + F_{21}A(I - F_{11}A)^{-1}F_{12} \tag{9}$$

  In the invertible case, application is associative (i.e., Church-Rosser):

  $$[F](A + B) = [[F]A]B \tag{10}$$

- $[F]A$ is a sort of functional application. In the same way *Modus Ponens* is better handled by cut, functional application finds a symmetrical formulation through the notion of a *cut-system* (σ, u) [8]: the *feedback* σ is a partial symmetry. (7) (with u, σ, σ^2 in the respective roles of $F, A, \boldsymbol{a}$) now becomes $(\sigma^2(x) = x, (I - \sigma^2)(y) = y, (I - \sigma^2)(y') = y')$:

  $$u(x + y) = \sigma(x) + y' \tag{11}$$

  if $u = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ w.r.t. the decomposition $I = \boldsymbol{s} + \boldsymbol{t} := \sigma^2 + (I - \sigma^2)$, the

  solution, when $\sigma - a$ is invertible, is the *normal form* $\sigma[\![u]\!]$:

  $$\sigma[\![u]\!] := c + b^*(\sigma - a)^{-1}b \tag{12}$$

  a straightforward reformulation of (9). The general feedback equation (7, 8) can be reduced to the form (11), by means of a change of underlying Hilbert space: instead of opposing F with A, we oppose u (the direct sum $F \oplus A$) to σ (which flips inputs and outputs).

- Cut-systems are the right — and slightly illegible — approach to the feedback equation. The associativity of the normal form rewrites as:

  $$(\sigma + \tau)[\![u]\!] = \tau[\![\sigma[\![u]\!]]\!] \tag{13}$$

One can decompose a general feedback as the sum $\sigma = \sigma^+ + \sigma^-$ of a positive (σ^+ is a projection) and a negative ($-\sigma^-$ is a projection) feedback. For *lopsided* feedbacks (e.g., σ^+, σ^-), the normal form can be consistently generalised so as to commute with directed suprema (σ^+) or infima (σ^-). The (non-trivial) achievement of [8] is that:

$$\sigma^-[\![\sigma^+[\![u]\!]]\!] = \sigma^+[\![\sigma^-[\![u]\!]]\!] \tag{14}$$

thus enabling to define $u[\![\sigma]\!]$ through (13). This induces a general definition of the functional application $[F]A$, which heavily relies on the detour via the symmetrical framework of cut-systems — which enables the decomposition $\sigma = \sigma^+ + \sigma^-$, without analogue in the functional case $[F]A$.

1.3 Associativity

We replace $\det(\cdot)$ with its cologarithm $\mathrm{ldet}(\cdot)$; when $\|u\| < 1$ and $u = u^*$:

$$\mathrm{ldet}(I - u) = \mathrm{tr}(u) + \mathrm{tr}(u^2)/2 + \mathrm{tr}(u^3)/3 + \ldots \tag{15}$$

In general, if $\|u\| < 1$, then $\mathrm{ldet}(I-u)$ is not real, thus fails to be the cologarithm of $\det(I - u)$. However, if $\|u\|, \|v\| < 1$ and u, v are hermitian, then $\mathrm{ldet}(I-uv) = \mathrm{ldet}(I-vu) = \mathrm{ldet}(I-uv)^*$, whence $\mathrm{ldet}(I-uv) \in \mathbb{R}$; furthermore, if $u, v \geq 0$, then $\mathrm{ldet}(I - uv) = \mathrm{ldet}(I - \sqrt{v}u\sqrt{v}) \geq 0$.

Assume that, *modulo* a block decomposition $I = \boldsymbol{a} + \boldsymbol{b}$:

$$F := \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \qquad G := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} (= A + B)$$

Theorem 2

$$\mathrm{ldet}(I - F(A + B)) = \mathrm{ldet}(I - [F]A \cdot B) + \mathrm{ldet}(I - FA) \tag{16}$$

Proof : the proof requires cut-systems: assume that, $I = \boldsymbol{s} + \boldsymbol{t} = \sigma^2 + \tau^2$; then:

$$\mathrm{ldet}(\sigma + \tau - u) = \mathrm{ldet}(\boldsymbol{s} + \tau - \sigma[\![u]\!]) + \mathrm{ldet}(\sigma + \boldsymbol{t} - \boldsymbol{s}u\boldsymbol{s}) \tag{17}$$

If, modulo the block decomposition $I = \boldsymbol{s} + \boldsymbol{t}$, $u = \begin{pmatrix} a & b^* \\ b & c \end{pmatrix}$, then (17) rewrites:

$$\mathrm{ldet}(\sigma + \tau - u) = \mathrm{ldet}(\tau - \sigma[\![u]\!]) + \mathrm{ldet}(\sigma - a) \tag{18}$$

Observe that $(\sigma + \tau - u)(I - \boldsymbol{z}) = \sigma + \tau - u$, $(\sigma + \boldsymbol{t} - \boldsymbol{s}u\boldsymbol{s})(I - \boldsymbol{z}) = \sigma + \boldsymbol{t} - \boldsymbol{s}u\boldsymbol{s}$, with $\boldsymbol{z} := \ker(\sigma - a)$): if $\sigma - a$ is not injective, $\mathrm{ldet}(I - \boldsymbol{z}) = \infty$ and (18) holds.

Assuming $\sigma \geq 0$, i.e., $\sigma = \pi$and $\pi - a$ injective, then $\begin{pmatrix} \pi - a & -b^* \\ -b & \tau - c \end{pmatrix} =$

$$\begin{pmatrix} \sqrt{\pi - a} & 0 \\ 0 & \boldsymbol{t} \end{pmatrix} \begin{pmatrix} \pi & 0 \\ -b(\pi - a)^{-1/2} & \boldsymbol{t} \end{pmatrix} \begin{pmatrix} \pi & 0 \\ 0 & \tau - \sigma[\![u]\!] \end{pmatrix} \begin{pmatrix} \pi & -(\pi - a)^{-1/2}b^* \\ 0 & \boldsymbol{t} \end{pmatrix} \begin{pmatrix} \sqrt{\pi - a} & 0 \\ 0 & \boldsymbol{t} \end{pmatrix}$$

The coefficient $-(\pi-a)^{-1/2}b^*$, which involves the *unbounded* operator $(\pi-a)^{-1/2}$, is indeed bounded, see [8], with adjoint $\overline{-b(\pi-a)^{-1/2}}$, the closure of $-b(\pi-a)^{-1/2}$. From the multiplicativity of the determinant and the fact that triangular matrices are of the form $I-u$ with u nilpotent, we get (18).

The same holds if $\sigma = -\nu \leq 0$. The full (17) follows from the associativity of the normal form (13) and the lopsided cases $\pi, -\nu$:

$$\begin{aligned}
\mathrm{ldet}(\sigma+\tau-u) &= \mathrm{ldet}(\pi-\nu+\tau-\pi[\![u]\!]) + \mathrm{ldet}(I-\pi u\pi) \\
&= \mathrm{ldet}(\boldsymbol{s}+\tau-\sigma[\![u]\!]) + \mathrm{ldet}(\sigma+\boldsymbol{t}-\nu\pi[\![u]\!]\nu) + \mathrm{ldet}(I-\pi u\pi) \\
&= \mathrm{ldet}(\boldsymbol{s}+\tau-\sigma[\![u]\!]) + \mathrm{ldet}(\sigma+\boldsymbol{t}-\boldsymbol{s}u\boldsymbol{s})
\end{aligned}$$

□

Observe that $\mathrm{ldet}(I-FA) = \mathrm{ldet}(I-F_{11}A)$.

1.4 The adjunction

Compared with (1), equation (16) suffers from a want of homogeneity, due to the term $\mathrm{ldet}(I-FA)$; in practice (e.g., when interpreting logic), FA is often nilpotent, which may explain why this term has no analogue in (1).

In order to obtain a satisfactory adjunction, one must homegeneise: instead of an operator, one introduces the pair of a *wager* $w \in]-\infty, +\infty]$, the set of possible values for the cologarithm of a positive real, and an operator, notation $w+U$. Define $[f+F](a+A) := f+a+\mathrm{ldet}(I-FA)+[F]A$, then:

$$(a+b)+f+\mathrm{ldet}(I-F(A+B)) = (f+a+\mathrm{ldet}(I-FA))+b+\mathrm{ldet}(I-[F]A\cdot B) \tag{19}$$

thus, defining $\ll c+C \,|\, d+D \gg := c+d+\mathrm{ldet}(I-CD)$:

Theorem 3 (Adjunction)
The application $[f+F](a+A)$ *is characterised by:*

$$\ll f+F \,|\, (a+b)+(A+B) \gg \quad = \quad \ll [f+F](a+A) \,|\, b+B \gg \tag{20}$$

Proof : (20) is theorem 2. It remains to show that $d+D \rightsquigarrow \ll c+C \,|\, d+D \gg$ determines $d+D$. First, $\ll c+C \,|\, 0+0 \gg = c$ determines c; then, $D \rightsquigarrow \mathrm{ldet}(I-CD)$ determines C: since $\mathrm{ldet}(I-(\lambda C)D) = \lambda(\mathrm{tr}(CD)+o(\lambda))$, $D \rightsquigarrow \mathrm{ldet}(I-CD)$ determines $D \rightsquigarrow \mathrm{tr}(CD)$. The latter dependency is linear; if $\mathrm{tr}(CD) = 0$ for all D, then $tr(D^2) = 0$, whence $D = 0$ by the faithfulness of the trace. □

The adjunction (19) can thus be used as an abstract definition of functional application in finite factors.

1.5 Traces and determinants

Trace and determinant make sense in any *finite* vN algebra. Three properties have been used:

Cyclicity: $\mathrm{tr}(uv) = \mathrm{tr}(vu)$ yields the mutiplicativity of the trace.

Positivity: $\mathrm{tr}(uu^*) \geq 0$, subsumed by $\mathrm{tr}(I) = 1$, yields the the monotonicity of the trace.

Normality: ultraweak continuity yields the extension to directed infima.

In a finite algebra, there are non-zero elements of the predual (section C.2) which are positive and cyclic. Indeed, the most general notion of trace for a finite algebra is that of a *central trace*, [11] ch. 8: a cyclic and normal[4] *conditional expectation* (section C.8) from $\mathcal{A}$ onto its center. In the particular case of an algebra with a finite dimensional center, that we can write $\mathcal{A} = \bigoplus_{i\in I} \mathcal{A}_i$ — I being the set of minimal projections of $\mathcal{A}$, so that $\mathcal{A}_i := i\mathcal{A}i$ —, the central trace associates to $u \in \mathcal{A}$ the element $\sum \mathrm{tr}_{a_i}(iui) \cdot i$ of the center.

Any normal and cyclic form on $\mathcal{A}$ writes $\varphi(u) = f(\mathrm{tr}(u))$, where f is a linear form on the center of $\mathcal{A}$: if $\mathcal{A} = \bigoplus_{i\in I} \mathcal{A}_i$, $\varphi(u) = \sum f_i \mathrm{tr}_{a_i}(iui)$.

For reasons that find their origin in ludics, especially the half-baked *bihaviours* [9] it is important to consider non positive traces, i.e., to replace *positivity* with the weaker:

Hermiticity: $\mathrm{tr}(u) = \mathrm{tr}(u^*)$.

Definition 1 (Pseudo-trace)
If $\mathcal{A}$ is a finite vN algebra, a pseudo-trace *is an element α of the predual of $\mathcal{A}$, which is hermitian, cyclic, faithful (see* infra*) and such that $\alpha(I) \neq 0$.*

Faithfulness generalises the notion of a faithful state: if α is hermitian, then $\mathcal{A}$ splits into a direct sum $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_0 \oplus \mathcal{A}_+$, with $\alpha(uu^*) < 0, = 0, > 0$ for $u \neq 0, u \in \mathcal{A}_-, \mathcal{A}_0, \mathcal{A}_+$; α is *faithful* when $\mathcal{A}_0 = 0$. There is small problem with the determinant: if u splits as $u_- \oplus u_+$, then $\det(u) := \det(u_-)\det(u_+)$ can take the undetermined value $(+\infty)0$. Indeed the sign of $\alpha(I)$ determines this ambiguous case while staying multiplicative. In terms of cologarithms, $\mathrm{ldet}(u) := \mathrm{ldet}(u_-) + \mathrm{ldet}(u_+)$, with:

$$\begin{array}{rcll} -\infty + (+\infty) & = & +\infty & (\text{if } \alpha(I) > 0) \\ -\infty + (+\infty) & = & -\infty & (\text{if } \alpha(I) < 0) \end{array}$$

Proposition 2
If the finite vN algebras $\mathcal{A}, \mathcal{B}$ are equipped with pseudo-traces α, β, then:

$$\mathrm{ldet}_{\lambda\alpha}(I-u) = \lambda \mathrm{ldet}_\alpha(I-u) \tag{21}$$

$$\mathrm{ldet}_{\alpha\oplus\beta}(I-u\oplus v) = \mathrm{ldet}_\alpha(I-u) + \mathrm{ldet}_\beta(I-v) \tag{22}$$

$$\mathrm{ldet}_\beta(I-\varphi(u)) = \mathrm{ldet}_\alpha(I-u) \tag{23}$$

$$\mathrm{ldet}(I-(u\otimes\pi)) = \mathrm{ldet}(I-u)\cdot\beta(\pi) \tag{24}$$

[4] Ultraweakly continuous.

With $u \in \mathcal{A}, v \in \mathcal{B}$; in (21) $\lambda \in \mathbb{R}$, in (24) π is a projection of $\mathcal{B}$, in (23) φ is a normal *$*$-isomorphism from $\mathcal{A}$ to $\mathcal{B}$ such that $\alpha = \beta \circ \varphi$.*

Proof : obvious. In (23), φ need not be unital, i.e., enjoy $\varphi(I_{\mathcal{A}}) = I_{\mathcal{B}}$: we only use $\alpha(I_{\mathcal{A}})\beta(I_{\mathcal{B}}) > 0$. Modulo this remark, (24) follows from (21) and (23), using $\varphi(u) := (\beta(\pi))^{-1} \cdot u \otimes \pi$. □

1.6 Idioms

GoI is now *idiomatic*; operators dwell in tensors products of the form $\mathcal{R} \otimes \mathcal{D}$, where $\mathcal{D}$ is a finite dimensional algebra and $\mathcal{R}$ is the hyperfinite factor[5]. When relating two operators trough a tensor or a cut, the idioms must be tensorised: from $A \in \mathcal{R} \otimes \mathcal{A}$ and $B \in \mathcal{R} \otimes \mathcal{B}$, we form $A^{\ddagger}, B^{\dagger} \in \mathcal{R} \otimes (\mathcal{A} \otimes \mathcal{B})$:

$$(a \otimes b)^{\ddagger} := a \otimes (b \otimes I_{\mathcal{B}}) \tag{25}$$

$$(a \otimes c)^{\dagger} := a \otimes (I_{\mathcal{A}} \otimes c) \tag{26}$$

If A, B are equipped with *pseudo-traces* (definition 1) α, β, then $\mathcal{A} \otimes \mathcal{B}$ is equipped with the pseudo-trace $\alpha \otimes \beta$.

Moreover, operators are given together with wagers: for reasons of homogeneity, when changing the idioms, $a + A, b + B$ must be replaced with $a \cdot \beta(I_{\mathcal{B}}) + A^{\ddagger}, b \cdot \alpha(I_{\mathcal{A}}) + B^{\dagger}$. Which explains the restriction $\alpha(I) \neq 0$ on pseudo-traces. By the way, the final restriction on wagers is that $a \in \mathbb{R} \cup \{\alpha(I_{\mathcal{A}}) \cdot \infty\}$.

Of course, we could have followed the alternative way, and formed the idiom $\mathcal{B} \otimes \mathcal{A}$, with a stricty isomorphic result; indeed, the canonical $*$-isomorphism $\varphi : \mathcal{R} \otimes (\mathcal{A} \otimes \mathcal{B}) \mapsto \mathcal{R} \otimes (\mathcal{B} \otimes \mathcal{A})$, combined with proposition 2 (23) shows that $\mathrm{ldet}(I - A^{\ddagger}B^{\dagger}) = \mathrm{ldet}(I - B^{\ddagger}A^{\dagger})$. This last formula is very hard to read — not to speak of writing it! Although its contents is rather trivial: a common idiom has been created by tensorisation, period.

Idioms are, so to speak, the *bound variables* of GoI. In logic, a bureaucratic discipline called *α-conversion* and specially devoted to the handling of bound variables, has been introduced. α-conversion is so boring, so devoid of interest, that I never paid any attention to it, thus writing $(\lambda xx)\lambda xx$ instead of the correct $(\lambda xx)\lambda yy$. I propose to do the same with idioms, thus ignoring the superscripts $A^{\ddagger}, B^{\dagger}$. But we need first to indulge in some α-conversion[6], GoI-style!

Definition 2 (Projects)
Let $\mathcal{R}$ be the hyperfinite factor. A project *$\mathfrak{a} = a \cdot\!+\!\cdot \alpha + A$ of* idiom *$\mathcal{A}$, where $\mathcal{A}$ is a finite dimensional vN algebra, consists in the following data:*

- *A pseudo-trace α on $\mathcal{A}$.*
- *A « real » number $a \in \mathbb{R} \cup \{\alpha(I_{\mathcal{A}}) \cdot \infty\}$, the* wager.

[5] Of type $\mathbf{II}_1$, called « matricial » in [11], ch. 12.

[6] The analogy between idioms and bound variables is helpful but technically incorrect: actual bound variables can be handled *without* the help of idioms, just by naming them after the location of their binder, De Bruijn-style!

- *A hermitian operator $A \in \mathcal{R} \otimes \mathcal{A}$ of norm ≤ 1, the* plot.

The notation $a \cdot\!+\!\cdot\, \alpha + A$ is incorrect: it mentions neither $\mathcal{A}$, which is however determined as the source space of α, nor the carrier $\boldsymbol{a}$, to be introduced below.

Definition 3 (α-conversion)
If $\mathfrak{a} = a \cdot\!+\!\cdot\, \alpha + A$ is a project of idiom $\mathcal{A}$, if φ is a $*$-isomorphism of $\mathcal{A}$ into another idiom $\mathcal{B}$ such that $\beta \circ \varphi = \lambda\alpha$ ($\lambda \in \mathbb{R}$), then $\varphi(\mathfrak{a}) := \lambda a \cdot\!+\!\cdot\, \beta + \varphi(A)$ is a variant *of $\mathfrak{a}$, an* isovariant *if $\lambda = 1$. More generally, two projects are* variants *when they have a common variant in the previous sense.*

If $\mathcal{B} \subset \mathcal{A}$ is a (unital) subalgebra of $\mathcal{A}$ such that $A \in \mathcal{R} \otimes \mathcal{B}$, then $\mathfrak{a}$ is a variant of $a \cdot\!+\!\cdot\, \alpha \upharpoonright \mathcal{B} + A \upharpoonright \mathcal{R} \otimes \mathcal{B}$.

Proposition 3
Among all unital subalgebras $\mathcal{B} \subset \mathcal{A}$ such that $A \in \mathcal{R} \otimes \mathcal{B}$, there is a smallest one, the minimal idiom *of $\mathfrak{a}$.*

Proof : let $\mathcal{A}_0$ be the subalgebra generated by the $\theta(A)$, where $\Theta : \mathcal{R} \otimes \mathcal{A} \mapsto \mathcal{A}$ is induced by an element θ of the predual of $\mathcal{R}$, i.e., $\Theta(u \otimes v) := \theta(u) \otimes v$. $\square$

Two variants have therefore isomorphic minimal idioms.

GoI is built so as to be variant-independent; this is why A, B are replaced with their variants $A^\ddagger, B^\dagger$; one might as well have chosen the variants $A^\dagger, B^\ddagger$, or variants involving a bigger idiom, e.g., some $\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C}$.

Definition 4 (Extraneousness)
Two projects $\mathfrak{a}, \mathfrak{b}$ with the same idiom and pseudo trace $\mathcal{A}, \alpha$ are alien *when their respective minimal idioms $\mathcal{A}_0, \mathcal{B}_0 \subset \mathcal{A}$ commute to each other and are such that, for all $u \in \mathcal{A}_0, v \in \mathcal{B}_0$:*

$$\alpha(u) \cdot \alpha(v) = \alpha(uv) \cdot \alpha(I_{\mathcal{A}}) \tag{27}$$

The typical example is that of $\mathcal{A} \otimes I_{\mathcal{B}}, I_{\mathcal{A}} \otimes \mathcal{B} \subset \mathcal{A} \otimes \mathcal{B}$:

Proposition 4
With the hypotheses and notations of definition 4, $\mathcal{A}_0 \otimes \mathcal{B}_0$ is isomorphic to the algebra generated by $\mathcal{A}_0 \cup \mathcal{B}_0$, the isomorphism φ being such that $\varphi(u \otimes I_{\mathcal{B}_0}) = u, \varphi(I_{\mathcal{A}_0} \otimes v) = v$ for $u \in \mathcal{A}_0, v \in \mathcal{B}_0$.

The construction of $A^\ddagger, B^\dagger$ is thus a way to built alien variants of A, B. Extraneousness is a sophisticated version of α-conversion, whose technical contents is the absence of interference, of « comprehension », between the idioms.

We shall therefore work, not quite with projects, but with equivalent classes (w.r.t. variance). When combining projects in a multiplicative way (which includes cut), we shall select alien elements in the respective classes. The resulting object will be well-defined up to variance.

2 Conducts

From now on, $\mathcal{R}$ is the hyperfinite factor of type $\mathbf{II}_\infty$. The reason for this minor modification is explained in the next section.

2.1 Carriers

Definition 5 (Carriers)
A carrier $\boldsymbol{a} \in \mathcal{R}$ is a finite *projection. If $\boldsymbol{a}$ is a carrier, then one defines $\mathcal{R}_{\boldsymbol{a}} : \boldsymbol{a}\mathcal{R}\boldsymbol{a} = \{u \in \mathcal{R} \,;\, \boldsymbol{a}u = u\boldsymbol{a} = u\}$. A project $\mathfrak{a} = a \cdot\!+\!\cdot \alpha + A$ of idiom $\mathcal{A}$ is of carrier $\boldsymbol{a}$ when $A \in \mathcal{R}_{\boldsymbol{a}} \otimes \mathcal{A} = (\boldsymbol{a} \otimes I_{\mathcal{A}})(\mathcal{R} \otimes \mathcal{A})(\boldsymbol{a} \otimes I_{\mathcal{A}})$. Two carriers $\boldsymbol{a}, \boldsymbol{b}$ are* disjoint *when $\boldsymbol{ab} = 0$ $(= \boldsymbol{ba})$.*

Carriers take into account the *locative* aspects of GoI. The replacement of type $\mathbf{II}_1$ with $\mathbf{II}_\infty$ ensures that we have no worry about the existence of « enough » disjoint carriers.

The hyperfinite factor of type $\mathbf{II}_\infty$ is unique up to isomorphism (yet another result of Connes [1]). It admits a *semi-finite* trace, unique up to renormalisation: $\mathrm{tr}' = \lambda \mathrm{tr}$ for some $\lambda > 0$; one chooses such a trace once for all. When $\boldsymbol{a} \neq 0$ is a carrier, then $\mathcal{R}_{\boldsymbol{a}}$ is of type $\mathbf{II}_1$, thus isomorphic to the hyperfinite factor of that type; the only minor detail is that $\mathrm{tr} \restriction \mathcal{R}_{\boldsymbol{a}}$ is not normalised, since $\mathrm{tr}(\boldsymbol{a}) > 0$ has no reason to be equal to 1, but this hardly matters!

Although we should write expressions of the form $\mathrm{ldet}(\boldsymbol{a} \otimes \alpha(I_{\mathcal{A}}) - AB)$, etc., we shall content ourselves with $\mathrm{ldet}(I - AB)$, which is less pedantic and, anyway, perfectly correct if we think twice.

2.2 Duality

Definition 6 (Duality)
Let $\mathfrak{a} := a \cdot\!+\!\cdot \alpha + A$, $\mathfrak{b} := b \cdot\!+\!\cdot \alpha + B$ be alien *projects of carrier $\boldsymbol{a}$; one defines:*

$$\ll \mathfrak{a} \,|\, \mathfrak{b} \gg := a + b + \mathrm{ldet}(I - AB) \tag{28}$$

$\mathfrak{a}$ and $\mathfrak{b}$ are polar*, notation $\mathfrak{a} \underset{\sim}{\downarrow} \mathfrak{b}$ iff $\ll \mathfrak{a} \,|\, \mathfrak{b} \gg \neq 0, \pm\infty$.*

The determinant is relative to the pseudo-trace $(\mathrm{tr} \restriction \mathcal{R}_{\boldsymbol{a}}) \otimes \alpha$.

Proposition 5

$$\mathfrak{a} \underset{\sim}{\downarrow} \mathfrak{b} \quad \Leftrightarrow \quad \mathfrak{b} \underset{\sim}{\downarrow} \mathfrak{a}$$

An explicit formulation of (28), when $\mathfrak{a} := a \cdot\!+\!\cdot \alpha + A$, $\mathfrak{b} := b \cdot\!+\!\cdot \beta + B$, still of the same carrier $\boldsymbol{a}$, are not supposed to be alien:

$$\ll \mathfrak{a} \,|\, \mathfrak{b} \gg := a\beta(I_{\mathcal{B}}) + b\alpha(I_{\mathcal{A}}) + \mathrm{ldet}(I - A^{\ddagger}B^{\dagger}) \tag{29}$$

The equivalence between the two notions follows from the obvious:

Proposition 6

If $\mathfrak{a} \underset{\sim}{\downarrow} \mathfrak{b}$ (in the sense of (29)) and $\mathfrak{a}', \mathfrak{b}'$ are variants of $\mathfrak{a}, \mathfrak{b}$, then $\mathfrak{a}' \underset{\sim}{\downarrow} \mathfrak{b}'$.

In (28) $\ll \mathfrak{a} \,|\, \mathfrak{b} \gg \in \mathbb{R} \cup \{\alpha(I_{\mathcal{A}}) \cdot \infty\}$; polarity thus excludes the two values $0, \infty$. One should see this exclusion as the analogue of the correctness property of proof-nets ([10], ch. 11): connectedness and acyclicity respectively corresponding to the exclusion of the outputs 0 and ∞.

Definition 7 (Conducts)

A conduct $\mathbf{A}$ *of carrier* $\boldsymbol{a}$ *is a « set » of projects of carrier* $\boldsymbol{a}$ *equal to its bipolar.*

Of course, due to the use of arbitrary idioms, a conduct *cannot* be a set, but this remark is pure nonsense. Up to variance, conducts do form a set.

2.3 Partial projects

Besides the standard duality, there is a coarser one, based upon $\ll \mathfrak{a} \,|\, \mathfrak{b} \gg \neq \infty$, and whose antagonists are styled *partial*. Indeed, making full use of non-positive pseudo-traces, a conduct generates a vector space and the map $\ll \cdot \,|\, \cdot \gg$ extends into a bilinear form.

In what follows, $\mathbf{A}$ is a conduct of carrier $\boldsymbol{a}$.

Definition 8 (Partial projects)

If we relax faithfulness and the condition $\alpha(I_{\mathcal{A}}) \neq 0$ (we can thus even afford to have $\mathcal{A} = 0$), we obtain partial projects. *If $\mathfrak{a}_i := a_i \cdot\!+\!\cdot \alpha_i + A_i$ are partial projects of idioms $\mathcal{A}_i$, if $\lambda_i \in \mathbb{R}$ $(i = 1, \ldots, n)$, we define:*

$$\sum_1^n \lambda_i \cdot \mathfrak{a}_i \;:=\; \sum_1^n \lambda_i a_i \cdot\!+\!\cdot \bigoplus_1^n \lambda_i \alpha_i + \bigoplus_1^n A_i \tag{30}$$

of idiom $\bigoplus_1^n \mathcal{A}_i$. The set $\wp\mathbf{A}$ of partial projects of $\mathbf{A}$ *is the closure of* $\mathbf{A}$ *under finite linear combinations*[7]*.*

The binary function $\ll \cdot \,|\, \cdot \gg$ naturally extends into a function from $\wp\mathbf{A} \times \wp\!\sim\!\mathbf{A}$ into $\mathbb{R}$, for instance by means of the formula (29). We define the equivalence relation $\equiv_{\mathbf{A}}$ on $\wp\mathbf{A}$:

$$\mathfrak{a} \equiv_{\mathbf{A}} \mathfrak{b} \quad :\Leftrightarrow \quad \forall \mathfrak{c} \in \sim\!\mathbf{A} \;\; \ll \mathfrak{a} \,|\, \mathfrak{c} \gg = \ll \mathfrak{b} \,|\, \mathfrak{c} \gg \tag{31}$$

The typical case is that of an isovariant (definition 3): $\mathfrak{a} \equiv_{\mathbf{A}} \varphi(\mathfrak{a})$.

Theorem 4 (Linearisation)

The quotient $\ell\mathbf{A} := \wp\mathbf{A}/\!\equiv_{\mathbf{A}}$ is a real vector space. The application $\ll \cdot \,|\, \cdot \gg$ from $\ell\mathbf{A} \times \ell\!\sim\!\mathbf{A}$ to $\mathbb{R}$ is bilinear.

Proof : in (31), one can replace « $\forall \mathfrak{d} \in \sim\!\mathbf{A}$ » with « $\forall \mathfrak{d} \in \wp\!\sim\!\mathbf{A}$ ». □

[7] $\mathbf{A}$ is anyway closed under non-zero homotheties.

Definition 9 (Internal completeness)
An ethics *of carrier* $\boldsymbol{a}$ *is any « set »* $\mathbf{E}$ *of projects of carrier* $\boldsymbol{a}$*;* $\mathbf{E}$ *generates a conduct, namely the bipolar* $\mathbf{A} := \sim\sim\mathbf{E}$*. The ethics* $\mathbf{E}$ *is said to be* complete *when any equivalence class of projects in* $\mathbf{A}$ *has a witness in* $\mathbf{E}$*:*

$$\forall \mathfrak{a} \in \mathbf{A}\ \exists \mathfrak{e} \in \mathbf{E} \quad \mathfrak{a} \equiv_{\mathbf{A}} \mathfrak{e} \tag{32}$$

Theorem 5 (Ethic lemma)
If $\alpha(I_{\mathcal{A}}) = \beta(I_{\mathcal{B}})$*, one can replace in (31) «* $\forall \mathfrak{d} \in \sim\mathbf{A}$ *» with «* $\forall \mathfrak{d} \in \mathbf{E}$ *», where* $\mathbf{E}$ *is any ethics for* $\sim\mathbf{A}$*.*

Proof : let $\mathfrak{c} \in \mathbf{A}$; then $\mathfrak{a} \equiv_{\mathbf{A}} \mathfrak{b}$ iff for all $\lambda \in \mathbb{R}$ $\lambda\mathfrak{a} - \lambda\mathfrak{b} + \mathfrak{c} \in \mathbf{A}$, i.e., $\lambda\mathfrak{a} - \lambda\mathfrak{b} + \mathfrak{c} \in \sim\mathbf{E}$, thus, iff for all $\mathfrak{c} \in \mathbf{E}$ $\ll \mathfrak{a} \,|\, \mathfrak{c} \gg = \ll \mathfrak{b} \,|\, \mathfrak{c} \gg$. □

The condition « $\alpha(I_{\mathcal{A}}) = \beta(I_{\mathcal{B}})$ » makes sure that $\lambda\mathfrak{a} - \lambda\mathfrak{b} + \mathfrak{c}$ is a project; in the polarised case, *rescaling* (definition 17) renders this restriction pointless.

2.4 Images and projections

The inclusion $\mathbf{A} \subset \mathbf{A} \oplus \mathbf{B}$ cannot make sense *stricto sensu* for questions of carrier. However, a project of carrier $\boldsymbol{a}$ can be seen as a project of carrier $\boldsymbol{a} + \boldsymbol{b}$. Whence the notion of *injection*, which is not problematic at the level of projects or ethics. But, injection does not commute at all with negation; its converse, *projection*, is better behaved.

Let $\boldsymbol{a}, \boldsymbol{b}$ be carriers, then:

Definition 10 (Images)
If $\Phi \in \mathcal{R}$*,* $\|\Phi\| \leq 1$*, is such that* $\Phi = \boldsymbol{b}\Phi$*, the* image *under* Φ *of a project* $\mathfrak{a} = a \cdot\!+\!\cdot \alpha + A$ *of carrier* $\boldsymbol{a}$ *is the project* $\Phi(\mathfrak{a}) := a \cdot\!+\!\cdot \alpha + (\Phi \otimes I_{\mathcal{A}})A(\Phi^* \otimes I_{\mathcal{A}})$ *of carrier* $\boldsymbol{b}$*. If* $\mathbf{E}$ *is an ethics of carrier* $\boldsymbol{a}$*, its* image *under* Φ *is the ethics* $\Phi(\mathbf{E}) := \{\Phi(\mathfrak{a}); \mathfrak{a} \in \mathbf{E}\}$ *of carrier* $\boldsymbol{b}$*.*

Example 1
The natural example is that of a projection $\Phi := \boldsymbol{b}$; two subcases are of interest:

Projection: if $\boldsymbol{b} \subset \boldsymbol{a}$, then $\boldsymbol{b}(\mathfrak{a})$, noted $\mathfrak{a}_{\boldsymbol{b}}$, is the *projection* of $\mathfrak{a}$ on $\boldsymbol{b}$; if $\mathbf{E}$ is an ethics of carrier $\boldsymbol{a}$, its *projection* on $\boldsymbol{b}$ is $\mathbf{E}_{\boldsymbol{b}} := \{\mathfrak{a}_{\boldsymbol{b}}; \mathfrak{a} \in \mathbf{E}\}$.

Injection: if $\boldsymbol{a} \subset \boldsymbol{b}$, then $\boldsymbol{b}(\mathfrak{a})$ $(= \mathfrak{a})$, noted $\mathfrak{a}_{\boldsymbol{b}}$ is the *injection* of $\mathfrak{a}$ in $\boldsymbol{b}$; if $\mathbf{E}$ is an ethics of carrier $\boldsymbol{a}$, its *injection* in $\boldsymbol{b}$ is $\mathbf{E}_{\boldsymbol{b}} := \{\mathfrak{a}_{\boldsymbol{b}}; \mathfrak{a} \in \mathbf{E}\}$.

Definition 11 (Faithfulness)
A subcarrier $\boldsymbol{b} \subset \boldsymbol{a}$ *is* $\mathbf{A}$-faithful *(w.r.t. a conduct* $\mathbf{A}$ *of carrier* $\boldsymbol{a}$*) when, for all* $\mathfrak{a} \in \mathbf{A}$*,* $\mathfrak{a}_{\boldsymbol{b}} \in \mathbf{A}$ *and* $\mathfrak{a}_{\boldsymbol{b}} \equiv \mathfrak{a}$*.*

Proposition 7
Let $\boldsymbol{b} \subset \boldsymbol{a}$ *be* $\mathbf{A}$*-faithful; then:*

(i) For all $\mathfrak{a} \in \wp\mathbf{A}$*,* $\mathfrak{a}_{\boldsymbol{b}} \in \wp\mathbf{A}$ *and* $\mathfrak{a}_{\boldsymbol{b}} \equiv \mathfrak{a}$*.*

(ii) $\boldsymbol{b}$ *is* $\sim\mathbf{A}$*-faithful.*

(iii) $\mathbf{A}_{\boldsymbol{b}}$ *is a conduct.*

(iv) $(\sim\mathbf{A})_{\boldsymbol{b}} = \sim(\mathbf{A}_{\boldsymbol{b}})$.

Proof : (i) Immediate.

(ii) Using *(i)* and the general property:

$$\ll \mathfrak{a}_{\boldsymbol{b}} \,|\, \mathfrak{b} \gg \;=\; \ll \mathfrak{a} \,|\, \mathfrak{b}_{\boldsymbol{b}} \gg \;=\; \ll \mathfrak{a}_{\boldsymbol{b}} \,|\, \mathfrak{b}_{\boldsymbol{b}} \gg \tag{33}$$

(iii) By the $\sim\mathbf{A}$-faithfulness of $\boldsymbol{b}$ and (33), the polar of $(\sim\mathbf{A})_{\boldsymbol{qp}}$, i.e., the set of projects of carrier $\boldsymbol{a}$ polar to the $\mathfrak{b}_{\boldsymbol{b}}$ ($\mathfrak{b} \in \sim\mathbf{A}$) is equal to $\mathbf{A}$. Whence the polar of $(\sim\mathbf{A})_{\boldsymbol{b}}$ is equal to $\mathbf{A}_{\boldsymbol{b}}$. This also proves *(iv)*. □

Proposition 8

$$\mathfrak{a}_{\boldsymbol{b}} \overset{\downarrow}{\sim} \mathfrak{b} \quad\Leftrightarrow\quad \mathfrak{a} \overset{\downarrow}{\sim} \mathfrak{b}_{\boldsymbol{b}} \quad\Leftrightarrow\quad \mathfrak{a}_{\boldsymbol{b}} \overset{\downarrow}{\sim} \mathfrak{b}_{\boldsymbol{b}}$$

Example 2
If $\boldsymbol{b} \subset \boldsymbol{a}$, a conduct $\mathbf{A}$ of carrier $\boldsymbol{b}$ induces the two dual injections of carrier $\boldsymbol{a}$: $\sim\sim\mathbf{A}_{\boldsymbol{a}}$ and $\sim(\sim\mathbf{A})_{\boldsymbol{a}}$. However, $\boldsymbol{b}$ faithfully projects both injections onto $\mathbf{A}$.

3 The social life of conducts

3.1 Multiplicatives

Let $\boldsymbol{a}, \boldsymbol{b}$ be disjoint carriers.

Definition 12 (Application)
If $\mathfrak{a} := a \cdot\!+\!\cdot \alpha + A$ *and* $\mathfrak{f} := f \cdot\!+\!\cdot \alpha + F$ *are alien projects of respective carriers* $\boldsymbol{a}, \boldsymbol{a}+\boldsymbol{b}$ *and idiom* $\mathcal{A}$*, one defines the project* $[\mathfrak{f}]\mathfrak{a}$ *of carrier* $\boldsymbol{b}$ *and idiom* $\mathcal{A}$*:*

$$[\mathfrak{f}]\mathfrak{a} := f + a + \mathrm{ldet}(I - FA) \cdot\!+\!\cdot \alpha + [F]A \tag{34}$$

where $[F]A$ *has been been defined in section 1.*

An explicit formulation of the same thing, when $\mathfrak{a} := a \cdot\!+\!\cdot \alpha + A$ and $\mathfrak{f} := f \cdot\!+\!\cdot \varphi + F$ are not assumed to be alien, is the project of idiom $\mathcal{F} \otimes \mathcal{A}$:

$$[\mathfrak{f}]\mathfrak{a} := f\alpha(I_{\mathcal{A}}) + a\varphi(I_{\mathcal{F}}) + \mathrm{ldet}(I - F^{\ddagger}A^{\dagger})) \cdot\!+\!\cdot (\varphi \otimes \alpha) + [F^{\ddagger}]A^{\dagger} \tag{35}$$

Definition 13 (Multiplicatives)
If $\mathbf{A}, \mathbf{B}$ *are conducts of carriers* $\boldsymbol{a}, \boldsymbol{b}$*, one defines the conducts of carrier* $\boldsymbol{a}+\boldsymbol{b}$*:*

$$\mathbf{A} \multimap \mathbf{B} \;:=\; \{\mathfrak{f} \in \mathbf{A}\ ;\ \forall \mathfrak{a} \in \mathbf{A} \quad [\mathfrak{f}]\mathfrak{a} \in \mathbf{B}\} \tag{36}$$

$$\mathbf{A} \otimes \mathbf{B} \;:=\; \sim(\mathbf{A} \multimap \sim\mathbf{B}) \tag{37}$$

$$\mathbf{A} \parr \mathbf{B} \;:=\; \sim\mathbf{A} \multimap \mathbf{B} \tag{38}$$

THEOREM 6 (ADJUNCTION)

$$\mathbf{A} \otimes \mathbf{B} \;=\; \sim\sim(\mathbf{A} \odot \mathbf{B}) \;:=\; \sim\sim\{\mathfrak{a} \otimes \mathfrak{b}\;;\; \mathfrak{a} \in \mathbf{A},\, \mathfrak{b} \in \mathbf{B}\} \tag{39}$$

with $\mathfrak{a} \otimes \mathfrak{b} := a{+}b \cdot{+}\cdot \alpha{+}(A{+}B)$ *when* $\mathfrak{a} \in \mathbf{A}, \mathfrak{b} \in \mathbf{B}$ *are alien.*

Proof : not quite a surprise: this is a by-product of theorem 3. □

COROLLARY 6.1
The tensor product is commutative, associative, with neutral element the conduct $\top := \{0 \cdot{+}\cdot \alpha + 0\;;\; \alpha \,\text{pseudo} - \text{trace on some idiom}\, \mathcal{A}\}$ *of carrier* 0.

The neutral element of $⅋$ is $\mathbf{0} := \{a \cdot{+}\cdot \alpha + 0\;;\; a \neq 0,\, \alpha\,\text{pseudo} - \text{trace}\ldots\}$.

REMARK 1
It is useful to rephrase the previous results in terms of *ethics*: $\mathbf{E} \odot \mathbf{F}$ and $\mathbf{E} \multimap \mathbf{F}$ can still be defined when $\mathbf{E}, \mathbf{F}$ are ethics. Observe that:

$$\sim\sim(\mathbf{E} \odot \mathbf{F}) \;=\; (\sim\sim\mathbf{E}) \otimes (\sim\sim\mathbf{F}) \tag{40}$$

$$\mathbf{E} \multimap (\sim\sim\mathbf{F}) \;=\; (\sim\sim\mathbf{E}) \multimap (\sim\sim\mathbf{F}) \tag{41}$$

PROPOSITION 9

$$\wp(\mathbf{A} \multimap \mathbf{B}) = \wp\mathbf{A} \multimap \wp\mathbf{B}$$

3.2 Quantifiers

Let $\mathbb{I} \neq \emptyset$ be a non empty index set (usually uncountable).

DEFINITION 14 (UNIVERSAL QUANTIFICATION)
If $\mathbf{A}[i] (i \in \mathbb{I})$ *is a family of conducts of carrier* $\boldsymbol{a}$*, then* $\forall i \!\in\! \mathbb{I}\; \mathbf{A}[i]$ *is the conduct of carrier* $\boldsymbol{a}$ *defined by:*

$$\forall i \!\in\! \mathbb{I}\; \mathbf{A}[i] \quad := \quad \bigcap_i \mathbf{A}_i \tag{42}$$

The definition makes sense because of:

PROPOSITION 10
Any intersection of conducts of carrier $\boldsymbol{a}$ *is a conduct of carrier* $\boldsymbol{a}$.

Proof : since an intersection of polars is the polar of a union:

$$\bigcap_i \sim\mathbf{E}[i] \quad = \quad \sim\bigcup_i \mathbf{E}[i]$$

□

PROPOSITION 11

$$\wp\forall i \!\in\! \mathbb{I}\; \mathbf{A}[i] \;=\; \bigcap_i \wp\mathbf{A}_i$$

THEOREM 7 (DISTRIBUTIVITY)

$$\mathbf{A} \multimap \forall i \in \mathbb{I}\ \mathbf{B}[i] \ = \ \forall i \in \mathbb{I}\ (\mathbf{A} \multimap \mathbf{B}[i]) \tag{43}$$

REMARK 2
Existential quantification is defined dually as $\exists i \in \mathbb{I}\ \mathbf{A}[i] := \sim\sim\bigcup_{i\in\mathbb{I}} \mathbf{A}[i]$. In terms of ethics, the following remark is useful:

$$\sim\sim\bigcup_{i\in\mathbb{I}} \mathbf{E}_i \quad = \quad \exists i \in \mathbb{I} \sim\sim\mathbf{E}[i] \tag{44}$$

Second order quantification is treated in appendix, section A.

3.3 Additives

Let $\boldsymbol{a}, \boldsymbol{b}$ be disjoint carriers.

DEFINITION 15 (ADDITIVES)
If $\mathbf{A}, \mathbf{B}$ are conducts of respective carriers $\boldsymbol{a}, \boldsymbol{b}$, we define:

$$\mathbf{A} \oplus \mathbf{B} \ := \ \sim\sim(\mathbf{A}_{\boldsymbol{a}+\boldsymbol{b}} \cup \mathbf{B}_{\boldsymbol{a}+\boldsymbol{b}}) \tag{45}$$
$$\mathbf{A} \,\&\, \mathbf{B} \ := \ \sim(\sim\mathbf{A}_{\boldsymbol{a}+\boldsymbol{b}}) \cap \sim(\sim\mathbf{B}_{\boldsymbol{a}+\boldsymbol{b}}) \tag{46}$$

PROPOSITION 12
The two definitions are dual, i.e.:

$$\sim(\mathbf{A} \oplus \mathbf{B}) \quad = \quad \sim\mathbf{A} \,\&\, \sim\mathbf{B}$$

Additives are commutative, associative, with as respective neutrals, the void conduct ($\oplus$) and the full conduct ($\&$) of carrier 0.

Little more can be said; a good transition towards *polarisation*.

4 Polarised conducts

4.1 Polarisation

DEFINITION 16 (DAIMON)
If $a \in \mathbb{R}$, the project $\mathfrak{Dai}_a := a \cdot\!+\!\cdot 1 + 0$, of idiom $\mathbb{C}$ and pseudo-trace $1(z) := z$ is called a daimon*;* proper *if $a \neq 0$.*

DEFINITION 17 (POLARISED CONDUCTS)
A conduct $\mathbf{A}$ is positive *when it enjoys the following:*

Daimon: $\mathbf{A}$ *contains all proper* daimons $\mathfrak{Dai}_a, a \neq 0$.

Rescaling: *if $a, b \neq 0$ and $a \cdot\!+\!\cdot \alpha + A \in \mathbf{A}$, then $b \cdot\!+\!\cdot \alpha + A \in \mathbf{A}$.*

Negative conducts are defined as the polars of positive conducts; a conduct is polarised *when it is either positive or negative.*

Proposition 13
A conduct $\mathbf{A}$ *is* negative *iff it enjoys the following:*

Wager: *all projects of* $\mathbf{A}$ *are* wager-free, *i.e., with a null wager.*

Rescaling: *if* $\mathfrak{a} \in \mathbf{A}$ *and* $\lambda \neq -\alpha(I_{\mathcal{A}})$*, then* $\mathfrak{a} + \lambda\mathfrak{Dai}_0 \in \mathbf{A}$*.*

Proof : $\ll a \cdot\!+\!\cdot 1 + 0 \,|\, b \cdot\!+\!\cdot \beta + B \gg = a\beta(I_{\mathcal{B}}) + b$; since $\beta(I_{\mathcal{B}}) \neq 0$, it turns out that $b \cdot\!+\!\cdot \beta + B$ is polar to all proper *daimons* iff $b = 0$. If $\mathfrak{a} := a \cdot\!+\!\cdot \alpha + A \in \mathbf{A}$ with $a \neq 0$, then $\ll \mathfrak{a} \,|\, \mathfrak{b} + \lambda\mathfrak{Dai}_0 \gg = \ll \mathfrak{a} \,|\, \mathfrak{b} \gg + \lambda a \neq 0$ for all $\lambda \neq -\beta(I_{\mathcal{B}})$ iff $\ll \mathfrak{a} \,|\, \mathfrak{b} \gg = a\beta(I_{\mathcal{B}})$, i.e., iff the component $\mathrm{ldet}(I - A^{\ddagger}B^{\dagger})$ is null. Whence the equivalence between the two rescaling conditions. □

Remark 3
Negative rescaling can be understood as the closure under non-unital *variants* (definition 3); by the way, negative rescaling holds for positive conducts too.

Corollary 13.1
If an ethics $\mathbf{A}$ *is positive (in the obvious sense), so is its bipolar.*

Proof : the conditions « daimon » and « rescaling » induce by duality conditions « wager » and « rescaling » on $\sim\mathbf{A}$ which, in turn, induce conditions « daimon » and « rescaling » on $\sim\sim\mathbf{A}$. □

Corollary 13.2
All non wager-free projects of a positive conduct $\mathbf{A}$ *are homothetic as elements of the vector space* $\ell\mathbf{A}$*.*

Proof : if $a \neq 0$, then $\ll \mathfrak{a} \mid \mathfrak{b} \gg = a\beta(I_{\mathcal{B}}) = \ll \mathfrak{Dai}_a \mid \mathfrak{b} \gg$ (proof of proposition 13), whence $\mathfrak{a} \equiv_{\mathbf{A}} \mathfrak{Dai}_a$. The $\mathfrak{Dai}_a$ are pairwise homothetic. □

Definition 18 (Proper conducts)
A positive conduct $\mathbf{A}$ *is* proper *when it does not contain the improper* daimon $\mathfrak{Dai}_0$*. A negative conduct* $\mathbf{A}$ *is* proper *when non empty.*

Proposition 14
The polar of a proper polarised conduct is proper.

Proof : $\ll 0 \cdot\!+\!\cdot 1 + 0 \,|\, 0 \cdot\!+\!\cdot \beta + B \gg = 0$, whence a mutual exclusion. □

4.2 Polarisation of multiplicatives

Polarisation is reasonably compatible with multiplicative constructions, although their « table of polarities » is quite unexpected.

Proposition 15
Assume that $\mathbf{A}, \mathbf{B}$ *are polarised conducts with disjoint carriers* $\boldsymbol{a}, \boldsymbol{b}$*:*

(i) If both are negative, $\mathbf{A} \otimes \mathbf{B}$ *is negative; and proper if both are proper.*

(ii) If $\mathbf{A}$ *is positive, if* $\mathbf{B}$ *is negative and proper, then* $\mathbf{A} \otimes \mathbf{B}$ *is positive; and proper in case* $\mathbf{A}$ *is proper.*

(iii) If both are positive, then $\mathbf{A} \otimes \mathbf{B}$ *is positive and unproper.*

Proof : $- \otimes - = -$**:** immediate.

$+ \otimes - = +$**:** if $\mathbf{A}$ is positive, if $\mathbf{B}$ is proper negative, let $\mathfrak{b} := b \cdot\!+\!\cdot \beta + B \in \mathbf{B}$; then $\mathfrak{Dai}_a \otimes (\mathfrak{b} + \lambda \mathfrak{Dai}_0) \in \mathbf{A} \otimes \mathbf{B}$. If $\mathfrak{c} := c \cdot\!+\!\cdot \gamma + C \in \sim\!(\mathbf{A} \otimes \mathbf{B})$, then $\ll \mathfrak{Dai}_a \otimes (\mathfrak{b} + \lambda \mathfrak{Dai}_0) \mid \mathfrak{c} \gg = a\gamma(I_{\mathcal{C}}) + c\beta(I_{\mathcal{B}} + \lambda) + \mathrm{ldet}(I - B^{\ddagger} C^{\dagger})$ can be nullified by an appropriate choice of a and λ, unless $c = 0$: this proves « daimon ». Moreover, since $\mathbf{A} \odot \mathbf{B}$ (theorem 6) obviously enjoys rescaling, so does its bipolar. If $\mathbf{A}$ is proper and $\mathfrak{c} \in \sim\!\mathbf{A}$ is total, then it is immediate (this is indeed *weakening*, section 5.1 *infra*) that $\mathfrak{c} \in \sim\!(\mathbf{A} \otimes \mathbf{B})$.

$+ \otimes + = u$**:** $\mathbf{A} \otimes \mathbf{B}$ contains all $(a + b) \cdot\!+\!\cdot 1 + 0$. The tensor product contains all the $(a+b) \cdot\!+\!\cdot 1+0$, with $a, b \neq 0$, hence $0 \cdot\!+\!\cdot 1 + 0$. Dually, $[\mathfrak{f}]\cdot$ cannot send all the $a \cdot\!+\!\cdot 1 + 0$ into something wager-fre. □

Consistently proposition 14, the neutral $\top$ of corollary 6.1 is negative.

Let us restate the polarity table for linear implication in the *proper* case:

(i) If $\mathbf{A}$ is negative and $\mathbf{B}$ is positive, then $\mathbf{A} \multimap \mathbf{B}$ is positive.

(ii) If $\mathbf{A}, \mathbf{B}$ have the same polarity, then $\mathbf{A} \multimap \mathbf{B}$ is negative.

(iii) If $\mathbf{A}$ is positive and $\mathbf{B}$ is negative, then $\mathbf{A} \multimap \mathbf{B}$ is unproper.

In terms of cotensor, the important thing to memorise is that a n-ary « par » $\mathbf{A}_1 \parr \ldots \parr \mathbf{A}_n$ of proper polarised conducts is proper iff at most one of them is negative, consistently with the maintenance of *sequents* in ludics [9].

4.3 Additives

Let $\boldsymbol{a}, \boldsymbol{b}$ be disjoint carriers; if $\mathfrak{f}, \mathfrak{g}$ are wager-free projects of carrier $\boldsymbol{a} + \boldsymbol{b}$, define $\mathfrak{f} \,\&\, \mathfrak{g} := \mathfrak{f} + \mathfrak{g}$, provided $\alpha(I_{\mathcal{A}}) + \beta(I_{\mathcal{B}}) \neq 0$.

Proposition 16

If $\mathbf{A}, \mathbf{B}$ *are positive conducts of respective carriers* $\boldsymbol{a}, \boldsymbol{b}$*, then:*

$$\sim\!\{\mathfrak{f} \,\&\, \mathfrak{g} \;;\; \mathfrak{f} \in \sim\!\mathbf{A}, \mathfrak{g} \in \sim\!\mathbf{B}\} = \sim\!\sim\!\mathbf{A}_{\boldsymbol{a}+\boldsymbol{b}} \,\cup \sim\!\sim\!\mathbf{B}_{\boldsymbol{a}+\boldsymbol{b}} \tag{47}$$

Proof : if $\mathfrak{a} \underset{\sim}{\downarrow} \mathfrak{f} \,\&\, \mathfrak{g}$, then $\ll \mathfrak{a} \mid x\mathfrak{f} + \lambda \mathfrak{Dai}_0 \gg + \ll \mathfrak{a} \mid y\mathfrak{g} + \mu \mathfrak{Dai}_0 \gg \neq 0$, for any $x, y \neq 0$ and *ad hoc* λ, μ. Then $x \ll \mathfrak{a} \mid \mathfrak{f} \gg + y \ll \mathfrak{a} \mid \mathfrak{g} \gg \neq 0$ for all $x, y \neq 0$, whence one and only one among $\ll \mathfrak{a} | \mathfrak{f} \gg, \ll \mathfrak{a} | \mathfrak{g} \gg$ is nonzero. Since $\mathfrak{f}, \mathfrak{g}$ are not related, the choice is always the same, i.e., either $\mathfrak{a} \in \sim\!\sim\!\mathbf{A}_{\boldsymbol{a}+\boldsymbol{b}}$ or $\mathfrak{a} \in \sim\!\sim\!\mathbf{B}_{\boldsymbol{a}+\boldsymbol{b}}$. The converse inclusion is almost immediate. □

Theorem 8 (Disjunction property)

$\mathbf{A}_{\boldsymbol{a}+\boldsymbol{b}} \,\cup\, \mathbf{B}_{\boldsymbol{a}+\boldsymbol{b}}$ *is a complete ethics for* $\mathbf{A} \oplus \mathbf{B}$*.*

Proof : by the proposition and example 2. □

THEOREM 9 (MYSTERY OF INCARNATION)
$\{\mathfrak{a}\,\&\,\mathfrak{b}\ ;\ \mathfrak{a}\in\mathbf{A},\mathfrak{b}\in\mathbf{B}\}$ *is a complete ethics for* $\mathbf{A}\,\&\,\mathbf{B}$.

Proof : both results are immediate, *modulo* the *ethic lemma* (theorem 5). □

COROLLARY 9.1
If $\mathbf{A},\mathbf{B}$ *are positive (resp. negative) and proper, so is* $\mathbf{A}\oplus\mathbf{B}$ *(resp.* $\mathbf{A}\,\&\,\mathbf{B}$*).*

PROPOSITION 17
$\&$ *is (literally) commutative, associative, with as unit the tensor unit* $\top$*.*

REMARK 4
The theorem was named « mystery of incarnation » in view of its obvious analogy with the *mystery of incarnation* of ludics [9]. The general idea of incarnation corresponds to a *conditional expectation* mapping a conduct onto another one. Such a theory of incarnations (no longer « the » incarnation) is still to be written.

4.4 Distributivity

For questions of carrier, $\invamp$ cannot *literally* distribute over $\&$. However, If $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{e},\boldsymbol{f},\boldsymbol{g}$ are disjoint carriers and u (resp. v) is a partial isometry from $\boldsymbol{a}+\boldsymbol{b}$ (resp. $\boldsymbol{a}+\boldsymbol{c}$) onto $\boldsymbol{d}+\boldsymbol{e}$ (resp. $\boldsymbol{f}+\boldsymbol{g}$) s.t. $u\boldsymbol{a}=\boldsymbol{d}u$ (resp. $v\boldsymbol{a}=\boldsymbol{f}v$), consider $\mathfrak{Distr}:=0\cdot\!+\!\cdot(1\oplus 1)+((u+u^*)\oplus(v+v^*))$ of idiom $\mathbb{C}\oplus\mathbb{C}$. Then, if $\mathbf{A},\mathbf{B},\mathbf{C}$ are negative conducts of respective carriers $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$:

(i) $\mathfrak{Distr}\in(\mathbf{A}\multimap\mathbf{B}\,\&\,\mathbf{C})\ \multimap\ (u(\mathbf{A}\multimap\mathbf{B})\,\&\,v(\mathbf{A}\multimap\mathbf{C}))$.

(ii) $\mathfrak{Distr}\in u(\mathbf{A}\multimap\mathbf{B})\,\&\,v(\mathbf{A}\multimap\mathbf{C})\ \multimap\ ((\mathbf{A}\multimap\mathbf{B}\,\&\,\mathbf{C}))$.

(iii) If $\mathfrak{f}=f\cdot\!+\!\cdot\varphi+F\in(\mathbf{A}\multimap\mathbf{B}\,\&\,\mathbf{C})$, then $[\mathfrak{Distr}]([\mathfrak{Distr}]\mathfrak{f})=$
$f\cdot\!+\!\cdot(\varphi\oplus\varphi\oplus\varphi\oplus\varphi)+((\boldsymbol{a}+\boldsymbol{b})F(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{a}F\boldsymbol{a}+\boldsymbol{a}F\boldsymbol{a}+(\boldsymbol{a}+\boldsymbol{c})F(\boldsymbol{a}+\boldsymbol{c}))$, which is $\equiv$ to $\mathfrak{f}$ when $f=0$.

(iv) If $\mathfrak{g}=g\cdot\!+\!\cdot\psi+G\in u(\mathbf{A}\multimap\mathbf{B})\,\&\,v(\mathbf{A}\multimap\mathbf{C})$, then $[\mathfrak{Distr}]([\mathfrak{Distr}]\mathfrak{g})=$
$g\cdot\!+\!\cdot(\psi\oplus\psi\oplus\psi\oplus\psi)+((\boldsymbol{d}+\boldsymbol{e})G(\boldsymbol{d}+\boldsymbol{e})+\boldsymbol{d}G\boldsymbol{d}+\boldsymbol{f}G\boldsymbol{f}+(\boldsymbol{f}+\boldsymbol{g})G(\boldsymbol{f}+\boldsymbol{g}))$, which is $\equiv$ to $\mathfrak{g}$ when $g=0$.

$\mathfrak{Distr}$ therefore implements a sort of distributivity, up to $\equiv$. By the way, the possibility to neglect the parasitic expressions $\boldsymbol{a}F\boldsymbol{a}$, $\boldsymbol{d}G\boldsymbol{d}$, $\boldsymbol{f}G\boldsymbol{f}$ is a pure result of polarisation: for instance, if $f=0$ and $\mathfrak{a}=0\cdot\!+\!\cdot\alpha+A\in\mathbf{A}$, the wager of $[\mathfrak{f}]\mathfrak{a}$ must be 0, whence $\mathrm{ldet}(I-A^{\ddagger}F^{\dagger})=0$.

5 Exponentials

The polarised exponentials turn out to be **ELL**-style ([10], ch. 16).

5.1 Structural rules

Polarisation enables weakening in the positive case.

PROPOSITION 18
If $\mathfrak{c} \in \mathbf{A} \otimes \mathbf{B}$, where $\mathbf{B}$ is of carrier $\boldsymbol{b}$ and $\mathbf{A}$ is negative, then $\mathfrak{c}_{\boldsymbol{b}} \in \mathbf{B}$.

Proof : a project $\mathfrak{f} \in \sim\mathbf{B}$ induces a « function » $\mathfrak{f}_{\boldsymbol{a}+\boldsymbol{b}} \in \mathbf{A} \multimap \sim\mathbf{B}$: since $\mathfrak{a} \in \mathbf{A}$ is wager-free, $[\mathfrak{f}_{\boldsymbol{a}+\boldsymbol{b}}]\mathfrak{a} = \mathfrak{f}$. Now, $\ll\mathfrak{f}\,|\,\mathfrak{c}_{\boldsymbol{b}}\gg = \ll\mathfrak{f}_{\boldsymbol{a}+\boldsymbol{b}}\,|\,\mathfrak{c}\gg \neq 0, \infty$. □

But contraction fails in presence of idioms: $\mathfrak{a}' \otimes \mathfrak{a}$" cannot be written $[\mathfrak{f}]\mathfrak{a}$: this would require something like $\mathcal{F} \otimes \mathcal{A} \simeq \mathcal{A} \otimes \mathcal{A}$ for all A, hopeless!

5.2 Perennial conducts

DEFINITION 19 (PERENNIALITY)
A project is perennial *when of the form $0 \cdot + \cdot 1 + A$. A* perennial *ethics is a negative ethics made of perennial projects. A* perennial *conduct is the bipolar of a perennial ethics; it is therefore negative.*

Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ be disjoint carriers and let u, v be partial isometries between $\boldsymbol{a}$ and (respectively) $\boldsymbol{b}, \boldsymbol{c}$. Consider the idiom $\mathcal{M}_2(\mathbb{C})$ and, as pseudo-trace, the normalised trace tr on $\mathcal{M}_2(\mathbb{C})$, so that $\mathcal{R} \otimes \mathcal{M}_2(\mathbb{C}) \simeq \mathcal{M}_2(\mathcal{R})$. We define the project $\mathfrak{Contr} := 0 \cdot + \cdot \operatorname{tr} + \begin{bmatrix} u + u^* & v \\ v^* & 0 \end{bmatrix}$. Then:

THEOREM 10 (CONTRACTION)
If $\mathbf{A}$ is a perennial conduct of carrier $\boldsymbol{a}$, then $\mathfrak{Contr}$ implements a rescaling of the map $\mathfrak{a} \rightsquigarrow u(\mathfrak{a}) \otimes v(\mathfrak{a})$. In particular $\mathfrak{Contr} \in \mathbf{A} \multimap (u(\mathbf{A}) \otimes v(\mathbf{A}))$.

Proof : if $\mathfrak{a} = 0 \cdot + \cdot 1 + A$, then $[\mathfrak{Contr}]\mathfrak{a} = 0 \cdot + \cdot \operatorname{tr} + \begin{bmatrix} uAu^* + vAv^* & 0 \\ 0 & 0 \end{bmatrix}$, which is a rescaling of $u(\mathfrak{a}) \otimes v(\mathfrak{a})$ by means of the map $z \rightsquigarrow \begin{bmatrix} z & 0 \\ 0 & 0 \end{bmatrix}$. □

Notice the use of a non-commutative idiom.

5.3 An amenable group

A type $\mathbf{II_1}$ vN algebra $\mathcal{A}[\mathfrak{G}]$ is hyperfinite iff $\mathfrak{G}$ is *amenable* (section C.8).

Commutative groups, locally finite groups are amenable. Amenability is stable under most constructions, in particular semi-direct products, with one major exception, free groups, see [11], 8.7.30.

PROPOSITION 19
There exists an amenable group $\mathfrak{G}$ containing a copy of the free monoid generated by two elements `l`, `r`

Proof : since $\mathbb{Z}$ is amenable, the semi-direct product $\mathfrak{G} = \mathbb{Z}^{|\mathbb{Z}|} \rtimes \mathbb{Z}$ is amenable. $\mathfrak{G}$ is the set of all $((s_k), m)$, where (s_k) is a function from $\mathbb{Z}$ to $\mathbb{Z}$ almost always null, and $m \in \mathbb{Z}$, with $((s_k), m)((t_k), n) := ((s_k + t_{m+k}), n + m)$. It admits $\mathtt{l} := ((\delta_{0k}), 0)$ and $\mathtt{r} := ((0_k), 1)$ generate a free monoid: let $c_k := n_k$ for $0 \leq k \leq p$, let $c_k = 0$ otherwise; then $\mathtt{l}^{n_0}\mathtt{r}\mathtt{l}^{n_1}\mathtt{r}\ldots\mathtt{r}\mathtt{l}^{n_p} = ((c_k), p)$. $\square$

5.4 Perennialisation

Here $\mathcal{H}$ stands for the hyperfinite factor (of type $\mathbf{II}_1$). The idea of perennialisation is first to standardise idioms, replacing them — when possible — with the hyperfinite factor, then to exploit the isomorphism $\mathcal{R} \otimes \mathcal{H} \simeq \mathcal{R}$. There is a first difficulty, namely that idioms are finite-dimensional ; howevever, one could easily accept idioms which are both finite and hyperfinite.

Definition 20 (Extended projects)
An extended project *is a sort of project where the idiom space $\mathcal{A}$ is both finite and hyperfinite, i.e., embeddable in $\mathcal{H}$. The polarity between extended projects is defined in the obvious way. If $\mathbf{A}$ is a conduct, then the extended project $\mathfrak{a}$ is* affiliated *to $\mathbf{A}$, notation $\mathfrak{a} \ \eta \ \mathbf{A}$ iff $\mathfrak{a}$ is polar to $\sim\mathbf{A}$.*

Definition 21 (Connectedness)
A project, extended or not, $\mathfrak{a} := a \cdot\!+\!\cdot \alpha + A$ is connected *when its pseudo-trace is positive. An extended project is* standard *when its idiom is $\mathcal{H}$, the pseudo-trace being* tr. *Any connected project $\mathfrak{a}$ can be* standardised, *i.e., replaced with the extended $\varphi(\mathfrak{a})$, where φ is any $*$-isomorphism of $\mathcal{A}$ into $\mathcal{H}$ such that $\mathrm{tr} \circ \varphi = \alpha$. If $\mathfrak{a} \in \mathbf{A}$ is connected, then $\varphi(\mathfrak{a}) \ \eta \ \mathbf{A}$.*

Definition 22 (Perennialisation)
Let Φ be a normal $*$-isomorphism from $\mathcal{R} \otimes \mathcal{H}$ into $\mathcal{R}$. If $\mathfrak{a} = a \cdot\!+\!\cdot \mathrm{tr} + A$ is a standardised project, one defines the project $!_\Phi \mathfrak{a} := a \cdot\!+\!\cdot 1 + \Phi(A)$ of carrier $\Phi(\boldsymbol{a} \otimes I_{\mathcal{H}})$. If $\mathbf{A}$ is a negative conduct of carrier $\boldsymbol{a}$, one defines the ethics $\sharp_\Phi \mathbf{A} := \{!_\Phi \mathfrak{a}; \mathfrak{a} = a \cdot\!+\!\cdot \mathrm{tr} + A \in \mathbf{A}\}$ and the negative conduct $!_\Phi \mathbf{A} := \ \sim\sim\sharp_\Phi \mathbf{A}$, both of carrier $\Phi(\boldsymbol{a} \otimes I_{\mathcal{H}})$.

Definition 23 (Exponentials)
$\mathfrak{G}$ being the amenable group of proposition 19, let $M \subset |\mathfrak{G}|$ be the monoid generated by $\mathtt{l}, \mathtt{r}$. If $\mathcal{H}^{[X]}$ denotes the X-fold tensor power of $\mathcal{H}$ and the crossed product $\mathcal{H}^{[|\mathfrak{G}|]} \rtimes \mathfrak{G}$ refers to the automorphic action $g(\bigotimes_h u_h) := \bigotimes_h u_{gh}$, define:

$$\Omega : \mathcal{R} \otimes \mathcal{H} \simeq \mathcal{R} \otimes \mathcal{H}^{[M]} \subset \mathcal{R} \otimes (\mathcal{H}^{[|\mathfrak{G}|]} \rtimes \mathfrak{G}) \simeq \mathcal{R}$$

We define the exponential $!\mathbf{A} := !_\Omega \mathbf{A}$*; and* $?\mathbf{A} := \sim!\sim\mathbf{A}$.

Theorem 11 (Exponentiation)

$$!(\mathbf{A} \,\&\, \mathbf{B}) = !\mathbf{A} \otimes !\mathbf{B} \tag{48}$$

Proof : using remark 1 (40), the right hand side can be replaced with $\sim\!\sim(\sharp\mathbf{A}\odot\sharp\mathbf{B})$. The elements of $\sharp\mathbf{A}\odot\sharp\mathbf{B}$ are of the form $0\cdot\!+\!\cdot 1+\Omega(A+B)$, where $0\cdot\!+\!\cdot \mathrm{tr}+A\ \eta\ \mathbf{A}, 0\cdot\!+\!\cdot \mathrm{tr}+B\ \eta\ \mathbf{B}$, i.e., $0\cdot\!+\!\cdot \mathrm{tr}+(A+B)\in\mathbf{A}\ \&\ \mathbf{B}$, whence $\sharp\mathbf{A}\odot\sharp\mathbf{B}\subset\sharp(\mathbf{A}\ \&\ \mathbf{B})$ and $!\mathbf{A}\otimes!\mathbf{B}\subset!(\mathbf{A}\ \&\ \mathbf{B})$. Indeed, if $\boldsymbol{a},\boldsymbol{b}$ are the carriers of $!\mathbf{A},!\mathbf{B}$, $0\cdot\!+\!\cdot 1+\Omega(C)\in\sharp(\mathbf{A}\ \&\ \mathbf{B})$ iff $0\cdot\!+\!\cdot 1+(\boldsymbol{a}\Omega(C)\boldsymbol{a}+\boldsymbol{b}\Omega(C)\boldsymbol{b})\in\sharp\mathbf{A}\odot\sharp\mathbf{B}$; we show that, in such a case, $0\cdot\!+\!\cdot 1+\Omega(C)\in\sim\!\sim(\sharp\mathbf{A}\odot\sharp\mathbf{B})=!\mathbf{A}\otimes!\mathbf{B}$, from which we shall get the converse inclusion $!(\mathbf{A}\ \&\ \mathbf{B})\subset!\mathbf{A}\otimes!\mathbf{B}$.
Consider the canonical map $\Theta:\mathcal{H}\mapsto\mathcal{R}\otimes(\mathcal{H}^{[|\mathfrak{G}|]}\rtimes\mathfrak{G})\simeq\mathcal{R}\otimes\mathcal{H}$ sending $\mathcal{H}$ to the component of index $1\in\mathfrak{G}$ of the tensor product $\mathcal{H}^{[|\mathfrak{G}|]}$; if $\nu\in\mathcal{H}$ is a projection of trace $1/2$, let $\varphi:=\Omega(\Theta(\nu)\cdot(\boldsymbol{a}\otimes\mathtt{l}))$; it is immediate that $0\cdot\!+\!\cdot 1+\Omega(A)\in\sharp\mathbf{A}$ iff $0\cdot\!+\!\cdot 1+\varphi\Omega(A)\varphi^*\in\sharp\mathbf{A}$, similarly for $\mathbf{B}$ and $\psi:=\Omega(\Theta(I-\nu)\cdot(\boldsymbol{b}\otimes l))$. Whence $0\cdot\!+\!\cdot 1+\Omega(C)\in\sharp\mathbf{A}\odot\sharp\mathbf{B}$ iff $0\cdot\!+\!\cdot 1+(\varphi\Omega(C)\varphi^*+\psi\Omega(C)\psi^*)\in\sharp\mathbf{A}\odot\sharp\mathbf{B}$. Since $\varphi\Omega(C)\varphi^*+\psi\Omega(C)\psi^*=(\varphi+\psi)\Omega(C)(\varphi^*+\psi^*)$, it follows that, dually speaking, $a\cdot\!+\!\cdot\alpha+A\in?\!\sim\!\mathbf{A}$ iff $a\cdot\!+\!\cdot\alpha+(\varphi^*+\psi^*)A(\varphi+\psi)\in?\!\sim\!\mathbf{A}$; bidually speaking, $0\cdot\!+\!\cdot 1+D\in!\mathbf{A}\otimes!\mathbf{B}$ iff $0\cdot\!+\!\cdot 1+(\varphi+\psi)D(\varphi^*+\psi^*)\in!\mathbf{A}\otimes!\mathbf{B}$. Since $(\varphi+\psi)\Omega(C)(\varphi^*+\psi^*)=(\varphi+\psi)(\boldsymbol{a}\Omega(C)\boldsymbol{a}+\boldsymbol{b}\Omega(C)\boldsymbol{b})(\varphi^*+\psi^*)$, we conclude that $0\cdot\!+\!\cdot 1+\Omega(C)\in!\mathbf{A}\otimes!\mathbf{B}$ iff $0\cdot\!+\!\cdot 1+(\boldsymbol{a}\Omega(C)\boldsymbol{a}+\boldsymbol{b}\Omega(C)\boldsymbol{b})\in!\mathbf{A}\otimes!\mathbf{B}$: this proves our claim. □

Remark 5
The specific perennialisation $\Phi=\Omega$ seems the most natural, but alternative exponentials $!_\Phi\mathbf{A}$, for suitable choices of Φ, may have interesting properties.

Remark 6
In order to get (48), the carriers of $\mathbf{A},\mathbf{B}$ must be disjoint, whence $\mathbf{A}\ \&\ \mathbf{B}$ cannot be defined when the carriers intersect like in $(\mathbf{A}\parr\mathbf{C})\ \&\ (\mathbf{B}\parr\mathbf{C})$. Whence the loss of literal distibutivity.

5.5 Promotion

Exponentiation enables *contextual* promotion « from $\Gamma\vdash A$, get $!\Gamma\vdash!A$ ».

Theorem 12 (Promotion)
The principle $!A,!(A\multimap B)\vdash!B$ can be implemented in GoI.

Proof : let $\boldsymbol{a},\boldsymbol{a'},\boldsymbol{b},\boldsymbol{b'}$ be disjoint carriers, let u,v be partial isometries from $\boldsymbol{a}$ to $\boldsymbol{a'}$ and from $\boldsymbol{b}$ to $\boldsymbol{b'}$. If $\mathbf{A},\mathbf{B}$ are negative conducts of respective carriers $\boldsymbol{a},\boldsymbol{b}$, we are seeking an inhabitant of $(!\mathbf{A}\otimes!(u(\mathbf{A})\multimap\mathbf{B})\multimap!v(\mathbf{B}))$. Indeed, $\mathfrak{c}:=0\cdot\!+\!\cdot 1+(u+u^*+v+v^*)$ inhabits $(\mathbf{A}\otimes(u(\mathbf{A})\multimap\mathbf{B})\multimap v(\mathbf{B}))$ and sends $0\cdot\!+\!\cdot\alpha+A, 0\cdot\!+\!\cdot\varphi+F$ to $0\cdot\!+\!\cdot\alpha\otimes\varphi+v([F^\dagger]u(A^\ddagger))$.
« Banging » $\mathfrak{c}$ basically means internalising the operations $(\cdot)^\dagger,(\cdot)^\ddagger$. For this, observe that the sets $\mathtt{l}M$ and $\mathtt{r}M$ are disjoint, because M is a free monoid. In particular $\mathcal{H}^{[\mathtt{l}M\cup\mathtt{r}M]}\simeq\mathcal{H}^{[\mathtt{l}M]}\otimes\mathcal{H}^{[\mathtt{r}M]}$. It is therefore possible to internalise $(\cdot)^\dagger,(\cdot)^\ddagger$ by the conjugations w.r.t. the unitaries $\mathtt{l},\mathtt{r}$.
We thus define $!\mathfrak{c}:=0\cdot\!+\!\cdot 1+\Omega(u\otimes\mathtt{r}^*\mathtt{l}+u^*\otimes\mathtt{l}^*\mathtt{r}+v\otimes\mathtt{r}+v^*\otimes\mathtt{r}^*)$. □

Corollary 19.1
Contextual promotion works for « ! ».

Proof : assume that, say $A, B \vdash C$; then one gets $A \multimap (B \multimap C)$, and, since context-free promotion is free of charge, $!(A \multimap (B \multimap C))$. Now, the theorem yields $!A, !B, !(A \multimap (B \multimap C)) \vdash !C$, whence, by a cut, $!A, !B \vdash !C$. $\square$

6 Lateralised logic

6.1 The witness theorem

Definition 24 (Witnesses)
If $\boldsymbol{p} \in \mathcal{R}$ is a carrier, one defines the sets:

$$\begin{array}{rcl} \mathbf{Z}_{\boldsymbol{p}} & := & \{a \cdot\!+\!\cdot (\lambda \oplus -\mu) + (\boldsymbol{u} \oplus \boldsymbol{v}) \ ; \ \boldsymbol{u}, \boldsymbol{v} \subset \boldsymbol{p}, \lambda \mathrm{tr}(\boldsymbol{u}) = \mu \mathrm{tr}(\boldsymbol{v}), \lambda, \mu > 0, a \neq 0\} \\ \mathbf{P}_{\boldsymbol{p}} & := & \{0 \cdot\!+\!\cdot 1 + \boldsymbol{u} \ ; \ 0 \neq \boldsymbol{u} \subset \boldsymbol{p}\} \end{array}$$

and the conduct $¥\boldsymbol{p} := \sim\sim(\mathbf{Z}_{\boldsymbol{p}} \cup \mathbf{P}_{\boldsymbol{p}})$ of carrier $\boldsymbol{p}$.

Theorem 13 (Witness)
The conducts $¥\boldsymbol{p}$ are positive and proper; moreover:

(i) $¥0$ *is the positive neutral* $\mathbf{0} := \sim\top$.

(ii) *If $\boldsymbol{p} \neq 0$, then $\mathbf{Z}_{\boldsymbol{p}} \cup \{0 \cdot\!+\!\cdot x + \boldsymbol{p} \ ; \ x \neq 0\}$ is a complete ethics for $¥\boldsymbol{p}$.*

(iii) *If $0 \cdot\!+\!\cdot \alpha + A \in ¥\boldsymbol{p}$ is positive, then A is a (nonzero) projection.*

(iv) *If $\boldsymbol{p}, \boldsymbol{q}$ are disjoint, then $¥\boldsymbol{p} \oplus ¥\boldsymbol{q} = ¥\boldsymbol{p} \parr ¥\boldsymbol{q}$.*

Proof : for $b \neq 0$, $\mathfrak{b} := b \cdot\!+\!\cdot (2 \oplus -1) + (0 \oplus 0) \in \mathbf{Z}_{\boldsymbol{p}}$; if $\mathfrak{a} = a \cdot\!+\!\cdot \alpha + A \in \sim\mathbf{Z}_{\boldsymbol{p}}$, $\ll \mathfrak{a} \,|\, \mathfrak{b} \gg = a + b\alpha(I_{\mathcal{A}}) \neq 0$ for all $b \neq 0$: this forces $\mathfrak{a}$ to be wager-free. Moreover, since $\mathbf{Z}_{\boldsymbol{p}} \cup \mathbf{P}_{\boldsymbol{p}}$ enjoys positive rescaling, so does its bipolar: we conclude that $¥\boldsymbol{p}$ is positive. Moreover, observe that $\sim\mathbf{Z}_{\boldsymbol{p}} \cap \sim\mathbf{P}_{\boldsymbol{p}}$ is proper:
$\ll 0 \cdot\!+\!\cdot 1 + \boldsymbol{p}/2 \mid 0 \cdot\!+\!\cdot \beta + \boldsymbol{q} \gg = \lambda \mathrm{tr}(\boldsymbol{q}) \beta(I_{\mathcal{B}}) \log 2$; from this, it follows that $0 \cdot\!+\!\cdot 1 + \boldsymbol{p}/2 \in \sim\mathbf{Z}_{\boldsymbol{p}} \cap \sim\mathbf{P}_{\boldsymbol{p}}$. Thus, $¥\boldsymbol{p} = \sim(\sim\mathbf{Z}_{\boldsymbol{p}} \cap \sim\mathbf{P}_{\boldsymbol{p}})$ is proper as well.

(i) There are not that many conducts of empty carrier: $¥0$ being positive and proper, it must be equal to $\mathbf{0}$. The remaining items being either vacuous or trivial in the case of null carriers, we assume that $\boldsymbol{p}, \boldsymbol{q} \neq 0$.

(ii) Since $¥\boldsymbol{p} \subset \sim\mathbf{Z}_{\boldsymbol{p}}$, $0 \cdot\!+\!\cdot \mathrm{tr}(\boldsymbol{v}) + \boldsymbol{u} \equiv_{¥\boldsymbol{p}} 0 \cdot\!+\!\cdot \mathrm{tr}(\boldsymbol{u}) + \boldsymbol{v}$ ($\boldsymbol{u}, \boldsymbol{v} \subset \boldsymbol{p}$ nonzero), so $¥\boldsymbol{p} = \sim\sim(\mathbf{Z}_{\boldsymbol{p}} \cup \{0 \cdot\!+\!\cdot x + \boldsymbol{p} \ ; \ x \neq 0\})$. If $\mathfrak{b} \in ¥\boldsymbol{p}$ is wager-free, if $\mathfrak{a}, \mathfrak{a}' \in \sim¥\boldsymbol{p}, \lambda \in \mathbb{R}$, then $\mathfrak{a} + \lambda\mathfrak{a}' + \lambda c\,\mathfrak{Dai}_0 \in \sim\mathbf{Z}_{\boldsymbol{p}}$, with $c := -\alpha(I_{\mathcal{A}})$. If $\ll \mathfrak{b} \,|\, \mathfrak{a} \gg + \lambda \ll \mathfrak{b} \,|\, \mathfrak{a}' \gg = 0$, then $\ll \mathfrak{b} \,|\, \mathfrak{a} + \lambda\mathfrak{a}' + \lambda c\,\mathfrak{Dai}_0 \gg = 0$ and $\ll 0 \cdot\!+\!\cdot 1 + \boldsymbol{p} \,|\, \mathfrak{a} + \lambda\mathfrak{a}' + \lambda c\mathfrak{Dai}_0 \gg = \ll 0 \cdot\!+\!\cdot 1 + \boldsymbol{p} \,|\, \mathfrak{a} \gg + \lambda \ll 0 \cdot\!+\!\cdot 1 + \boldsymbol{p} \,|\, \mathfrak{a}' \gg = 0$ whence $\mathfrak{b} \equiv_{¥\boldsymbol{p}} 0 \cdot\!+\!\cdot x + \boldsymbol{p}$ for some $x \neq 0$.

(iii) If $\mathfrak{a} = 0 \cdot\!+\!\cdot \alpha + A \in \yen\boldsymbol{p}$ ($\alpha > 0$), then $\ll \mathfrak{a} \,|\, 0 \cdot\!+\!\cdot 1 + \lambda \gg = c\,\mathrm{colog}(1-\lambda)$, whence $(\mathrm{tr} \otimes \alpha)(A^n) = c$. Since A is hermitian, $0 \leq A^2 \leq I$, whence $0 \leq A^4 \leq A^2$; since $\mathrm{tr} \otimes \alpha$ is faithful and positive, $(\mathrm{tr} \otimes \alpha)(A^4 - A^2) = 0$ yields $A^4 = A^2$: A^2 is a projection and the partial symmetry A is the difference $A = A^+ - A^-$ of two projections s.t. $A^+A^- = 0$; then $A^2 = A^+ + A^-$ and $(\mathrm{tr} \otimes \alpha)(A - A^2) = 0$ yields $A^- = 0$, i.e., $A^2 = A$.

(iv) Let $\mathfrak{f} = f \cdot\!+\!\cdot \varphi + F \in \yen\boldsymbol{p} \invamp \yen\boldsymbol{q}$, $\mathfrak{a} = 0 \cdot\!+\!\cdot \alpha + A \in \sim\!\yen\boldsymbol{p}$, $\mathfrak{b} = 0 \cdot\!+\!\cdot \beta + B \in \sim\!\yen\boldsymbol{q}$, with $\varphi(I_{\mathcal{F}}) = \alpha(I_{\mathcal{A}}) = \beta(I_{\mathcal{B}}) = 1$; write $\ll \mathfrak{f} \,|\, \mathfrak{a} \otimes \mathfrak{b} \gg = f + g_{\mathfrak{a}} + h_{\mathfrak{b}} + k_{\mathfrak{ab}}$, with $g_{\mathfrak{a}} := \mathrm{ldet}(I - FA)$, $h_{\mathfrak{b}} := \mathrm{ldet}(I - FB)$. If $x, y \neq 0$, then $\ll \mathfrak{f} \,|\, \mathfrak{a}_\lambda + (x-1)\mathfrak{Dai}_0 \otimes \mathfrak{b}_\mu + (y-1)\mathfrak{Dai}_0 \gg = fxy + g_{\mathfrak{a}}y + h_{\mathfrak{b}}x + k_{\mathfrak{ab}} \neq 0$, whence one and only one of the four reals $f, g_{\mathfrak{a}}, h_{\mathfrak{b}}, k_{\mathfrak{ab}}$ is $\neq 0$. If $\mathfrak{a}' \in \sim\!\yen\boldsymbol{p}$ with $\alpha'(I_{\mathcal{A}'}) = 1$, define $\mathfrak{a}" := \mathfrak{a} + \mathfrak{a}' - \mathfrak{Dai}_0$ or $\mathfrak{a}" := \mathfrak{a} - \mathfrak{a}' + \mathfrak{Dai}_0$ so that $\mathfrak{a}" \in \sim\!\yen\boldsymbol{p}$; the three cases $\mathfrak{a}/\mathfrak{b}$, $\mathfrak{a}'/\mathfrak{b}$, $\mathfrak{a}"/\mathfrak{b}$ select one among f, g, h, k, therefore always the same. Same remark for the argument $\mathfrak{b}$, whence we conclude that the departure $f/g/h/k$ is independent of the arguments $\mathfrak{a}, \mathfrak{b}$. Assume $k \neq 0$; let $\mathfrak{x} := 0 \cdot\!+\!\cdot 1 + x\boldsymbol{p} \in \sim\!\yen\boldsymbol{p}$, $\mathfrak{y} = 0 \cdot\!+\!\cdot 1 + y\boldsymbol{q} \in \sim\!\yen\boldsymbol{q}$ ($x, y \in]0,1[$). Then $k_{xy} = \mathrm{ldet}(I - xyF_{12}(I - yF_{22})^{-1}F_{21}(I - xF_{11})^{-1})$. The convergence radius of the analytical function $k_x : y \rightsquigarrow k_{xy}$ tends to infinity when $x \to 0$; but $k_x(y) = \ll [\mathfrak{f}]\mathfrak{x} \mid \mathfrak{y} \gg = \mathrm{colog}(I - c_x y)$ has the convergence radius 1, contradiction. Three cases remain:

f: then $\mathfrak{f} \equiv \mathfrak{Dai}_f$.

g: then $\mathfrak{f} \in \sim\sim(\yen\boldsymbol{p})_{\boldsymbol{p}+\boldsymbol{q}}$.

h: then $\mathfrak{f} \in \sim\sim(\yen\boldsymbol{q})_{\boldsymbol{p}+\boldsymbol{q}}$.

Whence, using weakening, $\yen\boldsymbol{p} \invamp \yen\boldsymbol{q} = \yen\boldsymbol{p} \oplus \yen\boldsymbol{q}$. □

6.2 The first action

In ludics [9], an essential role is devoted to *actions*: thus, in a behaviour $\mathbf{A} \oplus \mathbf{B}$, the first action of a proper design chooses between $\mathbf{A}$ and $\mathbf{B}$. In GoI, the role of first actions is played by positive projects of the form $0 \cdot\!+\!\cdot \alpha + A$, with A a nonzero projection. If $\mathbf{B} \subset \yen\boldsymbol{p}$ is a conduct of carrier $\boldsymbol{p}$, its positive projects are of the required form by theorem 13 *(iii)*: such a conduct may be seen as a « space of first actions ». It must be noticed that, whereas $\yen\boldsymbol{p}$ admits, up to equivalence, at most one first action, it is no longer the case with $\mathbf{B} \subset \yen\boldsymbol{p}$ whose equivalence is usually coarser than the one induced by $\equiv_{\yen\boldsymbol{p}}$. A few examples may help:

(i) When $\boldsymbol{p} \neq 0$, $\yen\boldsymbol{p}$ admits (up to equivalence) exactly one first action.

(ii) If $\boldsymbol{p} \cdot \boldsymbol{q} = 0$, the first actions of $\sim\sim(\yen\boldsymbol{p})_{\boldsymbol{p}+\boldsymbol{q}} \subset \yen(\boldsymbol{p}+\boldsymbol{q})$ are those of $\yen(\boldsymbol{p})$. Indeed, if $\mathfrak{a} = 0 \cdot\!+\!\cdot \alpha + A \in \sim\sim(\yen\boldsymbol{p})_{\boldsymbol{p}+\boldsymbol{q}}$ with $A \cdot (\boldsymbol{q} \otimes I_{\mathcal{A}}) \neq 0$, it is easy to find a project in $\sim(\yen\boldsymbol{p})_{\boldsymbol{p}+\boldsymbol{q}}$ not polar to $\mathfrak{a}$.

(iii) If $\boldsymbol{p} \cdot \boldsymbol{q} = 0$, $\boldsymbol{p}, \boldsymbol{q} \neq 0$, then $\yen\boldsymbol{p} \oplus \yen\boldsymbol{q} = \sim\sim(\yen\boldsymbol{p})_{\boldsymbol{p}+\boldsymbol{q}} \cup \sim\sim(\yen\boldsymbol{q})_{\boldsymbol{p}+\boldsymbol{q}}$ by theorem 8. Thus the first actions $0 \cdot + \cdot \alpha + A$ of $\yen\boldsymbol{p} \oplus \yen\boldsymbol{q}$ split into two equivalence classes: either $A \subset \boldsymbol{p} \otimes I_{\mathcal{A}}$ or $A \subset \boldsymbol{q} \otimes I_{\mathcal{A}}$.

(iv) The case of $\yen\boldsymbol{p} \parr \yen\boldsymbol{q}$ is reduced to the previous one by theorem 13 *(iv)*.

6.3 Lateralisation

The remarkable stability of positive subconducts of witnesses is the missing link between GoI and ludics; it is now possible to define *behaviours*, which are sorts of « conducts with a first action », thus allowing *changes of polarity*.

Definition 25 (Behaviours)
If $\boldsymbol{p} \subset \boldsymbol{a} \in \mathcal{R}$ are carriers, a right behaviour of base *$\boldsymbol{p}$ and carrier $\boldsymbol{a}$ is a positive conduct $\mathbf{A}$ of carrier $\boldsymbol{a}$ such that $\mathbf{A}_{\boldsymbol{p}} \subset \yen\boldsymbol{p}$.*
Polars of right behaviours are called left behaviours*; indeed a left behaviour of base $\boldsymbol{p}$ is a negative conduct containing $(\sim\yen\boldsymbol{p})_{\boldsymbol{a}}$.*

Consistently with the change of expression (left/right $\rightsquigarrow$ negative/positive), this refined form of polarisation is styled *lateralisation*.

Example 3
The simplest example of a right behaviour of base $\boldsymbol{p}$ and carrier $\boldsymbol{p}$ is $\yen\boldsymbol{p}$, in particular $\mathbf{0} := \yen 0$, the disjunctive neutral.

Proposition 20
Let $\mathbf{E}, \mathbf{F}$ be ethics of respective carriers $\boldsymbol{a} \supset \boldsymbol{p}$; then:

$$\mathbf{E}_{\boldsymbol{p}} \subset \mathbf{F} \quad \Leftrightarrow \quad (\sim\mathbf{F})_{\boldsymbol{a}} \subset \sim\mathbf{E} \tag{49}$$

Proof : both sides are equivalent to $\forall \mathfrak{a} \in \mathbf{E}\, \forall \mathfrak{b} \in \sim\mathbf{F}\ \ \mathfrak{a} \underset{\sim}{\downarrow} \mathfrak{b}$. $\square$

Corollary 20.1
Let $\mathbf{E}$ be an ethics for the positive conduct $\mathbf{A}$ of carrier $\boldsymbol{a}$ and assume that $\boldsymbol{p} \subset \boldsymbol{a}$ is such that $\mathbf{E}_{\boldsymbol{p}} \subset \yen\boldsymbol{p}$. Then $\mathbf{A}$ is a right behaviour of base $\boldsymbol{p}$.

Proof : $\mathbf{E}_{\boldsymbol{p}} \subset \yen\boldsymbol{p} \quad \Leftrightarrow \quad \sim(\yen\boldsymbol{p})_{\boldsymbol{a}} \subset \sim\mathbf{E} = \sim\mathbf{A} \quad \Leftrightarrow \quad \mathbf{A}_{\boldsymbol{p}} \subset \yen\boldsymbol{p}$. $\square$

7 The social life of behaviours

7.1 Multiplicatives

7.1.1 Right case

Definition 26 (Right tensor product)
If $\mathbf{A}$, $\mathbf{B}$ are right and left behaviours of bases $\boldsymbol{p}, \boldsymbol{q}$ and disjoint carriers $\boldsymbol{a}, \boldsymbol{b}$, then $\mathbf{A} \otimes \mathbf{B}$ is the positive conduct of carrier $\boldsymbol{a} + \boldsymbol{b}$, indeed a right behaviour of base $\boldsymbol{p}$, of definition 13.
One defines, mutatis mutandis, *the « par » of two behaviours of opposite lateralities, which is a negative conduct, indeed a left behaviour of base $\boldsymbol{p}$.*

Proposition 21
$\mathbf{A} \otimes \mathbf{B}$ is a right behaviour of base $\boldsymbol{p}$.

Proof : by weakening, $\sim\mathbf{A} \subset \sim\mathbf{A} \parr \sim\mathbf{B}$; hence $(\mathbf{A} \otimes \mathbf{B})_{\boldsymbol{a}}$ is included in (thus, equal to) $\mathbf{A}$. Then $(\mathbf{A} \otimes \mathbf{B})_{\boldsymbol{p}} = (\mathbf{A}_{\boldsymbol{a}})_{\boldsymbol{p}} = \mathbf{A}_{\boldsymbol{p}} \subset \yen\boldsymbol{p}$. □

Remark 7
The result persists when $\mathbf{B}$ is a plain negative conduct.

7.1.2 Left case

Definition 27 (Left tensor product)
If $\mathbf{A}$, $\mathbf{B}$ are left behaviours of bases $\boldsymbol{p}, \boldsymbol{q}$ and disjoint carriers $\boldsymbol{a}, \boldsymbol{b}$, then $\mathbf{A} \otimes \mathbf{B}$ is the negative conduct of carrier $\boldsymbol{a} + \boldsymbol{b}$, indeed a left behaviour of base $\boldsymbol{p} + \boldsymbol{q}$, of definition 13. One defines, mutatis mutandis, *the « par » of two right behaviours, which is indeed a right behaviour of base $\boldsymbol{p} + \boldsymbol{q}$.*

Lemma 22.1
If $\mathbf{E}, \mathbf{F}$ are ethics of disjoint carriers $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{p} \subset \boldsymbol{b}$, then $(\mathbf{E} \multimap \mathbf{F})_{\boldsymbol{a}+\boldsymbol{p}} \subset \mathbf{E} \multimap \mathbf{F}_{\boldsymbol{p}}$.

Proof : immediate; see remark 1 for the definition of $\mathbf{E} \multimap \mathbf{F}$. □

Proposition 22
$\mathbf{A} \otimes \mathbf{B}$ is a left behaviour of base $\boldsymbol{p}$.

Proof : dually, assume that $\mathbf{A}$, $\mathbf{B}$ are right behaviours; the lemma yields $(\mathbf{A} \parr \mathbf{B})_{\boldsymbol{p}+\mathbf{c}} \subset \mathbf{A}_{\boldsymbol{p}} \parr \mathbf{B}_{\boldsymbol{q}} \subset \yen\boldsymbol{p} \parr \yen\mathbf{c} \subset \yen(\boldsymbol{p} + \boldsymbol{q})$. □

7.2 Delateralisation

Definition 28 (Shift)
If $\mathbf{A}$ is a left behaviour of carrier $\boldsymbol{a}$ and base $\boldsymbol{p}$, if $\boldsymbol{s}$ is a non zero carrier disjoint from $\boldsymbol{a}$, one defines the right shift $\downarrow_{\boldsymbol{s}}\mathbf{A} := \yen\boldsymbol{s} \otimes \mathbf{A}$, *a right behaviour of base $\boldsymbol{s}$ and carrier $\boldsymbol{a} + \boldsymbol{s}$. One defines,* mutatis mutandis, *the left shift $\uparrow_{\boldsymbol{s}}\mathbf{a} := \sim\yen\boldsymbol{s} \parr \mathbf{A}$ of a right behaviour.*

Theorem 14 (Delateralisation)
The usual logical principles of the shift can be implemented in behaviours.

Proof : we treat the case of a context $\mathbf{\Gamma} = \mathbf{B}, \mathbf{C}$, where $\mathbf{B}, \mathbf{C}$ are right behaviours of bases $\boldsymbol{q}, \boldsymbol{r}$. We assume that $\mathbf{A}$ is a a behaviour of base $\boldsymbol{p}$. We assume that the three carriers and $\boldsymbol{s}$ are pairwise disjoint.

Right case: if $\mathfrak{a} = 0 \cdot\!+\!\cdot\, \alpha + A \in \mathbf{A} \parr \mathbf{B} \parr \mathbf{C}$, where $\mathbf{A}$ is a left behaviour, then $\downarrow_{\boldsymbol{s}}\mathfrak{a} := 0 \cdot\!+\!\cdot\, \alpha + (\boldsymbol{s} \otimes I_{\mathcal{A}} + A) \in \downarrow_{\boldsymbol{s}}\mathbf{A} \parr \mathbf{B} \parr \mathbf{C}$.

Left case: if $\mathfrak{a} = 0 \cdot\!+\!\cdot\, \alpha + A \in \mathbf{A} \parr \mathbf{B} \parr \mathbf{C}$ is positive, then:

$$(\mathbf{A} \parr \mathbf{B} \parr \mathbf{C})_{\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}} = \mathbf{A}_{\boldsymbol{p}} \parr \mathbf{B}_{\boldsymbol{q}} \parr \mathbf{C}_{\boldsymbol{r}} \subset \yen\boldsymbol{p} \parr \yen\boldsymbol{q} \parr \yen\boldsymbol{r} = \yen\boldsymbol{p} \oplus \yen\boldsymbol{q} \oplus \yen\boldsymbol{r}$$

By theorem 13, $\mathfrak{a}_{\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}} = 0 \cdot\!+\!\cdot\, \alpha + c$, with c a nonzero projection included in one of $\boldsymbol{p} \otimes I_{\mathcal{A}}$, $\boldsymbol{q} \otimes I_{\mathcal{A}}$ or $\boldsymbol{r} \otimes I_{\mathcal{A}}$. Let n be such that $n \cdot \mathrm{tr}(\boldsymbol{s}) \geq \mathrm{tr}(\boldsymbol{p}), \mathrm{tr}(\boldsymbol{q}), \mathrm{tr}(\boldsymbol{r})$, and let φ be the (non-unital) $*$-isomorphism from $\mathcal{A}$ to $\mathcal{A}' := \mathcal{M}_n(\mathbb{C}) \otimes \mathcal{A} = \mathcal{M}_n(\mathcal{A})$: $\varphi(u) := \begin{bmatrix} u & 0 & \dots \\ 0 & 0 & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$, the pseudo-trace on $\mathcal{A}'$ being $\mathrm{Tr} \otimes \mathcal{A}$.
Then, by remark 3, $\varphi(\mathbf{a}) := 0 \cdot\!+\!\cdot\, \alpha' + (I \otimes \varphi) A (I \otimes \varphi^*) \in \mathbf{A} \parr \mathbf{B} \parr \mathbf{C}$. Let $u \in \mathcal{R} \otimes \mathcal{A}'$ be a partial isometry of domain c and image $\boldsymbol{s'} \otimes I_{\mathcal{A}'}$, with $\boldsymbol{s'} \subset \boldsymbol{s}$. Then $\uparrow_{\boldsymbol{s}} \mathfrak{a} := 0 \cdot\!+\!\cdot\, \alpha' + ((I \otimes \varphi) A (I \otimes \varphi^*) + u + u^*) \in \uparrow_{\boldsymbol{r}} \mathbf{A} \parr \mathbf{B} \parr \mathbf{C}$. Indeed, if $\mathfrak{b} \in \sim\!\mathbf{A}$, then $[\uparrow_{\boldsymbol{r}} \mathfrak{a}](\downarrow_{\boldsymbol{r}} \mathfrak{b}) = [\mathfrak{a}]\mathfrak{b}$.
□

7.3 Quantifiers

Definition 29 (Quantifiers)
If $\mathbf{A}[i] (i \in \mathbb{I})$ is a non empty family of behaviours of the same carrier $\boldsymbol{a}$, the same base $\boldsymbol{p}$ and the same laterality, one defines $\forall i \in \mathbb{I}\ \mathbf{A}[i] := \bigcap_{i \in \mathbb{I}} \mathbf{A}[i]$, which turns out to be another behaviour of the same carrier, base and lateralisation. One defines dually $\exists i \in \mathbb{I}\ \mathbf{A}[i] := \sim\sim\bigcup_{i \in \mathbb{I}} \mathbf{A}[i]$.

Proposition 23
$\forall i \in \mathbb{I}\ \mathbf{A}[i]$ is a behaviour.

Proof : **Right:** if $i_0 \in \mathbb{I}$, then $(\forall i \in \mathbb{I}\ \mathbf{A}[i])_{\boldsymbol{p}} \subset \mathbf{A}[i_0]_{\boldsymbol{p}} \subset \yen\boldsymbol{p}$.

Left (dually): if the $\mathbf{A}[i]$ are right behaviours, then $\bigcap_i \mathbf{A}[i] \subset \yen\boldsymbol{p}$. By the corollary to proposition 20, $\exists i \in \mathbb{I} \mathbf{A}[i]$ is a right behaviour. □

Remark 8
« $\uparrow_{\boldsymbol{s}}$ » being an instance of « par », it distributes over universal quantification.

7.4 Additives

Definition 30 (Additives)
If $\mathbf{A}, \mathbf{B}$ are behaviours of the same lateralisation, of bases $\boldsymbol{p}, \boldsymbol{q}$ and disjoint carriers $\boldsymbol{a}, \boldsymbol{b}$, definition 15 yields conducts of carrier $\boldsymbol{a} + \boldsymbol{b}$, indeed a behaviour of base $\boldsymbol{p} + \boldsymbol{q}$ with the same lateralisation as $\mathbf{A}, \mathbf{B}$.

Proposition 24
If $\mathbf{A}, \mathbf{B}$ are right behaviours, so is $\mathbf{A} \oplus \mathbf{B}$.

Proof : if $\mathbf{E} := \mathbf{A}_{\boldsymbol{a}+\boldsymbol{b}} \cup \mathbf{B}_{\boldsymbol{a}} + \boldsymbol{b}$, then $\mathbf{E}_{\boldsymbol{p}+\boldsymbol{q}} \subset \yen\boldsymbol{p} \oplus \yen\boldsymbol{q} \subset \yen(\boldsymbol{p}+\boldsymbol{q})$. We conclude using the corollary to proposition 20. □

7.5 Du côté de chez Gustave

From the Gustave function (see, e.g., [10], ch. 12), we shall only remember the ternary structure $(A \oplus B) \parr (A' \oplus B') \parr (A" \oplus B")$ that coherent spaces cannot disentangle into something simpler, e.g., $(A \oplus B) \parr B' \parr (A" \oplus B")$.

Assuming $(\mathbf{A} \oplus \mathbf{B}) \parr \mathbf{B}' \parr (\mathbf{A}" \oplus \mathbf{B}")$ of base $\boldsymbol{r} := \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{a}' + \boldsymbol{b}' + \boldsymbol{a}" + \boldsymbol{b}"$:

$$
\begin{aligned}
((\mathbf{A} \oplus \mathbf{B}) \parr (\mathbf{A}' \oplus \mathbf{B}') \parr (\mathbf{A}" \oplus \mathbf{B}"))_{\boldsymbol{r}} &\subset (\yen\boldsymbol{a} \oplus \yen\boldsymbol{b}) \parr (\yen\boldsymbol{a}' \oplus \yen\boldsymbol{b}') \parr (\yen\boldsymbol{a}" \oplus \yen\boldsymbol{b}") \\
&\subset \yen\boldsymbol{a} \oplus \yen\boldsymbol{b} \oplus \yen\boldsymbol{a}' \oplus \yen\boldsymbol{b}' \oplus \yen\boldsymbol{a}" \oplus \yen\boldsymbol{b}"
\end{aligned}
$$

from which we get the existence of a « first action ».

7.6 Secularisation

If $\mathbf{A}$ is a left behaviour of carrier $\boldsymbol{a}$ and base $\boldsymbol{p}$, then $!\mathbf{A}$ (definition 22) is a negative conduct, but not a left behaviour. This problem is perhaps the explanation for the other iconoclast logic, indeed the original one, **LLL** ([10], ch. 16). In case, this would definitely show the soundness of the present approach, which manages to explain both light logics out of natural geometric constraints, and not in the usual *Deus ex machina*, i.e., essentialist, way. I just put together a few facts:

(i) Conducts may socialise with behaviours: when $\mathbf{A}$ is a right behaviour and $\mathbf{B}$ is a negative conduct, $\mathbf{A} \otimes \mathbf{B}$ is a right behaviour (remark 7). It is therefore possible to use ! on the left of an implication: if $\mathbf{A}, \mathbf{B}$ are left behaviours, so is $!\mathbf{A} \multimap \mathbf{B}$.

(ii) In terms of sequent calculus, this requires a special maintenance for « ! », e.g., through the familiar *underlining* technique ([10], ch. 15).

(iii) However, due to the want of *dereliction*, it is not reasonable to represent implication by $!\mathbf{A} \multimap \mathbf{B}$, and $!\mathbf{A} \multimap !\mathbf{B}$ is still not a behaviour. The idea is to use a lateralised *subrogate* for « ! », the *secularisation* §.

(iv) Promotion subsists under the weaker form « from $\Gamma \vdash A$, get $!\Gamma \vdash \S A$ » (Γ, A left lateralised).

Definition 31 (Semi-standard projects)
In the spirit of definition 21, an extended project is semi-standard when of the form $\mathfrak{a} := a \cdot\!+\!\cdot (\mathrm{tr} \oplus -\mathrm{tr}) + A$, the « idiom » being now $\mathcal{H} \oplus \mathcal{H}$ and the « pseudo-trace » being $\mathrm{tr} \oplus -\mathrm{tr}$. If $\mathfrak{a} \in \mathbf{A}$, choose φ such that $(\mathrm{tr} \oplus -\mathrm{tr}) \circ \varphi = \alpha$; then $\varphi(\mathfrak{a})\ \eta\ \mathbf{A}$.

Definition 32 (Secularisation)
If $\mathfrak{a} := a \cdot\!+\!\cdot (\mathrm{tr} \oplus \mathrm{tr}) + (A \oplus B)$ is a standardised project of carrier $\boldsymbol{a}$, one defines the project $\bar{\S}\mathfrak{a} := a \cdot\!+\!\cdot (1 \ominus 1) + (\Omega(A) \oplus \Omega(B))$.
If $\mathbf{A}$ is a right behaviour of carrier $\boldsymbol{a}$ and base $\boldsymbol{p}$, one defines the right behaviour $\bar{\S}\mathbf{A} := \sim\sim\{\bar{\S}\mathfrak{a} : \mathfrak{a}\ \eta\ \mathbb{A}\}$ of carrier $\Omega(\boldsymbol{a} \otimes I_{\mathcal{H}})$ and base $\Omega(\boldsymbol{p} \otimes I_{\mathcal{H}})$.
One symmetrically defines $\S\mathbf{A} := \sim\bar{\S}\sim\mathbf{A}$; obviously $!\mathbf{A} \subset \S\mathbf{A}$.

Proposition 25
If $\mathbf{A}$ *is a left behaviour, then* $\S\mathbf{A}$ *is a left behaviour.*

Proof : follows from the lemma:

Lemma 25.1
$\bar{\S}\yen\boldsymbol{p} \subset \yen\Omega(\boldsymbol{p} \otimes I_{\mathcal{H}})$.

In this Kamchatka of the paper, I feel like skipping one proof, easy anyway. □

A Second order quantification

A purely locative approach would consist in defining, for $r > 0$, $\forall_r \mathbf{X}\, \mathbf{A}[\mathbf{X}]$, where $\mathbf{X}$ varies over all conducts of carrier $\boldsymbol{r}$, where $\boldsymbol{r}$ is a given carrier such that $\mathrm{tr}(\boldsymbol{r}) = r$. The problem is with the change of « size »: the replacement of $\forall_r \mathbf{X}$ with $\forall_s \mathbf{X}$ is a cinch — using projections — when $0 < s \leq r$, but is problematic when $s > r$. Defining[8] $\mathbf{nat} := \forall_1 \mathbf{X}(!(\mathbf{X} \multimap \mathbf{X}) \multimap !(\mathbf{X} \multimap \mathbf{X}))$, we see that $\mathbf{nat}$ has size $4 > 1$ and cannot be substituted for $\mathbf{X}$, thus barring any decent form of recurrence.

The correct definition is semi-locative; in a spirit loosely inspired from the coherent interpretation of second order quantification ([10], ch. 8), we shall « approximate » conducts by means of conducts of smaller size. One should thus define *variable* conducts and projects.

A.1 The negative universe

Instead of a general (and illegible) definition of variability, I will content myself with the case of those negative conducts arising from variables $\mathbf{X}, \mathbf{Y}, \ldots$, the constant $\top$ (conjunctive unit), $\otimes, \&, !$ and an ad hoc redefinition of implication (to be used throughout this section):

$$\mathbf{A} \multimap \mathbf{B} := \{\mathfrak{f} \in \mathbf{A} \multimap \mathbf{B}\ ;\ \mathfrak{f} \text{ wager} - \text{free}\} \tag{50}$$

which is such that $\mathbf{A} \multimap \mathbf{B}$ is negative when both $\mathbf{A}, \mathbf{B}$ are negative. It will turn out that second order universal quantification — still to be defined — is also part of those operations internal to negative conducts.

A sort of *negative universe* has thus been introduced, where no change of polarity is actually needed: an alternative to lateralisation and behaviours. Should we need disjunction, the second order definition:

$$A \oplus B \quad := \quad \forall X\ (A \multimap X) \multimap ((B \multimap X) \multimap X) \tag{51}$$

would provide a sort of *ersatz*.

The negative universe is most likely **ELL**-like: usual data translate as $\mathbf{bool} := \forall X((X \otimes X) \multimap X)$, $\mathbf{bin} := \forall X((!(X \multimap X) \otimes !(X \multimap X)) \multimap !(X \multimap X))$, $\mathbf{nat} := \forall X(!(X \multimap X) \multimap !(X \multimap X))$.

[8] The formula « forgets » the four delocations of $\mathbf{X}$.

A.2 Variability

I restrict myself to those conducts obtained by means of $\mathbf{X}, \top, \otimes, \&, !, \multimap, \forall X$. I try, as much as possible, to minimise the use of isomorphisms; typically, the carriers $\boldsymbol{a}$ under consideration are such that $\Omega(\boldsymbol{a} \otimes I) = \boldsymbol{a}$, where Ω is the perennilaisation of section 5.4.

Supports: we fix, once for all, a carrier $\boldsymbol{v}$ such that $\mathrm{tr}(\boldsymbol{v}) = 1$. Second order variables $\mathbf{X}, \mathbf{Y}, \ldots$ will stand for negative conducts of carrier $\boldsymbol{v}$. *Literals* $\varphi(\mathbf{X}), \sim\psi(\mathbf{Y}), \ldots$ are obtained by means of *delocations* $\varphi, \psi, \ldots$, i.e., partial isometries of domain I (the full space) and *pairwise disjoint* images $\varphi\varphi^*, \psi\psi^*, \ldots$, the *supports* of $\varphi(\mathbf{X}), \sim\psi(\mathbf{Y}), \ldots$ which are infinite projections containing the carriers $\varphi(\boldsymbol{v}), \sim\psi(\boldsymbol{v}), \ldots$: some « extra space » is needed to handle second order substitution. It is indeed the case that the carrier (resp. the support) of a compound negative conduct $\mathbf{A}$ is the sum of the carriers (resp. supports) of its literals. In particular, the carrier $\boldsymbol{a}$ of a conduct $\mathbf{A}[\mathbf{X}]$ depending on $\mathbf{X}$ can symbolically be written $(m+n)\cdot\boldsymbol{v}$, which means « m occurrences (= delocations) of $\mathbf{X}$ and n occurences of other literals, free or bound ».

Substitution: the substitution of $\mathbf{B}$ for $\mathbf{X}$ in $\mathbf{A}$ cannot keep the carrier constant, for the simple reason that the carrier $\boldsymbol{b}$ of $\mathbf{B}$ is a priori distinct from $\boldsymbol{v}$. However, the carrier $\boldsymbol{c}$ of $\mathbf{A}[\mathbf{B}/\mathbf{X}]$ is included in the support of $\mathbf{A}$. The carrier of $\mathbf{A}[\mathbf{B}/\mathbf{X}]$ can symbolically be written $m\cdot\boldsymbol{b}+n\cdot\boldsymbol{v}$; if $\boldsymbol{b}$ is symbolically written $p\cdot\boldsymbol{v}$, we get the expression $(mp+n)\cdot\boldsymbol{v}$: a perfectly incorrect — but legible — way to speak of the various isomorphisms at stake. Since $mp + n \leq (m+n)(p+1)$, there is a (non unital) $*$-isomorphism (noted $\cdot[\mathbf{B}/\mathbf{X}]$) from $\boldsymbol{c}\mathcal{R}\boldsymbol{c}$ into $\boldsymbol{a}\mathcal{R}\boldsymbol{a} \otimes \mathcal{M}_{p+1}(\mathbb{C})$.

Quantification: if $\mathfrak{a} \in \sim\mathbf{A}[\mathbf{B}/\mathbf{X}]$, then $\mathfrak{a}[\mathbf{B}/\mathbf{X}]$ is a project of carrier $\boldsymbol{a}$, provided we consider the component $\mathcal{M}_{p+1}(\mathbb{C})$ of the image of the isomorphism $\cdot[\mathbf{B}/\mathbf{X}]$ as *idiomatic*. $\forall \mathbf{X}\, \mathbf{A}$ is defined as the polar of all $\mathfrak{a}[\mathbf{B}/\mathbf{X}]$, when $\mathfrak{a} \in \sim\mathbf{A}[\mathbf{B}/\mathbf{X}]$ for some negative conduct $\mathbf{B}$.

A.3 An example: natural numbers

Proofs of $(X \multimap X), \ldots, (X \multimap X) \vdash X \multimap X$, yield matrices M_n; those matrices are plain, i.e., embody the delocations: this explains the coefficients $\boldsymbol{v}$. Typically:

$$M_0 := \begin{bmatrix} 0 & \boldsymbol{v} \\ \boldsymbol{v} & 0 \end{bmatrix} \qquad M_3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{v} \\ 0 & 0 & \boldsymbol{v} & 0 & 0 & 0 & 0 & 0 \\ 0 & \boldsymbol{v} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boldsymbol{v} & 0 & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{v} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{v} & 0 \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{v} & 0 & 0 \\ \boldsymbol{v} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Next, the M_n are then perennialised, yielding matrices N_n. The case $n = 3$ involves elements $\mathtt{d_1}, \mathtt{d_2}, \mathtt{d_3}$ of the free monoid, e.g., $\mathtt{r}, \mathtt{lr}, \mathtt{l^2r}$ which are incompatible as prefixes, i.e., such that $\mathtt{d_i a} = \mathtt{d_j b}$ implies $\mathtt{a} = \mathtt{b}$ and $i = j$. For legibility, let us introduce the notation $a := \Omega(\boldsymbol{v} \otimes \mathtt{d_1}), b := \Omega(\boldsymbol{v} \otimes \mathtt{d_2}), c := \Omega(\boldsymbol{v} \otimes \mathtt{d_3})$:

$$N_0 := \begin{bmatrix} 0 & \boldsymbol{v} \\ \boldsymbol{v} & 0 \end{bmatrix} \qquad N_3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & a^* \\ 0 & 0 & b^*a & 0 & 0 & 0 & 0 & 0 \\ 0 & a^*b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c^*b & 0 & 0 & 0 \\ 0 & 0 & 0 & b^*c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & c^* & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Finally comes the contraction/weakening, yielding 4×4 matrices P_n:

$$P_0 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{v} \\ 0 & 0 & \boldsymbol{v} & 0 \end{bmatrix} \qquad P_3 := \begin{bmatrix} 0 & A & 0 & B \\ A^* & 0 & C & 0 \\ 0 & C^* & 0 & 0 \\ B^* & 0 & 0 & 0 \end{bmatrix}$$

where $A, B, C, \ldots$ are the 4×4 matrices:

$$A := \begin{bmatrix} 0 & b^*a & 0 & 0 \\ 0 & 0 & c^*b & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B := \begin{bmatrix} 0 & 0 & 0 & a^* \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix}$$

The idiom of P_n is $\mathcal{M}_{n+1}(\mathbb{C})$, whence the 4×4 matrices A, B, C for $n = 3$. The map θ from $\{1, \ldots, 8\}$ to $\{1, \ldots, 4\} \times \{1, \ldots, 4\}$, which sends $1, 3, 5, 7$ to respectively $(1, 1), (1, 2), (1, 3), (3, 4)$ and $2, 4, 6, 8$ to $(2, 1), (2, 2), (2, 3), (4, 4)$ enables one to replace the 8×8 matrix N_3 with the 4×4 matrix P_3 whose coefficients are in turn 4×4 matrices.

What we just constructed can be noted $P_n[\boldsymbol{v}]$ to emphasise the dependency upon the carrier $\boldsymbol{v}$. Should we perform an extraction on $\mathbf{B}$ of carrier $\boldsymbol{b}$, then P_n should become $P_n[\boldsymbol{b}]$, an element of $!(\mathbf{B} \multimap \mathbf{B}) \multimap !(\mathbf{B} \multimap \mathbf{B})$. The important point is that this extraction can be implemented, using delocations, by a sort of contraction. For instance, if $\mathrm{tr}(\boldsymbol{b}) = 2$, an appropriate variant of the project $\mathfrak{Contr}$ of section 5.2 will do the job: if φ, ψ are partial isometries between $\boldsymbol{v}$ and $\boldsymbol{b}', \boldsymbol{b}$" such that $\boldsymbol{b} = \boldsymbol{b}' + \boldsymbol{b}$", define $u := \Omega(\varphi \otimes I), v := \Omega(\psi \otimes I)$, so that $\boldsymbol{b} = u\boldsymbol{v}u^* + v\boldsymbol{v}v^*$.

B Truth

B.1 Viewpoints

It $\mathbb{R}$ is equipped with the Lebesgue measure μ, if T is a partial measure-preserving bijection from $X \subset \mathbb{R}$ to $Y \subset \mathbb{R}$, then $\widetilde{T}(f) := f \circ T$ defines a bounded operator on $\mathcal{L}^2(\mathbb{R})$. Indeed, $\widetilde{TU} = \widetilde{T}\widetilde{U}$, $\widetilde{T}^* = \widetilde{T^{-1}}$, whence $\widetilde{T}$ is a partial isometry, of domain and image $\widetilde{X}, \widetilde{Y}$ $(= \mathcal{L}^2(X), \mathcal{L}^2(Y))$ where X and Y denote the identity maps of X and Y.

Definition 33 (Viewpoints)
A viewpoint *is a normal representation of $\mathcal{R}$ in $\mathcal{L}^2(\mathbb{R})$, what we (abusively) write $\mathcal{R} \subset \mathcal{B}(\mathcal{L}^2(\mathbb{R}))$ such that, for any $X \subset \mathbb{R}$, with $\mu(X) < \infty$, $\widetilde{X} \in \mathcal{R}$ and* $\mathrm{tr}(\widetilde{X}) = \mu(X)$.

Lemma 26.1
Let $a \in \boldsymbol{p}\mathcal{R}\boldsymbol{p}$ with $\boldsymbol{p}$ finite and $\|a\| \leq 1$ be such that $\mathrm{tr}(a^n) = 0$ for all $n > 0$; then $\mathrm{ldet}(I - a) = 0$ in the two following cases:

(i) a is hermitian.

(ii) a is a partial isometry.

Proof : $2\mathrm{ldet}(I - a) = \mathrm{ldet}((I - a)(I - a^*)) = -log4 + \mathrm{ldet}(I - b)$, with $b := 1/4(3I + a + a^* - aa^*)$. Since $\|aa^* - a - a^*\| \leq 3$, $0 \leq b \leq I$ and $\mathrm{ldet}(b) = \sum_{n>0} \mathrm{tr}(b^n)/n$. If a is hermitian, the b^n are polynomials in a and $\mathrm{tr}(b^n) = (3/4)^n$, whence $\mathrm{ldet}(b) = \mathrm{colog}(1 - 3/4) = \log 4$ and $\mathrm{ldet}(I - a) = 0$. If a is partial isometry, then $b^n = x_n I + y_n(a + a^*) + z_n aa^* + w_n a^*a$ and $\mathrm{tr}(b^n) = x_n + (z_n + w_n)\mathrm{tr}(aa^*)$. The coefficients x_n, z_n, w_n do not depend upon a; in particular, if $a^2 = 0$, then $\mathrm{ldet}(I - a) = 0$ and $\mathrm{ldet}(I - b) = \log 4$, whence the two series $\sum_{n>0} x_n = \log 4$ and $\sum_{n>0}(z_n + w_n)\mathrm{tr}(aa^*) = 0$ are absolutely convergent. The same holds for any partial isometry a. □

Lemma 26.2
If $\widetilde{T} \in \mathcal{R}$, where T is a partial measure-preserving bijection from $X \subset \mathbb{R}$ ($\mu(X) < \infty$) to $Y \subset \mathbb{R}$; then $\mathrm{tr}(\widetilde{T}) = \mu(\{x \,;\, T(x) = x\})$.

Proof : if $Z \subset X$ is measurable, let $T_Z : Z \mapsto T(Z)$ be the restriction of T to Z. If $A := \{x \in X \,;\, T(x) \neq x\}$, then $\mathrm{tr}(T) = \mathrm{tr}(T_A) + \mu(X \setminus A)$: it remains to prove that $\mathrm{tr}(T_A) = 0$; in other terms that $\mathrm{tr}(T) = 0$ when $T(x) \neq x$ for all $x \in X$. By the strong continuity of the trace, there is a maximal (up to a negligibility) $Z \subset X$ such that $\mathrm{tr}(T_Z) = 0$. If, up to negligibility, $Z \neq X$, there is a non-negligible $W \subset X \setminus Z$ such that $T(W) \cap W = \emptyset$; it is immediate that $\mathrm{tr}(T_{Z \cup W}) = 0$, contradicting the choice of Z. Whence $\mathrm{tr}(T) = \mathrm{tr}(T_Z) = 0$. □

Proposition 26
If $\widetilde{T} \in \mathcal{R}$, where T is a partial measure-preserving bijection from $X \subset \mathbb{R}$ ($\mu(X) < \infty$) to $Y \subset \mathbb{R}$, then $\mathrm{ldet}(I - \widetilde{T}) = 0$ or $\mathrm{ldet}(I - \widetilde{T}) = \infty$.

Proof : if the set $\{z \,;\, \exists n > 0\; T^n(z) = z\}$ is of measure 0, lemma 26.2 yields $\mathrm{tr}(\widetilde{T^n}) = 0$ for all $n > 0$, whence, by lemma 26.1 *(ii)*, $\mathrm{ldet}(I - \widetilde{T}) = 0$. Otherwise, let $N > 0$ be such that $Z := \{z \,;\, T^N(z) \neq 0\}$ is not negligible. Then, writing $T = T_Z \cup (T \restriction Z)$ and $\widetilde{T} = \widetilde{T_Z} + \widetilde{T \restriction Z}$, with $T_Z \cdot (T \restriction Z)$, we get $\mathrm{ldet}(I - \widetilde{T}) = \mathrm{ldet}(I - \widetilde{T_Z}) + \mathrm{ldet}(I - \widetilde{T \restriction Z})$. By lemma 26.2, the terms

$\mathrm{tr}(\widetilde{T \upharpoonright Z}^{kN})/kN$ are equal to $\mu(Z)/kN$. Whence $\mathrm{ldet}(I - \widetilde{T \upharpoonright Z}) = +\infty$ and $\mathrm{ldet}(I - \widetilde{T}) = +\infty$. $\square$

Let us come back to the feedback equation: if w.r.t. a decomposition $I = \boldsymbol{a} \oplus \boldsymbol{b}$, $F := \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$ and $A = \boldsymbol{a}A\boldsymbol{a}$, then $[F]A = F_{22} + F_{21}A(\boldsymbol{a} - F_{11}A)^{-1}F_{12}$, provided $\boldsymbol{a} - F_{11}A$ is invertible. My contention is that this formula is still valid when $\boldsymbol{a} - F_{11}$ is injective and $(\boldsymbol{a} - F_{11}A)^{-1}F_{12}$ is total, whence (since of closed graph) bounded; I prove it in a particular case:

Lemma 27.1
If $\boldsymbol{a} - F_{11}A$ is injective, $\mathrm{ldet}(I - F_{11}A) < +\infty$ *and $(\boldsymbol{a} - F_{11}A)^{-1}F_{12}$ is total, then $[F]A = F_{22} + F_{21}A(\boldsymbol{a} - F_{11}A)^{-1}F_{12}$.*

Proof : let B be such that $\mathbf{b}B\boldsymbol{b} = B$; a standard computation shows that $\mathrm{ldet}(I - F \cdot (A+B)) = \mathrm{ldet}(I - F_{11}A) + \mathrm{ldet}(I - (F_{22} + F_{21}A(\boldsymbol{a} - F_{11}A)^{-1}F_{12}B)$. Whence, using theorem 3, $[F]A = F_{22} + F_{21}A(\boldsymbol{a} - F_{11}A)^{-1}F_{12}$. $\square$

Proposition 27
If $\boldsymbol{a} = \widetilde{X}, \boldsymbol{b} = \widetilde{Y}$ are disjoint carriers and if the partial measure-preserving bijections A, F induce partial isometries $\widetilde{A}, \widetilde{F} \in \mathcal{R}$ of respective carriers $\boldsymbol{a}$ and $\boldsymbol{a} + \boldsymbol{b}$, then $[F]A = \widetilde{U}$ for some partial measure-preserving bijection U.

Proof : let $Z := \{x \in \mathbf{R}\ ;\ \exists n > 0\ (F_{11}A)^n(x) = x\}$; one easily reduces the problem to the case where $\mu(Z) = 0$. Consider the partial bijection
$U := F_{22} \cup F_{21}(A \cup AF_{11}A \cup AF_{11}AF_{11}A \cup \ldots)F_{12}$: $\boldsymbol{a} + \widetilde{F_{11}A} + \widetilde{F_{11}AF_{11}A} + \ldots$ is a left inverse of $\boldsymbol{a} - F_{11}A$ and $(\boldsymbol{a} + \widetilde{F_{11}A} + \widetilde{F_{11}AF_{11}A} + \ldots)\widetilde{F_{12}}$ comes from a partial bijection and is thus bounded. The result follows from the lemma. $\square$

Remark 9
If $\mu(Z) = 0$, the $(\widetilde{F_{11}A})^N$ tend to 0, strongly: a case of *strong nilpotency.*

B.2 Subjective truth

If $\mathcal{R} \subset \mathcal{B}(\mathcal{L}^2(\mathbb{R}))$ is a viewpoint, then a base $\mathbf{e_1}, \ldots, \mathbf{e_n}$ of the idiom $\mathcal{A}$ induces a viewpoint $\mathcal{R} \otimes \mathcal{A} \subset \mathcal{B}(\mathcal{L}^2(\mathbb{R} \times \{1, \ldots, n\}))$ $(\simeq \mathcal{B}(\mathcal{L}^2(\mathbb{R})))$.

Definition 34 (Success)
$\mathfrak{a} := a \cdot\!+\!\cdot \alpha + A$ *is* successful *w.r.t. a viewpoint $\mathcal{R} \subset \mathcal{B}(\mathcal{L}^2(\mathbb{R}))$ when:*

(i) The carrier $\boldsymbol{a}$ of $\mathfrak{a}$ is of the form $\widetilde{X}$.

(ii) $\mathfrak{a}$ is wager-free ($a = 0$) and positive ($\alpha > 0$).

(iii) W.r.t. a base $\mathbf{e_1}, \ldots, \mathbf{e_n}$ of the idiom $\mathcal{A}$, $A = \widetilde{T}$ for a certain partial measure-preserving map from a subset of $\mathbb{R} \times \{1, \ldots, n\}$ to a subset of $\mathbb{R} \times \{1, \ldots, n\}$.

Definition 35 (Truth)
A conduct $\mathbf{A}$ *of carrier* $\boldsymbol{a}$ *is* true w.r.t. *a viewpoint* $\mathcal{R} \subset \mathcal{B}(\mathcal{L}^2(\mathbb{R}))$ *when* $\boldsymbol{a}$ *is of the form* $\widetilde{X}$ *and* $\mathbf{A}$ *contains a project* $\mathfrak{a}$ *successful w.r.t. the viewpoint.* $\mathbf{A}$ *is* false *when* $\sim\mathbf{A}$ *is true.*

Theorem 15 (Compositionality of truth)
If $\boldsymbol{a} = \widetilde{X}$, $\boldsymbol{b} = \widetilde{Y}$, *if the conducts* $\mathbf{A}$ *and* $\mathbf{A} \multimap \mathbf{B}$ *of respective carriers* $\boldsymbol{a}, \boldsymbol{a} + \mathrm{b}$ *are true, then* $\mathbf{B}$ *is true.*

Proof : $\mathfrak{a} := 0 \cdot\!+\!\cdot\, \alpha + \widetilde{T} \in \mathbf{A}, \mathfrak{f} := 0 \cdot\!+\!\cdot\, \varphi + \widetilde{U} \in \mathbf{A} \multimap \mathbf{B}$, then $[\mathfrak{f}]\mathfrak{a} := \mathrm{ldet}(I - \widetilde{U}^{\ddagger}_{11}\widetilde{T^{\dagger}}) \cdot\!+\!\cdot\, [\widetilde{U^{\ddagger}}]\widetilde{T}^{\dagger}$. If $\sim\mathbf{B}$ is true, then $0 \cdot\!+\!\cdot\, 1 + \underline{0} \in \mathbf{B}$ is successful; if $\mathfrak{b} \in \sim\mathbf{B}$, then $\ll [\mathfrak{f}]\mathfrak{a} \mid \mathfrak{b} \gg \neq \infty$ implies $\mathrm{ldet}(I - \widetilde{U}^{\ddagger}_{11}\widetilde{T^{\dagger}}) \neq \infty$, using theorem 3. Now, if T, U are measure-preserving partial bijections of $\mathbb{R} \times \{1, \ldots, n\}$, $\mathbb{R} \times \{1, \ldots, m\}$, then $\widetilde{T}^{\dagger}, \widetilde{U}^{\ddagger}$ come from partial bijections $T^{\dagger}, U^{\ddagger}$ of $\mathbb{R} \times \{1, \ldots, n\} \times \{1, \ldots, m\}$. By proposition 26, $\mathrm{ldet}(\underline{I - \widetilde{U}^{\ddagger}_{11}\widetilde{T^{\dagger}}}) = 0$, since the value ∞ has just been excluded. By proposition27, $[\widetilde{U^{\ddagger}}]\widetilde{T}^{\dagger}$ is of the form $\widetilde{V}$, whence $[\mathfrak{f}]\mathfrak{a}$ is successful. □

Corollary 15.1 (Subjective consistency)
A conduct cannot be both true and false w.r.t. a given viewpoint.

Proof : $\sim\mathbf{A} = \mathbf{A} \multimap \mathbf{0}$, where $\mathbf{0} := \{a \cdot\!+\!\cdot\, \alpha + 0\ ;\ a \neq 0\}$ of carrier 0 is the neutral element of « par », which contains no wager-free project. □

B.3 The subjective paradox

A conduct can be true or false depending on the viewpoint:

Proposition 28
There exists a conduct $\mathbf{C}$ *and viewpoints* $\mathcal{P}_1, \mathcal{P}_{\in}$ *such that* $\mathbf{C}$ *is true w.r.t.* $\mathcal{P}_1$ *and* $\sim\mathbf{C}$ *is true w.r.t.* $\mathcal{P}_2$*.*

Proof : should we define truth in the finite dimensional case, then a viewpoint would become a plain base. Let $u, v \in \mathcal{M}_3(\mathbb{C})$ be the partial symmetries:

$$u := \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad v := \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix} \quad \text{whence } I - uv = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-\sqrt{2}/2 & \sqrt{2}/2 \\ 0 & 0 & 1 \end{bmatrix}$$

$\det(I - uv) = 1 - \sqrt{2}/2 \neq 0, 1$. $\mathfrak{u} := 0 \cdot\!+\!\cdot\, 1 + u$ is successful w.r.t. the base $\{(\sqrt{2}/2, \sqrt{2}/2, 0), (\sqrt{2}/2, -\sqrt{2}/2, 0), (0, 0, 1)\}$, while $\mathfrak{v} := 0 \cdot\!+\!\cdot\, 1 + v$ succeeds w.r.t. the canonical base. It suffices to define $\mathbf{C} := \sim\{\mathfrak{v}\}$. The argument is made rigourous by replacing $\mathcal{M}_3(\mathbb{C})$ with $\mathcal{M}_3(\boldsymbol{p}\mathcal{R}\boldsymbol{p}) = \mathcal{M}_3(\mathbb{C}) \otimes \boldsymbol{p}\mathcal{R}\boldsymbol{p}$. □

B.4 Subjectivity vs. subjectivism

Subjectiv*ity* has nothing to do with subjectiv*ism*; it is indeed its antidote. Let us start with a famous example: Ptolemaic astronomy was the most « objective » science ever, allowing no room for the subject; however, it produced the subjectivistic delirium of *epicycles*. Later on, Kepler, Galileo, were able to disantangle this mess by restoring some subjectivity: observations are realtive to a *viewpoint*, Earth.

A similar problem occurs in logic, especially when dealing with cognitive questions: here the subject — the cognitive process — is part of the data. Take the interesting remark that an absent information is false: a bank has no record of its non-clients; handled *subjectively*, this works swell, since one can easily determine whether or not the bank *considers* Mr. Girard as part of its clients. The same idea, handled *objectively*, would amount at deciding whether or not Mr. Girard is a client under, say, an assumed name, i.e., independently of any cognitive process. The replacement of « I don't know » by « one cannot know », i.e., « one cannot prove » led to those modern epicycles — non-monotonic logics, closed world assumption, etc. —, which were refuted long before their invention by *incompleteness*, which, in fine, exposes the limitations of a blunt objectivity.

A *constructive*[9] approach thus requires to rebuild the objectivity by taking into account the subjective aspects of logic: after all, logic is about about reasoning, language, etc. In contrast with the objectivistic fantasy known as *semantics*.

For instance, formulas do no proceed from the sky; they proceed from their own operationality. What can be internalised by means of the negation, which thus takes in charge logical *normativity*: before *refuting*, negation *forbids*. This idea of « negation as norm » was first implemented in *ludics* [9]: although the expression has a game-theoretic flavour, ludics is strongly antagonistic to « game semantics » which, as the name suggests, relies on a *ready-made* normativity, thus missing the point.

GoI is even more radical, since it introduces a doubt — absent from ludics — as to the underlying combinatoric universe. The idea being that, like in the quantum world, logical artifacts interact « wavelike », but that questions like truth are rather base-dependent, i.e., « measurement-like ».

This approach can hardly be considered as subjectivistic. Typically, the choice of a viewpoint is implicit in the statement of a problem: through the decomposition of a formula into its significant subformulas — a decomposition which suits our *own* analyticity, thus subjective. In practice, there will be a preferred viewpoint — like in astronomy, the geocentric viewpoint —, but the existence of other viewpoints — « non standard » ones, if the term were not too heavily connoted — introduces unexpected possibilities of interpretation.

[9] Forget the sectarian connotation taken nowadays by this expression, which basically means that object and subject must be constructed, do not preexist.

C Von Neumann algebras

C.1 Operator algebras

There are two main brands of operator algebras:

C^*-algebras: complex Banach involutive Banach algebras, whose norm satisfies $\|uu^*\| = \|u\|^2$.

Von Neumann algebras: sub-C^* algebras of some $\mathcal{B}(\mathbb{H})$ closed under the weak (equivalently the strong, the ultraweak) topology. The most standard definition is the equality to the bicommutant.

A commutative C^*-algebra is isomorphic to a space $C(X)$: continuous complex-valued functions on some compact set X. A commutative vN algebra is isomorphic to some $\mathcal{L}^\infty(X, \mu)$, where (X, μ) is a measure space. The two main brands of operator algebras can thus be described as non commutative topology or non commutative measure theory.

C.2 The predual

Von Neumann algebras are conveniently considered *implemented*, i.e., acting on some Hilbert space $\mathbb{H}$, whence the inclusion $\mathcal{A} \subset \mathcal{B}(\mathbb{H})$. But the theory does not depend upon any particular representation; when dealing with isomorphisms of vN algebras, one must consider *normal* $*$-isomorphisms, i.e., isomorphisms of C^*-algebras which are *utraweakly* continuous; ultraweak continuity is the same as weak continuity on the unit ball. It can also be characterised as the commutation to directed suprema of positive hermitians.

Up to isomorphism, von Neumann algebras are exactly the *dual C^**-algebras, i.e., those isomorphic to the dual of some Banach space. The *predual* of the vN algebra $\mathcal{A}$, unique up to isomorphism, consists of the *ultraweakly* continuous forms, often styled *normal*, i.e., weakly continuous on the unit ball of $\mathcal{A}$. Typically, the predual of ℓ^∞ is ℓ^1.

C.3 Factors

Definition 36 (Factors)
A factor *is a von Neumann algebra whose center is trivial, i.e., consists in the scalar multiples of the identity.*

The theory of von Neumann algebras reduces to the study of factors: $\mathcal{A}$ can be written as a sum of factors — discrete or continuous — indexed by its center.

Definition 37 (Comparison of projections)
Between the projections of a von Neumann algebra $\mathcal{A}$, one defines the preorder relation $\preccurlyeq$, with associated equivalence $\sim$:

$$\pi \sim \pi' \quad \Leftrightarrow \quad \exists u\ (u^*u = \pi \text{ and } uu^* = \pi') \tag{52}$$

$$\pi \preccurlyeq \pi' \quad \Leftrightarrow \quad \exists \pi''\ (\pi = \pi\pi'' \text{ and } \pi'' \sim \pi') \tag{53}$$

In a factor, the preorder $\sim$ is total; this induces a classification of factors over a separable Hilbert space:

- $\mathbf{I}_n$ (resp. $\mathbf{I}_\infty$): order type $\{0, \ldots, n\}$ (resp. $\mathbb{N} \cup \{+\infty\}$).
- $\mathbf{II}_1$(resp. $\mathbf{II}_\infty$): order type $[0, 1]$ (resp. $[0, +\infty]$).
- **III**: order type $\{0, +\infty\}$.

The symbol « $+\infty$ » has a special meaning; it denotes the class of the identity, when the identity is *infinite*, i.e., not alone in its equivalence class.

C.4 The trace

Definition 38 (Finiteness)
$\mathcal{A}$ is finite *when I stands alone in its equivalence class: $uu^* = I \Rightarrow u^*u = I$.*

The finite factors are those of type $\mathbf{I}_n (n < \infty)$ and $\mathbf{II}_1$. In a finite factor, given projections $\pi, \pi' \neq 0$, define an « euclidian division »: $\pi = \pi'_1 + \ldots + \pi'_n + \pi"$ with $\pi'_1 \sim \ldots \sim \pi'_n \sim \pi'$, $\pi" \precsim \pi', \pi" \not\sim \pi'$, what one writes $\pi \sim n.\pi' + \pi"$; n and the *remainder* $\pi"$ (up to $\sim$) are unique. This enables one to define the dimension of a projection by a continued fraction. Typically, π is of dimension $1/2$ when $\pi \sim I - \pi$. Dimension extends by linearity to linear combinations of projections, then to the full algebra by ultraweak continuity: this is the *trace*: finite algebras are algebras with a trace.

Definition 39 (Trace)
In the vN algebra $\mathcal{A}$, a trace *is an ultraweakly continuous state τ such that:*

$$\tau(uv) = \tau(vu)$$

The trace is thus an element of the *predual*.

Proposition 29
A factor is finite iff it admits a trace (necessarily unique).

For factors of type $\mathbf{I}_n$, the trace (in the sense of definition 39) is obtained by renormalising the usual algebraic trace: $\tau(u) = 1/n \cdot \mathrm{Tr}(u)$.

C.5 Algebra of a discrete group

If $\mathfrak{G}$ is a discrete (i.e., finite or denumerable) group, the space of complex linear combinations of elements of $\mathfrak{G}$ is the *convolution* ring $\mathcal{A}(\mathfrak{G})$ of $\mathfrak{G}$:

$$(\sum_g x_g \cdot g) * (\sum_h y_h \cdot h) := \sum_{gh=k} x_g y_h \cdot k \tag{54}$$

The convolution product can be extended to square-summable sequence: it sends $\ell^2(\mathfrak{G}) \times \ell^2(\mathfrak{G})$ into $\ell^\infty(\mathfrak{G})$.

Definition 40 (Algebra of a group)
The group algebra of $\mathfrak{G}$ *is defined as:*

$$\mathcal{A}[\mathfrak{G}] := \{x \in \ell^2(\mathfrak{G}) \; ; \; \forall y \in \ell^2(\mathfrak{G}) \quad x * y \in \ell^2(\mathfrak{G})\} \tag{55}$$

$x \in \mathcal{A}[\mathfrak{G}]$ induces an operator on the space space $\ell^2(\mathfrak{G})$; $\mathcal{A}[\mathfrak{G}]$ is thus identified with a subalgebra of $\mathcal{B}(\ell^2(\mathfrak{G}))$, indeed a vN algebra, since the commutant of the right convolutions $r_g(y) := y * g$.

$\mathcal{A}[\mathfrak{G}]$ admits the *trace*:

$$\mathrm{tr}(\sum_g x_g \cdot g) := x_1 \tag{56}$$

Proposition 30
$\mathcal{A}[\mathfrak{G}]$ *is a finite algebra.*

Definition 41 (i.c.c. groups)
$\mathfrak{G}$ *is with* infinite conjugacy classes *(i.c.c.) iff, for all* $g \in \mathfrak{G}, g \neq 1$, *the set* $\{h^{-1}gh; h \in \mathfrak{G}\}$ *of conjugates of* g *is infinite.*

Proposition 31
The algebra $\mathcal{A}[\mathfrak{G}]$ *of an i.c.c. group is a type* $\mathbf{II}_1$ *factor.*

C.6 Crossed products

Let $\mathfrak{G}$ be a discrete group and α be an automorphic representation of $\mathfrak{G}$ on $\mathcal{A}$, i.e., an homomorphism associating to any $g \in \mathfrak{G}$ an automorphism α_g of the vN algebra $\mathcal{A} \subset \mathcal{B}(\mathbb{H})$. On the Hilbert space $\mathbb{H} \otimes \ell^2(\mathfrak{G})$, we can consider:

- For $u \in \mathcal{A}$, the operators $\tilde{\alpha}(u)(x \otimes g) := \alpha_{g^{-1}}(u)(x) \otimes g$.
- For $g \in \mathfrak{G}$ the operators $\ell_g(x \otimes h) := x \otimes gh$.

Definition 42 (Crossed product)
The crossed product $\mathcal{A} \rtimes_\alpha \mathfrak{G}$ *is the vN subalgebra of* $\mathcal{A} \otimes \mathcal{A}[\mathfrak{G}]$ *generated by (i.e., the bicommutant of) the* $\tilde{\alpha}(u)$ *and the* ℓ_g.

One easily checks that:

$$\ell_g \tilde{\alpha}(u) \ell_g^* = \tilde{\alpha}(\alpha_g(u)) \tag{57}$$

The $\tilde{\alpha}(u)$ thus generate a vN algebra isomorphic with $\mathcal{A}$, and the conjugations $u \rightsquigarrow \ell_g u \ell_g^*$ act as the original α_g. In other terms, $\mathcal{A} \rtimes_\alpha \mathfrak{G}$ is the vN algebra obtained from $\mathcal{A}$ by « internalising » the α_g.

Proposition 32
If $\mathcal{A}$ *is a factor and the* α_g *are* outer *for* $g \neq 1$, *then* $\mathcal{A} \rtimes_\alpha \mathfrak{G}$ *is a factor.*

C.7 Hyperfiniteness

Definition 43
A vN algebra $\mathcal{A}$ is hyperfinite *when there exists a denumerable sequence $A_0 \subset A_1 \subset A_2 \ldots$ of finite-dimensional subalgebras s.t. $\mathcal{A}$ is the closure (weak, strong, or the bicommutant) of the union $\bigcup_n \mathcal{A}_n$.*

Hyperfiniteness has nothing to do with finiteness; in most types: $\mathbf{I}_n, \mathbf{I}_\infty, \mathbf{II}_1$, $\mathbf{II}_\infty, \mathbf{III}_\lambda (0 < \lambda \leq 1)$, there is exactly one hyperfinite factor. Typically:

Theorem 16 (Murray-von Neumann)
Up to isomorphism, there is only one hyperfinite factor of type $\mathbf{II}_1$, the one usually referred to as « the » hyperefinite factor.

C.8 Amenable groups

The most important characterisation of hyperfiniteness is due to Connes:

Theorem 17 (Injectivity)
A vN algebra $\mathcal{A}$ is hyperfinite iff it is injective, *i.e., if there is a linear projection Π of norm 1 of $\mathcal{B}(\mathbb{H})$ onto $\mathcal{A}$.*

Proposition 33 (Tomiyama, 1957)
If Π is a linear projection of $\mathcal{B}(\mathbb{H})$ onto $\mathcal{A}$ such that $\|\Pi(u)\| \leq \|u\| (u \in \mathcal{B}(\mathbb{H}))$, then Π is a conditional expectation, *i.e.:*

(i) Π is positive: $\Pi(u) \geq 0$ when $u \geq 0$.

(ii) $\Pi(I) = I$.

(iii) If $a, b \in \mathcal{A}, u \in \mathcal{B}(\mathbb{H})$, then $\Pi(aub) = a\Pi(u)b$.

Coming back to group algebras, $\mathcal{A}[\mathfrak{G}]$ is injective iff $\mathfrak{G}$ is *amenable*:

Definition 44 (Amenability)
An invariant mean *on $\mathfrak{G}$ is a state on $\ell^\infty(\mathfrak{G})$ which is left invariant:*

$$\mu(\sum_g x_g \cdot g) = \mu(\sum_g x_g \cdot hg) \tag{58}$$

$\mathfrak{G}$ is amenable *iff it admits an invariant mean.*

Remember that a *state* μ on a vN algebra $\mathcal{A}$ is a positive ($\mu(uu^*) \geq 0$) and normalised ($\mu(1) = 1$) linear form on A.

The typical non amenable group is the free group with two generators; fortunately for us, there is an amenable group containing a copy of the free monoid (proposition 19).

The crossed product of the hyperfinite factor $\mathcal{H}$ with an amenable group of outer atomorphisms remains hyperfinite, i.e., is isomorphic to $\mathcal{H}$.

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