Jean-Yves Girard
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  - For instance many isomorphic (standard !) versions of $\mathbb{N}$.
  - Non internally isomorphic.
  - Possibility of *subjective* truth.
- Got beyond the essential(ist) circularity of logic, the *blind spot*.
I - THE BLIND SPOT
2-Existence vs. Essence

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2-EXISTENCE VS. ESSENCE

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- « God created integers, everything else is the deed of man ». 
3-PERFECT VS. IMPERFECT

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► Set theory problematic in extreme situations (foundations).
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► Accept foundations with most of operations external.
6-THE ICONOCLAST PROGRAMME

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- The Murray-von Neumann factor \( \mathcal{R} \).
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  - Finite and hyperfinite, both notions of finiteness having noting to do with Hilbertian finitism.
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- The Murray-von Neumann factor $R$.
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- Forget the idea of creation in 7 days, from simple to complicated (sets, algebra, reals, function spaces) since it does not work anyway (Incompleteness theorem).
II-The CATEGORICAL LAYER
7-The three layers

- Foundations can be operated at three *layers* (undergrounds) :
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- Foundations can be operated at three layers (undergrounds):
  -2: Functions: categories, formulas as objects, proofs as morphisms.
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- Foundations can be operated at three layers (undergrounds):
  -1: **Truth**: consistency, models: bleak.
  -2: **Functions**: categories, formulas as objects, proofs as morphisms.
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    - Coherent spaces.
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- Level $-2$ not fit to go beyond the blind spot.
8-SCOTT DOMAINS

- A Scott domain $X$ is a set $\lvert X \rvert$ equipped with a consistent system of intuitionistic sequents $\Gamma \vdash \Delta$, $\Gamma, \Delta \subset \lvert X \rvert$. 

Keio 16/17 Mars 2006
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  - **Continuous map**: Functor preserving direct limits.
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9-Stability

- Pull-backs are the natural companion of direct limits.
- Correspond to $a \cap b$ provided $a \cup b$ is consistent.
- Preservation of pull-backs a.k.a. stability (Berry):
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  F(a \cap b) = F(a) \cap F(b) \quad (a \cup b \text{ consistent})
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  \[ F(a) = \{ y ; \exists x \in a \ (x, y) \in \text{Sq}(F) \} \]  
  \[ (5) \]
11-DESESSENTIALISATION

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  Cardinal: Replaced by bilinear form, or better, trace.
12-Finite dimensional Hermitian geometry

- Hilbert space $\mathbb{C}^n$ equipped with sesquilinear form:
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- If $h, k$ hermitian, then $\text{tr}(h \cdot k) \in \mathbb{R}$. 
The desessentialised version adapts *mutatis mutandis*:
13-QUANTUM COHERENT SPACES

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  **Web**: Finite dimensional Hilbert space $\mathbb{X}$. 
13-Quantum Coherent Spaces

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The desessentialised version adapts \textit{mutatis mutandis}: 

\textbf{Web} : Finite dimensional Hilbert space $X$. 
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    tr(\pi_a \cdot \pi_b) = \#(a \cap b)
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**Functional application** (involves $X \otimes Y$):
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- **Functional application (involves \( X \otimes Y \))**:
  \[
  \text{tr}(F(a) \cdot b)) = \text{tr}(\text{Sq}(F) \cdot (a \otimes b)) \tag{13}
  \]
14-SUBJECT AND OBJECT

- Hidden assumption: commutativity (diagonal).
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- The points of the diagonal correspond to atoms.
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- Difference between twist (identity) and its etaspansion:

\[ \sigma := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \eta := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \] (14)
15-QUANTUM BOOLEANS

- Spin, a two-state system, represented by $2 \times 2$ matrices:
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$$\text{true} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{false} := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

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15-QUANTUMBOOLEANS

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\begin{align*}
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\end{align*}
\]

Tilting the gyros: quantum booleans:
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  $z \in \mathbb{C} \cup \{+\infty\}$
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\frac{1}{1 + z\overline{z}} \begin{bmatrix} 1 & z \\ z & z\overline{z} \end{bmatrix} \quad z \in \mathbb{C} \cup \{+\infty\}
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(16)

- **Measurement** is operated by $\eta$-expansion:
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$$\frac{1}{1 + zz} \begin{bmatrix} 1 & z \\ z & zz \end{bmatrix} \quad z \in \mathbb{C} \cup \{+\infty\}$$  \hspace{1cm} (16)

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$$\eta(\begin{bmatrix} a & \bar{b} \\ b & c \end{bmatrix}) = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}$$  \hspace{1cm} (17)
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  \[
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  \]

- Chops off the antidiagonal coefficients; yields probabilistic boolean: $\lambda \cdot \text{true} + (1 - \lambda) \cdot \text{false}$, with $\lambda := \frac{1}{1 + z\bar{z}}$. 
III-Passage to infinity
16-THE UNFINISHED

- Infinite = perennial = duplicable = imperfect (unfinished).
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- Dedekind integers (system F version) :

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(18)
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- Heavily rely on exponentials. Four laws:

```
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  **Weakening:** $\forall A \vdash 1$. 

Keio 16/17 Mars 2006
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- Dedekind integers (system F version):
  \[ \text{nat} := \forall X \left( \! (X \rightarrow X) \rightarrow (X \rightarrow X) \right) \]

- Heavily rely on exponentials. Four laws:
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- These rules express our vision of infinity. Strongly influenced by Western theology (Thomas Aquinus).
- Just as opaque as integers. At least this is logic.
- Light logics ($LLL$, $ELL$...) ; not grounded. But some hope!
17-Quantum Coherent Spaces

- Can we use infinite dimensional Hilbert spaces?
17-QUANTUM COHERENT SPACES

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- Typical example: space $\ell^2$ of square-summable sequences:
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$$\langle (x_n) \mid (y_n) \rangle := \sum_n x_n \cdot \overline{y_n}$$ (19)
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- Something wrong with the methodology.
18-IMMANENT JUSTICE

- When God created the universe, he first defined the actual, then the potential.
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- Reflected in **Kripke models** : parallel universes like butterflies.
- Obviously, the potential should remain potential.
- The same is true of categories : composition **costs nothing**.
- Because operations have been performed **in advance**.
- This actualisation of potentialities is possible in finite dimension ; in infinite dimension, it **diverges**, yielding useless values, zero or infinite.
- **GoI** : a potential interpretation which remains potential.
19- THE DETERMINANT

- Other invariant (after $\#(a \cap b)$ and $\text{tr}(h \cdot k)$):
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- Other invariant (after \( ♯(a \cap b) \) and \( \text{tr}(h \cdot k) \)):
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  \det(I + u) = \text{tr}(\Lambda u) \quad (22)
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- In finite dimension, use exterior algebra (Fock space), and observe that:
  \[ \det(I + u) = \text{tr}(\Lambda u) \] (22)
- Actualisation is the functor $\Lambda_i h$ : it lists all cycles, all possibilities:
  \[ \det(I - h k) = \text{tr}((\Lambda_i h)(\Lambda_i k)) \] (23)
- Equation (22) does not pass infinite limits. Remains the determinant, i.e., GoI. One should remain potential.
20 - The Flush

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No Hilbert Hotel, since rooms have a **size** (trace, dimension).
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- No Hilbert Hotel, since rooms have a size (trace, dimension).
- Responsible for dereliction.
Another flush: fresh variables.
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Typical flush obtained by internalising the isometry:
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- Typical flush obtained by internalising the isometry:

\[
\mathbf{X} \otimes (\mathbf{X} \otimes \mathbf{X}) \sim (\mathbf{X} \otimes \mathbf{X}) \otimes \mathbf{X}
\]  

(26)
21-The flush (continued)

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- Typical flush obtained by internalising the isometry:
  \[ X \otimes (X \otimes X) \sim (X \otimes X) \otimes X \]  
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- Starting with \( u \otimes I = u \otimes (I \otimes I) \), one gets \( (u \otimes I) \otimes I \).
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- Not possible in the hyperfinite factor.
- The Murray-von Neumann factor (finite and hyperfinite) seems the appropriate space for true finitism.
IV-\textbf{C}*-\textbf{-ALGEBRAS}
22-Definition and Examples

- Complex involutive Banach algebra such that:
22-DEFINITION AND EXAMPLES

- Complex involutive Banach algebra such that:
  \[ \|uu^*\| = \|u\|^2 \]  (27)
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- Space \( \mathbb{C}(X) \) of complex continuous functions on compact \( X \).
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- Space \( \mathbb{C}(X) \) of complex continuous functions on compact \( X \).
  - Indeed the generic commutative example.
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- Complex involutive Banach algebra such that:
  \[ \|uu^*\| = \|u\|^2 \]  

- Space \( \mathbb{C}(X) \) of complex continuous functions on compact \( X \).
  - Indeed the generic commutative example.
  - If \( C \) commutative, take for \( X \) the space of characters.
22-Definition and Examples

- Complex involutive Banach algebra such that:
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  - B.t.w., character = pure (extremal) state.
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- Space \( \mathbb{C}(X) \) of complex continuous functions on compact \( X \).
  - Indeed the generic **commutative** example.
  - If \( C \) commutative, take for \( X \) the space of **characters**.
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  - State: linear form \( \rho \) such that \( \rho(uu^*) \geq 0 \), \( \rho(I) = 1 \).
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- Space \( \mathcal{B}(\mathbb{H}) \) of bounded operators on Hilbert space \( \mathbb{H} \).
  - Involution defined by \( \langle u^*(x) \mid y \rangle := \langle x \mid u(y) \rangle \).
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- Space \( \mathcal{B}(H) \) of bounded operators on Hilbert space \( H \).
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  - Involution defined by \( \langle u^*(x) \mid y \rangle := \langle x \mid u(y) \rangle \).
  - Subalgebras of \( \mathcal{B}(\mathbb{H}) \) are generic C*-algebras.
  - Non equivalent faithful representations on \( \mathbb{H} \).
23-SIMPLICITY

Morphisms of $C^*$-algebras defined algebraically.
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  - Use $\|uu^*\| = \|u\|^2$ to reduce to positive hermitians $uu^*$. 
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- Indeed bounded, $||\varphi(u)|| \leq ||u||$:
  - Use $||uu^*|| = ||u||^2$ to reduce to positive hermitians $uu^*$.
  - Use $||uu^*|| = r(\text{Sp}(uu^*))$ to define the norm algebraically.
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- Typical example: matrix algebras $\mathcal{M}_n(\mathbb{C})$.
- $\mathcal{B}(\mathbb{H})$ not simple (infinite dimension) : compact operators.
24-The CAR Algebra

 Canonical anticommutation relations, between creators $\kappa(a)$ and their adjoints, the annihilators $\zeta(b)$:
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  - The (exterior) Fock space : represent $\kappa(a)(x) := a \wedge x$. 
V-vN ALGEBRAS
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- Also: the commutant of a self-adjoint subset of $\mathcal{B}(\mathbb{H})$. 
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26-Commutative VN Algebras

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- In general: C*-algebra + faithful state \( \rho \) (i.e., \( \rho(uu^*) = 0 \) implies \( u = 0 \).) yields a vN completion.
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- In general: C*-algebra + faithful state $\rho$ (i.e., $\rho(uu^*) = 0$ implies $u = 0$.) yields a vN completion.
- The CAR-algebra admits completions of all types I, II, III.
27-The GNS Construction

- From a C*-algebra $C$ and a state $\rho$ construct a representation.
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- Typical case: simple algebras.
28-**THE CAR ALGEBRA**

- Indeed inductive limit of matrices $2^n \times 2^n$. 
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Up to isomorphism, only one such vN algebra, the Murray-von Neumann factor $\mathcal{R}$. 
VI-The finite/hyperfinite factor
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Each \( \mathcal{A}(x) \) is a factor, i.e., a vN algebra with trivial center.

Classification of vN algebras thus reduces to classification of factors.
30-COMPARISON OF PROJECTIONS

- Equivalence of projections:

\[ \pi \simeq \pi' \iff \exists u \ (u^*u = \pi \ \text{and} \ \text{uu}^* = \pi') \] (33)
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- Ordering of projections (inclusion + equivalence):
  \[ \pi \preceq \pi' \iff \exists \pi'' \ (\pi = \pi \pi'' \text{ and } \pi'' \simeq \pi') \]  
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30-Comparison of Projections

- **Equivalence of projections**: 
  \[ \pi \simeq \pi' \iff \exists u \ (u^*u = \pi \text{ and } uu^* = \pi') \]  
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31-Traces

- Finiteness is the same as the existence of a normal (weakly continuous on the unit ball) trace.
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- The completion of the CAR-algebra is finite and infinite-dimensional:
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- On a finite factor, the trace is unique.
32-Discrete Groups

$G$ denumerable induces a convolution algebra, obtained by linearisation.
**32-Discrete Groups**

- $G$ denumerable induces a *convolution* algebra, obtained by linearisation.
- The convolution:

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(x_g) \ast (y_g) := \left( \sum_{g=g' \cdot g''} x_{g'} \cdot y_{g''} \right)
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- Define $A(G) := \{(x_g); (x_g) \ast : \ell^2(G) \leadsto \ell^2(G)\}$. 
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- Define $\mathcal{A}(G) := \{(x_g); (x_g) \ast : \ell^2(G) \hookrightarrow \ell^2(G)\}$.

- $\mathcal{A}(G)$ is the commutant of the right convolutions $\ast (y_g)$.
**32-Discrete Groups**

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- If $G$ has infinite conjugacy classes (i.c.c.), then $\mathcal{A}(G)$ is a factor.
- B.t.w., $\text{tr}((x_g)) = x_1$. 
33-HYPERFINITISM

- If $G \subset G'$, then $\mathcal{A}(G) \hookrightarrow \mathcal{A}(G')$. 
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- If $G \subset G'$, then $\mathcal{A}(G) \hookrightarrow \mathcal{A}(G')$.
- If $G$ is \textit{locally finite}, the union $\bigcup_n \mathcal{A}(G_n)$ is weakly dense.
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- There are hyperfinite algebras of any type (close the CAR algebra w.r.t. appropriate state).
- But only one hyperfinite factor of type $\text{II}_1$. Murray-von Neumann factor $\mathcal{R}$. 
34-The hyperfinite factor

- The factor $\mathcal{R}$ is remarkably stable:
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34 - THE HYPERFINITE FACTOR

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  - But adding $\mathcal{M}_2(\mathcal{R}) \sim \mathcal{R}$ leads to a type III factor.
VII-Gol
35-The Feedback Equation

- Basic paradigm:
35-The feedback equation

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\[ h(x \oplus y) = x' \oplus \sigma(y) \]
35-THE FEEDBACK EQUATION

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  - Associativity: \( (\sigma + \tau)[h] = \sigma[\tau[h]] \).
36-The determinant

- In finite dimension:
36- THE DETERMINANT

In finite dimension:

$$\det \begin{bmatrix} I - a & b \\ b^* & c \end{bmatrix} = \det(I - a) \cdot \det(I - (c + b^*(I - a)^{-1}b))$$

(40)
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- Familiar manipulations on determinants accessible through (converging) power series.
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- Old style: interprets proofs by operators.
37-GoI in a VN algebra

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  - Are galaxies made of stars or is it the other way around?
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- New style: takes place in the Murray-vN factor R:
  - Finiteness forbids the primitives p, q, d.
  - In a finite algebra, \( pp^* = I \Rightarrow p^*p = I \).
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- **Old style**: interprets proofs by *operators*.
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- **New style**: takes place in the Murray-vN factor $\mathcal{R}$:
  - Finiteness forbids the primitives $p, q, d$.
    - In a finite algebra, $pp^* = I \Rightarrow p^*p = I$.
  - Hyperfiniteness forbids $t(u \otimes (v \otimes w))t^* = (u \otimes v) \otimes w$. 
VIII-Finite GoI
38-Finite GoI

- A base is the pair \((\xi, \xi')\) of two orthogonal projections of the same dimension \(\neq 0\) (default \(1/2\)).
38-FINITE GOI

- A base is the pair \((\xi, \xi')\) of two orthogonal projections of the same dimension \(\neq 0\) (default \(1/2\)).
- Design of base \((\xi, \xi')\): \((\delta, h) \in \mathbb{R} \times \mathcal{R} \otimes \mathcal{R}\) such that:
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  - \(\delta \in \mathbb{R}\) s.t. \(0 \leq \delta < 2^{1-\dim \xi}\) is the daimon.
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Duality on the same base : given \(h, k\) :
38-Finite GoI

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  - \(\delta \in \mathbb{R}\) s.t. \(0 \leq \delta < 2^{1-\text{dim} \xi}\) is the **daimon**.
- **Duality on the same base** : given \(h, k\):
  - Tensorise \(h, k\) with \(I\), swap the two \(\mathcal{R}\), to get \(h', k''\).
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Duality on the same base : given \(h, k\) :

- Tensorise \(h, k\) with \(I\), swap the two \(\mathcal{R}\), to get \(h', k''\) :
  \[* \cdot \otimes \mapsto \cdot \otimes \cdot \otimes I*\]
**38-Finite GOL**

- **A base** is the pair \((\xi, \xi')\) of two orthogonal projections of the same dimension \(\neq 0\) (default \(1/2\)).

- **Design of base** \((\xi, \xi')\) : \((\delta, h) \in \mathbb{R} \times \mathcal{R} \otimes \mathcal{R}\) such that:
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  - \(\delta \in \mathbb{R}\) s.t. \(0 \leq \delta < 2^{1-\dim \xi}\) is the daimon.

- **Duality on the same base** : given \(h, k\) :
  - Tensorise \(h, k\) with \(I\), swap the two \(\mathcal{R}\), to get \(h', k''\) :
    - \(*\otimes \cdot \mapsto \cdot \otimes \cdot \otimes I\)
    - \(*\otimes \cdot \mapsto \cdot \otimes I \otimes \cdot\)
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- A base is the pair $(\xi, \xi')$ of two orthogonal projections of the same dimension $\neq 0$ (default $1/2$).

- Design of base $(\xi, \xi') : (\delta, h) \in \mathbb{R} \times \mathcal{R} \otimes \mathcal{R}$ such that:
  - $h$ hermitian of support $\subset \xi \otimes I$ of norm $\leq 1$.
  - Second tensor component $\mathcal{R}$ is the dialect.
  - $\delta \in \mathbb{R}$ s.t. $0 \leq \delta < 2^{1-\dim \xi}$ is the daimon.

- Duality on the same base: given $h, k$:
  - Tensorise $h, k$ with $I$, swap the two $\mathcal{R}$, to get $h', k''$:
    - $\rightarrow \odot \otimes \rightarrow \odot \otimes I$
    - $\rightarrow \odot \otimes \rightarrow \otimes I \otimes$
  - $(\delta, h), (\epsilon, k)$ are polar, notation $(\delta, h) \perp (\epsilon, k)$ iff:
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**Duality on the same base** : given \(h, k\) :
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  * \(\cdot \otimes \cdot \mapsto \cdot \otimes I \otimes \cdot\)
  * \(\cdot \otimes \cdot \mapsto \cdot I \otimes \cdot\)
- \((\delta, h), (\epsilon, k)\) are polar, notation \((\delta, h) \perp (\epsilon, k)\) iff :
  \[
  r(h'k'') < 1 \quad \delta \cdot \epsilon \cdot \det(I - h'k'') \neq 1 \quad (41)
  \]
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Duality on the same base : given \(h, k\) :

- Tensorise \(h, k\) with \(I\), swap the two \(\mathcal{R}\), to get \(h', k''\) :
  - \(\odot \odot \rightarrow \odot \mathcal{I} \odot \odot \)
  - \(\odot \odot \rightarrow \odot \mathcal{I} \odot \odot \)
- \((\delta, h), (\epsilon, k)\) are polar, notation \((\delta, h) \lessdot (\epsilon, k)\) iff :
  \[ r(h'k'') < 1 \quad \delta \cdot \epsilon \cdot \det(I - h'k'') \neq 1 \] (41)
- Behaviour : set \(B\) of designs of given base s.t. \(B = \sim\sim B\).
39-SEQUENTS

- Heavy use of the cobase $\xi'$. 
Heavy use of the cobase $\xi'$.

Binary example $(\xi, \xi') \vdash (\eta, \eta')$:
39-Sequents

- Heavy use of the cobase $\xi'$.
- Binary example $(\xi, \xi') \vdash (\eta, \eta')$:
  - $2 \times 2$ matrix with entries in $\mathcal{R} \otimes \mathcal{R} \otimes \mathcal{R}$. 
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- Binary example $(\xi, \xi')\vdash (\eta, \eta')$:
  - $2 \times 2$ matrix with entries in $\mathcal{R} \otimes \mathcal{R} \otimes \mathcal{R}$.
  - Supports $\xi \otimes \eta' \otimes I$, $\eta \otimes \xi' \otimes I$. 
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- Let $(\gamma, h)$ and $(\delta, k)$ of respective bases $(\xi, \xi')$ replace:
  - In $h$, $\cdot \otimes \cdot$ with $\cdot \otimes \eta' \otimes \cdot \otimes I$ yields $h'$
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  - In $k$, $\cdot \otimes \cdot \otimes \cdot$ with $\cdot \otimes \cdot \otimes I \otimes \cdot$: yields $k''$
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  - In $k$, $\cdot \otimes \cdot \otimes$ with $\cdot \otimes \cdot \otimes I \otimes \cdot$: yields $k''$
- Apply GoI, which yields $l$. 
39-Sequents

- Heavy use of the cobase $\xi'$.
- Binary example $(\xi, \xi') \vdash (\eta, \eta')$:
  - $2 \times 2$ matrix with entries in $R \otimes R \otimes R$.
  - Supports $\xi \otimes \eta' \otimes I, \eta \otimes \xi' \otimes I$.
  - All supports have same dimension: no need for $p, q$.
- Let $(\gamma, h)$ and $(\delta, k)$ of respective bases $(\xi, \xi')$ replace:
  - In $h$, $\cdot \otimes \cdot \otimes \eta' \otimes \cdot \otimes I$: yields $h'$
  - In $k$, $\cdot \otimes \cdot \otimes I \otimes \cdot \otimes I \otimes \cdot$: yields $k''$
- Apply Gol, which yields $l$.
- Output: $(\gamma^{\dim(\eta)} \cdot \delta \cdot \det(I - h' \cdot k''), l)$
40-MULTIPLICATIVES

- The fax (identity axiom) :
**40-MULTIPLICATIVES**

- **The fax (identity axiom):**

\[
\begin{bmatrix}
0 & \xi \otimes \xi' \otimes I \\
\xi \otimes \xi' \otimes I & 0
\end{bmatrix}
\]  

(42)
The fax (identity axiom):

\[
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  \[\text{(42)}\]

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- Not an etaspansion.
- If \(\dim(\xi)\) rational, finite matrix with entries \(= 0, 1\).
- Tensor (cotensor) product replaces \((\xi, \xi'), (\eta, \eta')\) with
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Tensor (cotensor) product replaces \((\xi, \xi'), (\eta, \eta')\) with \((\xi \otimes \eta' + \xi' \otimes \eta, \xi \otimes \eta + \xi' \otimes \eta')\).

- Basically use an isometry \( \varphi \) between \( \xi' \otimes \eta \) and \( \eta \otimes \xi' \).
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- The fax (identity axiom):
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  \end{bmatrix}
  \]  
  (42)

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- Basically use an isometry \( \varphi \) between \( \xi' \otimes \eta \) and \( \eta \otimes \xi'. \)

- \( \varphi \) is part of the data.
The fax (identity axiom):
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\( \varphi \) is part of the data.

\( A \rightarrow A \) based on \((\xi \otimes \xi' + \xi' \otimes \xi, \xi \otimes \xi + \xi' \otimes \xi')\).
Additive situation: $\xi, \xi', \eta, \eta'$ pairwise orthogonal.
41 - THE ADDITIVE MIRACLE

- Additive situation: $\xi, \xi', \eta, \eta'$ pairwise orthogonal.
- Replace $(\xi, \xi'), (\eta, \eta')$ with $(\xi + \eta, \xi' + \eta')$. 
41-The additive miracle

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- The with rule (how to share contexts):
41-The additive miracle

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  - Premises are $2 \times 2$ matrices:
41-The Additive Miracle

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  - Premises are $2 \times 2$ matrices:
  - Their supports are $\xi \otimes \nu' \otimes I, \nu \otimes \xi' \otimes I$ and $\eta \otimes \nu' \otimes I, \nu \otimes \eta' \otimes I.$
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  - Just sum them: disjoint supports.
41-THE ADDITIVE MIRACLE

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  - Premises are $2 \times 2$ matrices:
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  - Just sum them: disjoint supports.
► Violently anti-$\eta$, like Quantum coherent spaces.
41-The additive miracle

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- Replace $(\xi, \xi'), (\eta, \eta')$ with $(\xi + \eta, \xi' + \eta')$.
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  - Premises are $2 \times 2$ matrices:
  - Their supports are $\xi \otimes \nu' \otimes I$, $\nu \otimes \xi' \otimes I$ and $\eta \otimes \nu' \otimes I$, $\nu \otimes \eta' \otimes I$.
  - Just sum them: disjoint supports.
- Violently anti-$\eta$, like Quantum coherent spaces.
- Summing up, perfect logic (in the linguistic sense) can be interpreted in the hyperfinite factor.
42-NOVELTIES

- $A \vdash B$ no longer maps $A$ into $B$. 
42-NOVELTIES

- $A \vdash B$ no longer maps $A$ into $B$.
- Maps $A \otimes \eta'$ into $B \otimes \xi'$. 
42-NOVELTIES

- $A \vdash B$ no longer maps $A$ into $B$.
- Maps $A \otimes \eta'$ into $B \otimes \xi'$.
- $A \otimes \eta' := \{(\gamma^{\dim(\eta)}, h \otimes \eta'); (\gamma, h) \in A\}$ (modulo some twisting). Basic fact:
42-NOVELTIES

- A ⊨ B no longer maps A into B.
- Maps A ⊗ η' into B ⊗ ξ'.
- A ⊗ η' := \{(\gamma^{\text{dim}(\eta)}, h \otimes \eta'); (\gamma, h) \in A\} (modulo some twisting). Basic fact:
  \[(\sim A) \otimes \eta' = \sim (A \otimes \eta')\] (43)
42-NOVELTIES

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- Which relies upon:
42-NOVELTIES

- \( A \downarrow \not\rightarrow B \) no longer maps \( A \) into \( B \).
- Maps \( A \otimes \eta' \) into \( B \otimes \xi' \).
- \( A \otimes \eta' := \{(\gamma^{\dim(\eta)}, h \otimes \eta'); (\gamma, h) \in A\} \) (modulo some twisting). Basic fact:
  \[
  (\sim A) \otimes \eta' = \sim (A \otimes \eta') \tag{43}
  \]
- Which relies upon:
  \[
  \det(I - h \otimes \eta') = \det(I - h)^{\dim(\eta')} \tag{44}
  \]
42-NOVELTIES

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- Maps $A \otimes \eta'$ into $B \otimes \xi'$.
- $A \otimes \eta' := \{ (\gamma^{\dim(\eta)}, h \otimes \eta'); (\gamma, h) \in A \}$ (modulo some twisting). Basic fact:
  \[ (\sim A) \otimes \eta' = \sim (A \otimes \eta') \tag{43} \]
- Which relies upon:
  \[ \det(I - h \otimes \eta') = \det(I - h)^{\dim(\eta')} \tag{44} \]
- The daimon, i.e., the scalar component.
42-NOVELTIES

- \( A \vdash B \) no longer maps \( A \) into \( B \).
- Maps \( A \otimes \eta' \) into \( B \otimes \xi' \).
- \( A \otimes \eta' := \{ (\gamma^{\dim(\eta)}, h \otimes \eta'); (\gamma, h) \in A \} \) (modulo some twisting). Basic fact:
  \[
  (\sim A) \otimes \eta' = \sim (A \otimes \eta')
  \]  \hspace{1cm} (43)
- Which relies upon:
  \[
  \det(I - h \otimes \eta') = \det(I - h)^{\dim(\eta')}
  \]  \hspace{1cm} (44)
- The daimon, i.e., the scalar component.
- Corresponds to failure, i.e., falsity, when \( \neq 1 \).
42-Novelty

- $A \vdash B$ no longer maps $A$ into $B$.
- Maps $A \otimes \eta'$ into $B \otimes \xi'$.
- $A \otimes \eta' := \{ (\gamma^{\dim(\eta)}, h \otimes \eta'); (\gamma, h) \in A \}$ (modulo some twisting). Basic fact:
  $$(\sim A) \otimes \eta' = \sim (A \otimes \eta')$$
  \hspace{1cm} (43)
- Which relies upon:
  $$\det(I - h \otimes \eta') = \det(I - h)^{\dim(\eta')}$$
  \hspace{1cm} (44)
- The daimon, i.e., the scalar component.
- Corresponds to failure, i.e., falsity, when $\neq 1$.
- In ludics (commutative), daimon cannot be created.
42-Novelties

- $A \vdash B$ no longer maps $A$ into $B$.
- Maps $A \otimes \eta'$ into $B \otimes \xi'$.
- $A \otimes \eta' := \{(\gamma^{\dim(\eta)}, h \otimes \eta'); (\gamma, h) \in A\}$ (modulo some twisting). Basic fact:
  \[
  (\sim A) \otimes \eta' = \sim (A \otimes \eta')
  \] (43)
- Which relies upon:
  \[
  \det(I - h \otimes \eta') = \det(I - h)^{\dim(\eta')}
  \] (44)
- The daimon, i.e., the scalar component.
- Corresponds to failure, i.e., falsity, when $\neq 1$.
- In ludics (commutative), daimon cannot be created.
- Professional losers, so to speak.
42-NOVELTIES

- $A \vdash B$ no longer maps $A$ into $B$.
- Maps $A \otimes \eta'$ into $B \otimes \xi'$.
- $A \otimes \eta' := \{ (\gamma^{\dim(\eta)}, h \otimes \eta'); (\gamma, h) \in A \}$ (modulo some twisting). Basic fact:
  $$(\sim A) \otimes \eta' = \sim (A \otimes \eta')$$  (43)
- Which relies upon:
  $$\det(I - h \otimes \eta') = \det(I - h)^{\dim(\eta')}$$  (44)
- The daimon, i.e., the scalar component.
- Corresponds to failure, i.e., falsity, when $\neq 1$.
- In ludics (commutative), daimon cannot be created.
- Professional losers, so to speak.
- Here the daimon is created by the determinant.
42-NOVELTIES

- A ⊢ B no longer maps A into B.
- Maps A ⊗ η' into B ⊗ ξ'.
- A ⊗ η' := \{(γ^{\dim(η)}, h ⊗ η'); (γ, h) ∈ A\} (modulo some twisting). Basic fact:
  \((\sim A) ⊗ η' = \sim (A ⊗ η')\) (43)
- Which relies upon:
  \(\det(I - h ⊗ η') = \det(I - h)^{\dim(η')}\) (44)
- The daimon, i.e., the scalar component.
- Corresponds to failure, i.e., falsity, when \(\neq 1\).
- In ludics (commutative), daimon cannot be created.
- Professional losers, so to speak.
- Here the daimon is created by the determinant.
- Truth (winning) not preserved by logical consequence.
Let us fix a subject, i.e., a maximal commutative subalgebra (= boolean algebra) $\mathcal{B} \subset \mathcal{R}$.
43-SUBJECTIVE TRUTH

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Subjective winners are closed under logical consequence; indeed the feedback equation is of the nilpotent type and no daimon can be created.
IX-An ICONOCLAST LOGIC
44-THE ICONOCLAST PROGRAMME

- Finite from **inside**, infinite from **outside**.
44-THE ICONOCLAST PROGRAMME

- Finite from inside, infinite from outside.
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44 - The Iconoclast Programme

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  - Use the geometrical constraints of factor $R$.
- B.t.w., logic in a factor of type $\Pi_1$ should correspond to ELL.
**45-Perennial Behaviours**

- **B is perennial when** \( B = \sim \sim (\{1\} \times C \otimes I) \).
PERENNIAL BEHAVIOURS

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  - Denumerable tensor product $\mathcal{R} \ldots \otimes \ldots \mathcal{R}$ crossed by a locally finite group $G$.
  - $G$ acts on integers by swapping bits in hereditary base $2$. 
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  Product: Type \( \text{nat}_Y \otimes \text{nat}_Y' \rightarrow \text{nat}_Y \sqcup Y' \).
  Square: Type \( !_X \text{nat}_{2Y} \rightarrow !_X \sqcup X' \text{nat}_{2Y+1} \text{nat}_{2Y+1} \).
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- Which complexity classes can be expressed?