Keio 16/17 Mars 2006

THE BLIND SPOT

Jean-Yves Girard

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- Got beyond the essential(ist) circularity of logic, the blind spot.

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- Set theory problematic in extreme situations (foundations).

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- Accept foundations with most of operations external.

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- Forget the idea of creation in 7 days, from simple to complicated (sets, algebra, reals, function spaces) since it does not work anyway (Incompleteness theorem).

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- Level -2 not fit to go beyond the blind spot.

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$$\sigma := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \eta := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}$$

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- **Chops off the antidiagonal coefficients ; yields probabilistic boolean :** $\lambda \cdot \text{true} + (1-\lambda) \cdot \text{false}$, **with** $\lambda := 1/(1+z\overline{z})$.

III-PASSAGE TO INFINITY

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- Dedekind integers (system $\mathbf{F}$ version) :

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- Light logics (LLL, ELL. . .) ; not grounded. But some hope !

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- **Something wrong with the methodology.**

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- This actualisation of potentialities is possible in finite dimension ; in infinite dimension, it diverges, yielding useless values, zero or infinite.
- GoI : a potential interpretation which remains potential.

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- Equation (22) does not pass infinite limits. Remains the determinant, i.e., GoI. One should remain potential.

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- The Murray-von Neumann factor (finite and hyperfinite) seems the appropriate space for true finitism.

Keio 16/17 Mars 2006

IV-C*-ALGEBRAS

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- **Indeed bounded,** $\|\varphi(u)\| \leqslant \|u\|$ **:**
  - **Use** $\|uu^*\| = \|u\|^2$ **to reduce to positive hermitians** uu^***.**
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- $\mathcal{B}(\mathbb{H})$ not simple (infinite dimension) : compact operators.

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  - The (exterior) Fock space : represent $\kappa(a)(x) := a \wedge x$.

Keio 16/17 Mars 2006

V-vN ALGEBRAS

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- Also : the commutant of a self-adjoint subset of $\mathcal{B}(\mathbb{H})$.

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- The CAR-algebra admits completions of all types $\mathbf{I}, \mathbf{II}, \mathbf{III}$.

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- Up to isomorphism, only one such vN algebra, the Murray-von Neumann factor $\mathcal{R}$.

VI-THE FINITE/HYPERFINITE FACTOR

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- Classification of vN algebras thus reduces to classification of factors.

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- The completion of the CAR-algebra is finite and infinite-dimensional :
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- There are hyperfinite algebras of any type (close the CAR algebra w.r.t. appropriate state).
- But only one hyperfinite factor of type $\mathbf{II}_1$. Murray-von Neumann factor $\mathcal{R}$.

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VII-GoI

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- Violently anti-η, like Quantum coherent spaces.
- Summing up, perfect logic (in the linguistic sense) can be interpreted in the hyperfinite factor.

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- **Subjective winners are closed under logical consequence ; indeed the feedback equation is of the nilpotent type and no daimon can be created.**

IX-An iconoclast logic

44-The iconoclast programme

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44-THE ICONOCLAST PROGRAMME

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- Reduces to search for light exponentials (BLL, LLL, ELL, . . .).

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- B.t.w., logic in a factor of type II_1 should correspond to ELL.

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