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Transcendental syntax I: deterministic case

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À Corrado, sans qui le λ-calcul
ne serait pas ce qu'il est.
Et donc, la logique non plus.

We study logic in the light of the Kantian distinction between *analytic* (untyped, meaningless, locative) answers and *synthetic* (typed, meaningful, spiritual) questions. Which is specially relevant to proof-theory: in a proof-net, the upper part is locative, whereas the lower part is spititual: *a posteriori* (explicit) as far as correctness is concerned, *a priori* (implicit) for questions dealing with consequence, typically cut-elimination. The divides locative/spiritual and explicit/implicit give rise to four blocks which are enough to explain the whole logical activity.

	ANALYTIC	SYNTHETIC
EXPLICIT	Constat	Usine
IMPLICIT	Performance	Usage

1. Under the sign of Herbrand

1.1. *Semantics as the prejudice* par excellence

Transcendental syntax[1] is the study of the *conditions of possibility* of language, e.g., the presuppositions implicit in simple acts like the writing of a proposition or deduction.

This means refusing the usual baloney "langague + interpretation", the interpretation being usually a *selfy* of the language: to say, like Tarski, that $\vee$ is interpreted by `or` is of limited interest. Especially since we know that there are several sorts of disjunctions, classical ($⅋$) vs. intuitionistic ($\oplus$)!

The prejudice — the untold presupposition — *par excellence* is the idea that a proposition A has a well-defined signification — a denotation would say Frege. This supposedly

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well-defined meaning renders possible the *reuse* of A, e.g., deduction: from $\Gamma \vdash A$ and $\Delta, A \vdash B$, deduce $\Gamma, \Delta \vdash B$. But are we that sure those two A are the same? The fact that we can both *create* and *use* word is at the very heart of syntheticity, i.e., *meaning.*

The balance between the creation of words and their use is the main presupposition of logic, and thus the heart of transcendental syntax. The goal of our program is not quite the removal of presuppositions, but rather their exposure: how come that we can use, believe, in such an identification? This is why transcendental syntax is so antagonistic to analytic philosophy and semantics which take this balance, the fact that A has a well-defined denotation, as an undisputable fact, one of those which go without saying, thus transforming the major presupposition of logic into a *prejudice.*

1.2. *First block: the constat*

The first step in the quest for the elusive meaning is to forget it, thus reaching a completely neutral (analytic, locative) state, which excludes any reference to a preexisting language — e.g., semantics, the language of "reality". We then proceed with a reconstruction of logic from scratch, consistently with its primal status: during this process, familiar logic artefacts, e.g., proofs, will progressively emerge.

A prefiguration of this approach can be found in the outstanding theorem of Herbrand (1930). A classical proof involves a disjunction (contraction rule) and the values of the existentials as functions of the universals; now, in order, to explain the dependencies between quantifiers, he expresses the universals as virtual functions of the *previous* existentials. Say he wants to prove $\exists y \forall x A[x, y]$: the answer may be a term t such that $A[x, t]$. In order to say that t cannot depend upon x, he writes $x = f(y)$; if $t = t[x]$ happens to contain x, then the unification $y = t[f(y)]$ fails.

The idea surfaced again with my *proof-nets* (1986) (Gir87; Gir11b): the failure of unification becomes a cycle in the proof, a vicious circle so to speak. Proof-nets contain all the lineaments of an approach to logic free of prejudices, i.e., of language: a proof-net is basically a drawing in which the role of formulas is extremely limited. The real logic artefact is at bay: it only remains to get rid of the remaining traces of language.

The upper part of a proof-net, the *vehicle*, is made of identity links $[\![\, A, \sim A \,]\!]$. In order to avoid the prejudice, the propositions $A, \sim A$ should be replaced with their *locations* $p_A(x), p_{\sim A}(x)$ which are but the (disjoint) spots where $A, \sim A$ are written and bear no special relation to the meaning of A, in the same way "Groenland" is understood as a location on the map, by no way as a green land. Of course, having lost their meaning, p_A and $p_{\sim A}$ bear no longer any special correlation: this is the difference with proof-nets which still use some linguistic prejudice. We could as well write links $[\![\, A, B \,]\!]$, e.g., $[\![\, A, A \,]\!]$. One of the immediate tasks of transcendental syntax will be to explain why $[\![\, A, A \,]\!]$ is "incorrect" without the usual compendium of semantic prejudices.

This upper part has strictly no meaning, it is *analytic*, in other terms *locative*: it concerns the locations of links, i.e., formulas and their subformulas. The location $p_A(x)$ can be split, using a binary function letter "$\cdot$" into various sublocations: for instance, if A is $B \vee C$, $p_B(x) := p_A(\mathtt{l} \cdot x)$, $p_C(x) := p_A(\mathtt{r} \cdot x)$ (left and right subformulas). Substituting $\mathtt{l} \cdot x$ and $\mathtt{r} \cdot x$ for x in the link $p_A(x), p_{\sim A}(x)$ yields $[\![\, p_A(\mathtt{l} \cdot x), p_{\sim A}(\mathtt{l} \cdot x) \,]\!]$

and $[\![\, p_A(\mathtt{r}\cdot x), p_{\sim A}(\mathtt{r}\cdot x)\,]\!]$, i.e., $[\![\, p_B(x), p_{\sim B}(x)\,]\!]$ and $[\![\, p_C(x), p_{\sim C}(x)\,]\!]$, a process known as η-expansion. The technique of substitution — indeed, Herbrand's unification — enables us to see our links as wires that can be split into subwires. These sublinks can, in turn, be split again, yielding, say, $[\![\, p_A(\mathtt{l}\cdot(\mathtt{r}\cdot x)), p_{\sim A}(\mathtt{l}\cdot(\mathtt{r}\cdot x))\,]\!]$.

1.3. *Second block: the performance*

The previous use of locativity is *constative*: it is but a static representation of proofs. The main aspect of proofs is however their *dynamics*, cut-elimination a.k.a. normalisation. This dynamicity is the *performative* aspect of locativity. Here, we start to be concerned with the *knitting* of our four blocks (constat/performance/usine/usage): since we want to eliminate the use of any sort of external reference, the compactness of the approach will compensate the absence of reference to "reality". Better, this knitting is the actual reality: analytic and synthetic aspects of knowledge are so densely mixed that this creates an impression of objectivity; but only an impression — what *realism* forgets.

The knitting thus denies any special status to the constat that would make it a sort of semantics of the performance: something is constative because we choose not to perform it. Typically an electric appliance is performative only if we decide to *plug* it. In our framework, we describe pluggings as as a matter of *painting*: a `green` location will be *pluged* with the same location painted with the complementary colour, `magenta`. Typically, $[\![\, p_A(x), \boxed{p_B(x)}\,]\!]$ and $[\![\, p_B(x), p_C(x)\,]\!]$ will thus merge into $[\![\, p_A(x), p_C(x)\,]\!]$. It must be observed that we could as well plug $[\![\, p_A(x), \boxed{p_B(x)}\,]\!]$ with $[\![\, p_B(\mathtt{l}\cdot x), p_C(x)\,]\!]$, in which case, the sole subwire $[\![\, p_A(\mathtt{l}\cdot x), \boxed{p_B(\mathtt{l}\cdot x})\,]\!]$ will actually be connected, yielding $[\![\, p_A(\mathtt{l}\cdot x), p_C(x)\,]\!]$. Unification (indeed, its technical variant, *matching*) enables one to use and reuse the same wire undefinitely by splitting it into smaller and smaller subwires... hence the possibility of divergence, i.e., of a never-ending performance.

Plugging through unification avoids the problem of external evaluation, at work in rewriting: λ-calculus, whatever its qualities, is not quite analytic, since not sufficiently knitted. The same applies to the *boxes* of proof-nets and their wedding-cake normalisation. Boxes, rewriting, etc., as *global* operations, can however be legitimate at a more elaborate stage. Take rewriting: the analytic process of normalisation knits constat and performance; its synthetic counterpart is the knitting usine/usage through cut-elimination theorems which replace (rewrite) cuts with "simpler" ones. This rewriting, which performs very little at the analytic level, involves drastic changes of meaning, i.e., typing, subjectivity: in other terms, rewriting makes sense as a *synthetic* operation.

This description of the locative layer (constat + performance) by means of plugging and unfication seems to be the converging point of various approches to computation. Typically, the *resolution method* of logic programming directly inspired from Herbrand, a purely analytic approach to computation, beyond any idea of programming language[2]. A clause $\Gamma \vdash A$ can be written $[\![\, \Gamma, \boxed{A}\,]\!]$ ($[\![\, \Gamma, A\,]\!]$ in case A is a goal). My own *Geometry*

[2] Which suggested the childish idea of computing without algorithmic ideas, the so-called *declarative programming* at work in the ill-fated language PROLOG.

of Interaction, although officially set within functional analysis, made indeed use of the same unification, see, e.g., (Gir95).

In terms of knitting, the performance is basic in matters of correctness, i.e., the *usine* block. It is thus essential that a clear notion of normal form — strong normalisation — can be devised. Indeed, the very form of correctness (normalisation of the plugging vehicle + ordeal) has been chosen the most analytic, i.e., general and neutral possible. The first version of this paper used a beautiful idea of Mogbil and de Naurois (MdN11), thus leading to "virtual switchings" which yielded the first box-free approach to exponentials. However, virtual switchings introduced a sort of subcondition within correctness, thus unknitting blocks 2 and 3, and had to be relinquished.

The performance is also related to the last block, *usage* which deals with consequence: the associativity of composition requires a Church-Rosser property, which belongs in the performative block: this is one of the strongest knitting between our four blocks.

1.4. *Third block: l'usine*

Let us come back to proof-nets: the lower part is made of formulas. In traditional approaches, e.g., natural deduction, the meaning of the proof is given by the semantics of the formulas, in other terms by prejudice. Proof-nets introduced a major twist, namely the *correctness criterion.* This condition is homogeneous in spirit with Herbrand's expression of the universals as functions of the existentials: when I write $x = f(y)$ I indeed say that y is an existential variable which does not depend upon x. In the same way, the *switchings* of proof-nets forbid certain dependencies.

When I do switch a proof-net, I create a finite set of links, an *ordeal.* To say that the proof is correct amounts at saying that the plugging of the *vehicle* — the upper part made of identity links — with such an ordeal normalises into a link of a specified form. The finite set of all ordeals is called a *gabarit*: the gabarit thus conveys the spirit, the sense of the formula we are proving.

There is no difference of nature between an ordeal and a vehicle: they use the same sort of links, both are finite, etc. However, whereas the vehicle is purely locative — p_A means the location of A — the gabarit is spiritual: depending upon the formula, distinct links will be drawn: a "$\otimes$" will not be "switched" like a "$⅋$".

This is why, although everything here remains analytic, locative, something essential occurred: we are no longer neutral, the formulas got their meaning. A meaning wholly contained in the choice of the gabarit, which can be seen as a set of factory tests: this is *l'usine*[3]. If we see the synthetic as dealing with names, l'usine is the place where they grow. Typically, in the Fiat factory, (Lingotto) the name *cinquecento* was bestowed on any vehicle complying with the factory tests.

[3] Usine = factory ; the french opposition *usine/usage* works too well to be translated.

1.5. *Fourth block: l'usage*

Blocks 2 and 4 are *implicit*: block 2 because we cannot be sure of the output of a performance, block 4 because we cannot foresee the possible uses of a logical artefact. If syntheticity is meaning, spirit, block 4 (*l'usage*) illustrates the expression of Wittgenstein *meaning as use*. In mathematics, meaning as use is deduction through *Modus Ponens*, i.e., the cut rule. We cannot foresee the possible uses of a formula — of a name —, this is why block 4 is far more implicit than block 2 which deals with a specific evaluation.

Coming back to our *Fiat 500*: once gotten its label, its *use* is governed by operating instructions. But there is never, in any situation, an *absolute* certainty that the factory test (l'usine) guarantees the use (l'usage); but, often, as in mathematics, a *reasonable* confidence. The gap between l'usine et l'usage is known as *incompleteness* — with an analytic conterpart, *undecidability*, the gap between constat and performance.

L'usine may perform the wrong tests: imagine an incompetent engineer concentrating on windscreen wipers and forgetting brakes[4]. In logic, the idea is thus to knit blocks 3 and 4 by showing that l'usine justifies l'usage. Indeed, if we can do such a thing with a reasonable certainty (by means of usual mathematics), then we have completed our non prejudiced approach to logic. *Derealism* (another expression for transcendental syntax, emphasising its opposition to semantics) turns out to be a powerful tool. It compels us into writing logical rules from the nets and not the other way around. A good rule, i.e., a good factory test, is a rule acceptable w.r.t. the *a prioris* of deduction, i.e., of the operating instructions.

The analytic substrate of l'usage is *plugging*; the adequation usine/usage is thus but a *cut-elimination* concerning the performances implicit in deduction, i.e., in the cut rule. Cut-elimination is the very heart of syntheticity: it asserts the adequation between the creation (usine) and the use (usage) of the words, between rights and duties, so to speak. It splits in two independent parts, the convergence of the normalisation process and the fact that the output (normal form) is correct in the usine sense (block 3). This second half is known as *consistency* which is thus not the full story; however, the impossibility of consistency proofs shows, *a fortiori*, the impossibility of any absolute certainty as to the knitting usine/usage.

By the way, the knitting between blocks 2 and 4 strongly relies on the *Church-Rosser* property of the performative block, which, translated in terms of use, yields the *compositionality* (associativity) of consequence, and thus the (pleasant) illusion of objectivity, stability, at work in what we perceive as "reality".

1.6. *The paper*

The general task is to reconstruct logic along our four blocks. Which amounts basically at finding the definite version of proof-nets. For editorial reasons (two distinct *Festschrifts*), the paper has been split in two parts, this one and (Gir16). A splitting along the divide

[4] Paraconsistent logicians commit the same mistake, but on purpose: for them, there is nothing like l'usage, only irrelevant tests. Never buy a paraconsistent car!

locative/spiritual would have made part 1 quite arid. This is why I preferred to split the analytics between its deterministic version which is almost self-contained as long as we deal with multiplicative and exponentials, and its general, non deterministic version (Gir16) which enables one to handle, say, additives.

Even if we want to construct logic from scratch, this can only be a *re*construction, which means that we roughly know what we are aiming at. For an obvious reason, by the way: logic is a very healthy activity, which only suffers from a metaphysical, prejudiced, approach, that of analytic philosophy. This is why we shall be compelled into translating usual proofs in analytics terms (the vehicles) and formulas in synthetic ones (ordeals, gabarits). These translations must be seen as examples of what we can do (and not as yet-another-semantics).

Now, the role of a formula is played by a gabarit, i.e., a finite set of constellations; and that of a proof by any vehicle (i.e., constellation) complying with the gabarit — which includes the examples obtained from syntax. It then remains to establish that, for specific choices of gabarits, e.g., $\otimes/\parr$, cut-elimination holds. This process involves a reconstruction of the language as system of abbreviation for our synthetic units, the gabarits.

Among the surprising outputs of the new technology, the fact that the exponentials $!A$ and $?A$ do not exist by themselves. This remark is consistent with the known impossibility at drawing proof-nets for the multiplicative neutral $\perp$. The reason is that $?A$ corresponds to *hidden* conclusions; such conclusions can never be made visible. Our new exponentials $A \otimes B := !A \otimes B$ and $A \ltimes B := ?A \parr B$ are De Morgan variants of intuitionistic implication: $A \otimes B = \sim(A \Rightarrow \sim B), A \ltimes B = \sim A \Rightarrow B$.

The multiplicative units $\mathbf{1}$ and $\perp$ therefore disappear, since one could otherwise recover $!A := A \otimes \mathbf{1}$. However, "semantically" speaking, units do make sense as neutrals. This is a typical conflict between derealism which relies upon an independent, tightly knitted approach, and realism which respects anything looking like a logical connective, even if this means loosening the screws. This dilemma can be described as the exposure of a prejudice.

2. The locative layer

We define an analytic (i.e., meaningless, locative) layer in which *answers* dwell.

2.1. *Unification and matching*

We consider, once for all, denumerably many functional symbols of any arity as well as variables $x_1, x_2, \ldots$. They can be used to build terms $x_1, f(x_3), g(f(x_1), a)$, etc.

An equation $t = u$ between terms can be solved by means of *substitutions*, i.e., values for all variables, even those not occurring in t, u: $x_1 = \theta_1, x_2 = \theta_2, \ldots$; if θ is the substitution $x_1 \to \theta_1, x_2 \to \theta_2, \ldots$, then $t\theta := t[\theta_1/x_1, \theta_2/x_2, \ldots]$ denotes the result of the replacement of the x_i with the θ_i. We say that t, u are *unifiable* when $t\theta = u\theta$ for some *unifier* (i.e., substitution) θ. The point is that substitutions do compose, hence Herbrand's result (1930):

Theorem 1 *Assume t, u unifiable; there is a mother of all unifiers for t, u, i.e., a substitution θ_0 such that any unifier θ for t, u can be uniquely written $\theta_0\theta'$.*

This *principal unifier* is therefore unique up to usual nonsense.

Matching is, so to speak, unification with bound variables: in order to match t and u, we first change their respective variables to make them distinct, then proceed with unification. Therefore, while $f(g(x_1))$ and $f(x_1)$ are not unifiable, they are matchable: $f(g(x_2)) = f(x_1)$ admits the principal unifier $x_2 := x_2$, $x_1 := g(x_2)$. Terms which do not match are styled *disjoint:* their associated projections (sec. A.1) are indeed orthogonal.

2.2. *Stars and constellations*

A *star* $[\![\, t_1, \ldots, t_{n+1} \,]\!]$ consists in $n+1$ terms with exactly the same variables; these terms, the *rays* of the star, must be pairwise non-matchable, i.e., disjoint. If $\sigma = [\![\, t_1, \ldots, t_{n+1} \,]\!]$ is a star and θ is a substitution, then $\sigma\theta := [\![\, t_1\theta, \ldots, t_{n+1}\theta \,]\!]$ is still a star.

A *constellation* is a finite set of stars; the variables occuring in a constellation are bound, i.e., local to each of its stars. The rays issued from the stars of the constellation must too be pairwise *disjoint,* i.e., not matchable.

Stars are a symmetric alternative to the *flows* of annex A.1; the dispute between the two versions is still not completely settled. The unreachability of a star with no ray explains the exclusion of closed (empty) stars. The fact that the variables must be exactly the same comes from endless complications coming from "irrelevant" variables that may disappear during evaluation. Finally, it is obvious that the rays of a star should be distinct; hence, since stars are bound to be substituted, not unifiable. But our request is the stronger "not matchable": this is because a star may be combined with itself to form a *diagram.*

Colours are nothing but specific unary symbols used in explicitation. We use pairs of complementary colours, typically `magenta`/`green`, `yellow`/`blue` and `cyan`/`red`[5]. We assume that all terms using colours are of of the form $c(t)$ where c is the colour symbol and t does not use any colour. Instead of writing `green`(t), we rather paint t in green, i.e., write t; in the same way, `magenta`(t), `blue`(t), `yellow`(t) become t, t, t.

Colouring being but a notational convention, two terms of different colours are disjoint.

2.3. *Normalisation*

A coloured constellation is *explicit* (cut-free) when uncoloured, *implicit* otherwise. The purpose of *evaluation* is to turn an implicit constellation into an explicit one by matching rays of complementary colours.

In order to normalise (i.e., evaluate) a constellation, we first form its *diagrams*, i.e., all

[5] Pairs of complementary colours are easily identified, since the additive ones are but boxings of their substractive complements : `blue` vs. `yellow`, `green` vs. `magenta`, `red` vs. `cyan`. Moreover additive colours, e.g., `green` and `blue`, are used as conclusions, substractive ones, e.g., `magenta` and `yellow`, as premises.

trees (in the topological acception) obtained by plugging $N+1$ stars of the constellation by means of N *edges*, i.e., formal equations between rays of complementary colours, e.g., $\boxed{t} = \colorbox{lightgray}{\textit{u}}$. All diagrams can be obtained in the following incremental way, starting with a single star: we select a *free* coloured ray in the diagram, i.e., a coloured ray not yet involved in an edge and pair it with any ray of the complementary colour occurring in the constellation, thus aggregating a new star to the diagram. Since diagrams may reuse stars, a constellation is likely to generate infinitely many diagrams.

The *actualisation* of a diagram consists in matching the uncoloured terms underlying each edge: $\boxed{t} = \colorbox{lightgray}{\textit{u}}$ becomes the actual equality $t\theta = u\theta$. Most actualisations fail; we are basically concerned with *correct* diagrams, those for which actualisation succeeds.

The following *strong normalisation* conditions knit blocks 2 and 1 by explaining how a performance may produce a constat (result, output):

1 There are only finitely many correct diagrams. In other terms, for an appropriate N, all diagrams of size $N+1$ fail. Since a diagram of size $N+2$ contains diagrams of size $N+1$, there is no point in forming diagrams of bigger sizes.

2 There is no *closed* (i.e., without free ray) correct diagram.

We define the *residual star* of a correct diagram as made of the actualised free rays of the diagram. And the *normal form* of a strongly normalising constellation as the (finite) set of its *uncoloured* residual stars. We must however prove that this is actually a constellation. Condition 2 excludes the presence of a star without ray; it remains to prove that all residual rays are pairwise disjoint.

Take two diagrams whose free rays are uncoloured, with (before actualisation) a common free ray s; starting from s, the two diagrams must first disagree on some edge, e.g., $\boxed{t} = \colorbox{lightgray}{\textit{u}}$ vs. $\boxed{t} = \colorbox{lightgray}{\textit{v}}$, thus inducing actualisations $t\theta, t\theta'$. Since u, v are disjoint, so are the eventual actualisations of s in both diagrams: here we use the fact that the variables are exactly the same. Take now the case where the two s (say, s_1, s_2) are part of the same diagram: there is a path leading from s_1 to s_2 through edges $T_1/U_1, \ldots, T_n/U_n$, with $T_i, U_i = \boxed{t_i}, \colorbox{lightgray}{u_i}$ or $\colorbox{lightgray}{$t_i$}, \boxed{u_i}$. We can handle this case as if s_1, s_2 were in distinct diagrams, unless there is an automorphism σ of the diagram such that $\sigma(s_1) = s_2$; if $s_3 := \sigma(s_2) \neq s_1$, then $s_1, s_2, s_3, \ldots, s_k = s_1$ would form a cycle in the diagram. If $\sigma(s_2) = s_1$, then $T_1 = U_n, T_2 = U_{n-1}, \ldots, T_n = U_1$. If $n = 2m$, then $T_m = U_m$, which is impossible, since T_m, U_m are of distinct colours; $n = 2m+1$ is impossible too, since there would be a star with equal rays U_m, T_{m+1}.

By the way, condition 1 implies the following "condition 3":

3 If $\boxed{t}, \colorbox{lightgray}{\textit{u}}$ are two free rays of complementary colours in an actualised diagram $\mathcal{D}$, there are no substitutions θ, θ' such that $t\theta = u\theta\theta'$.

One could otherwise plug $\mathcal{D}$ *ad libitum* with itself, by means of the edge $\boxed{t} = \colorbox{lightgray}{\textit{u}}$: $\mathcal{D}\theta + \mathcal{D}\theta\theta' + \mathcal{D}\theta\theta'\theta' + \ldots$ would provide actualisations for these arbitrarily large diagrams.

2.4. *The Church-Rosser property*

When dealing with normalisation, a single pair of colours is enough. However, in view of the synthetic aspects of logic, we may need to perform nomalisation in two steps, which requires a second pair of colours. Here remember that colours are just a convenience, therefore that the additional colours `blue` and `yellow` are but ordinary unary symbols that we may view as uncoloured on request. In presence of two pairs of colours, one can:

— Either normalise all edges `green`/`magenta` and `blue`/`yellow`.

— Or normalise the edges `green`/`magenta` so as to get a normal form still harbouring the colours `blue`/`yellow`, then normalise the edges `blue`/`yellow`.

The Church-Rosser property says that the two methods yield the same normal form; as a corollary, a third method (first `blue`/`yellow`, then `green`/`magenta`) is equivalent to those two. Also, if the first method is strongly normalising then the second one is (twice) strongly normalising. The converse fails in presence of stars of arity > 2: the lone star $[\![x, x, x]\!]$ contradicts condition 3 of strong normalisation; but normalisation in two steps, first `green`/`magenta` (yielding an empty constellation), then `blue`/`yellow` works.

A solution to this minor problem consists in introducing uncoloured duplicates `gr`, `mg` of the colours `green`,`magenta` and in adding the stars $[\![x, \mathtt{mg}(x)]\!], [\![x, \mathtt{gr}(x)]\!]$ to our constellation; since x matches all terms t, rays are no longer disjoint, but this relaxation poses no problem as we shall see in (Gir16). If we normalise this expanded constellation, the normal form contains copies of the coloured residual stars in which `green` and `magenta` have been replaced with `gr` and `mg`. In presence of two pairs of colours, we should add two more additional stars, say $[\![x, \mathtt{ye}(x)]\!], [\![x, \mathtt{bl}(x)]\!]$. The discrepancy between one and two steps disappears: $[\![x, x, x]\!] + [\![x, \mathtt{mg}(x)]\!] + [\![x, \mathtt{gr}(x)]\!] + [\![x, \mathtt{ye}(x)]\!] + [\![x, \mathtt{bl}(x)]\!]$ yields, after normalising the edges `green`/`magenta`, the normal form $[\![x, x, \mathtt{gr}(x)]\!] + [\![\mathtt{gr}(x), \mathtt{mg}(x)]\!] + [\![x, \mathtt{ye}(x)]\!] + [\![x, \mathtt{bl}(x)]\!]$ which does not normalise.

A satisfactory solution is therefore at hand, provided we use the version of constellations of (Gir16). If the additional stars $[\![x, \mathtt{mg}(x)]\!], [\![x, \mathtt{gr}(x)]\!], \ldots$ are systematically added to coloured stars, the normal forms thus obtained, the *explicit forms*, enjoy a full Church-Rosser property. Usual normal forms are just what remains of explicit forms when we remove all stars making use of the extra symbols `gr`, `mg`, ...

3. Proof-nets, vehicles and gabarits

We now use the analytic layer to revisit *proof-nets*, see e.g., (Gir11b). The point is to provide us with a couple of examples coming from multiplicative logic.

3.1. *Locativity*

In order to reach an analytic, i.e., meaningless layer, we must get rid of syntactical decorations so as to describe proof-nets in a purely *locative* way[6]: a proof of $\vdash A, B, C$ will be represented by means of unary functions p_A, p_B, p_C which distinguish between the basic *locations* available in the sequent, those of the subpropositions of A, B, C. We could emphasise the oblivion of the syntax by using p_1, p_2, p_3, but this would compel us into a systematic and pedantic indexing of sequents, e.g., $\vdash A_1, A_2, A_3$.

[6] Only the locations matter: *Locus Solum.*

3.2. *Vehicles*

Let us choose, once for all, distinct constants $\mathtt{l}, \mathtt{r}$ and a binary function letter "$\cdot$". To each proof π we associate its *vehicle*, i.e., a constellation $\pi^\bullet$; this constellation is uncoloured.

Identity axiom: if π is the axiom $\vdash A, \sim A$, then $\pi^\bullet := [\![\, p_A(x), p_{\sim A}(x) \,]\!]$.

⅋-rule: if the proof π of $\vdash \Gamma, A ⅋ B$ comes from a proof ν of $\vdash \Gamma, A, B$, then $\pi^\bullet := \nu^\bullet$ in which p_A and p_B are now *defined* by $p_A(x) := p_{A⅋B}(\mathtt{l} \cdot x)$, $p_B(x) := p_{A⅋B}(\mathtt{r} \cdot x)$.

⊗-rule: if the proof π of $\vdash \Gamma, \Delta, A \otimes B$ comes from proofs ν of $\vdash \Gamma, A$ and μ of $\vdash \Delta, B$, then $\pi^\bullet := \nu^\bullet + \mu^\bullet$, with p_A, p_B *defined* by $p_A(x) := p_{A\otimes B}(\mathtt{l} \cdot x)$, $p_B(x) := p_{A\otimes B}(\mathtt{r} \cdot x)$.

The vehicle is thus a constellation of axiom links, seen as stars. The rules ⅋, ⊗ have been used to relocate these links. For instance, the axiom $[\![\, p_A(x), p_{\sim A}(x) \,]\!]$ may relocate as $[\![\, p_{A⅋(\sim A\otimes B)}(\mathtt{l} \cdot x), p_{A⅋(\sim A\otimes B)}(\mathtt{r} \cdot (\mathtt{l} \cdot x)) \,]\!]$.

3.3. *Gabarits*

We must now make sense of the lower part of the proof-net, the one dealing with the ⅋ and ⊗ links. Each switching involved in the correctness condition induces an *ordeal*, i.e., a coloured constellation. The finite set of these ordeals, called the *gabarit*, depends on the sole conclusions of the net.

We already defined the unary functions $p_A(x)$ for each proposition and subproposition of the proof-net. We now introduce $q_A(x) := p_A(\mathtt{g} \cdot x)$, where $\mathtt{g}$ is another constant. The replacement of p_A with q_A is due to the fact that $p_{A\otimes B}(x)$ is not disjoint from $p_A(x) := p_{A\otimes B}(\mathtt{l} \cdot x)$, in constrast to the $q_{A\otimes B}(x)$ w.r.t. the $q_A(x)$: the q_A provide disjoint locations for the propositions occurring in the lower part of the proof-net.

Given a proof-net of conclusions Γ, a switch L/R of its ⅋ links induces an *ordeal*, namely the coloured constellation made of the following stars, written in the style premises/conclusion; conclusions are `green` or uncoloured, premises `magenta` or `yellow`. The various stars of an ordeal thus look like LEGO bricks, that we can superpose one on top on another.

$X, \sim X$**:** $[\![\, \frac{p_A(x)}{q_A(x)} \,]\!]$ when A is a *literal* $X, Y, \sim X, \sim Y, \ldots$

⊗: $[\![\, \frac{q_A(x), q_B(x)}{q_{A\otimes B}(x)} \,]\!]$.

⅋$_L$**:** $[\![\, \frac{q_A(x)}{q_{A⅋B}(x)} \,]\!] + [\![\, \frac{q_B(x)}{} \,]\!]$. In terms of graphs, $[\![\, q_B(x) \,]\!]$ "terminates" all $[\![\, q_B(t) \,]\!]$.

⅋$_R$**:** $[\![\, \frac{q_B(x)}{q_{A⅋B}(x)} \,]\!] + [\![\, \frac{q_A(x)}{} \,]\!]$.

Conclusion: $[\![\, \frac{q_A(x)}{p_A(x)} \,]\!]$ when $A \in \Gamma$, i.e., is a conclusion.

An ordeal thus normalises into a constellation `literals`/conclusions. By the way, our ordeals will react in the same way to a general identity link $[\![\, p_A(x), p_{\sim A}(x) \,]\!]$ and its η-expansion made of atomic identity links between the literals of A and $\sim A$.

4. The spiritual layer

Everything related to the *sense* is synthetic (spiritual, meaningful), in opposition to the purely locative, meaningless analyticity.

4.1. *Correctness, a.k.a. l'usine*

Let $\boxed{\mathcal{V}}$ be $\mathcal{V}$ painted `blue`. The correctness criterion thus writes as:

For any ordeal $\mathcal{S}$, $\boxed{\mathcal{V}} + \mathcal{S}$ *strongly normalises into* $[\![\, p_\Gamma(x) \,]\!] := [\![\, p_A(x)\ ;\ A \in \Gamma \,]\!]$.

This condition is obviously necessary; its sufficiency — which relates the symbolic testing by means of the ordeals with the proofs in a logical system — is the most elaborate form of *completeness* that one can dream of.

The main technical problem with correcteness is that usual proof-nets are, "preconstrained", prejudiced: identity links relate "complementary" propositions $A, \sim\!A$, whereas nothing of the kind makes any sense locatively speaking. In other terms, our treatment of literals is completely indistinct: $X, \sim\!X, Y$ are the same, up to their locations. How can we force an axiom link to relate X with a $\sim\!X$ (and not a Y, nay another X)?

Here, we must remember that predicate or propositional calculi are convenient and rather convincing structures, but that a marginal part of them is ill-written. Typically, the so-called propositional "constants" X, Y which mean nothing by themselves. In order to find their actual transcendental status, we must move to second order logic — say, to system **F** (Gir11b) — a system without constants in which propositions are *closable*; we can thus replace the "constants" with propositional variables and proceed with a universal quantification. What we call first order logic indeed deals with the universal closures $\forall X_1 \ldots \forall X_n A$ of quantifier-free propositions A: the behaviour of such propositions is extremely simple, especially in view of completeness issues, e.g., the subformula property. In practice, the universal prefix being compulsory, it is convenient to omit it; and since variables are treated in a static way, they are easily mistaken for constants.

To make the long story short, when dealing with a proof-net, we must take into account the *implicit* second order quantification $\forall X$ on all propositional "constants". What follows is part of the treatment of second order logic dealing with the quantifier $\forall X$. For this, we must introduce another kind of ordeal, styled *cancelling.* The correctness relation for a cancelling ordeal $\mathcal{S}$ writes as the *cancellation condition:*

$\boxed{\mathcal{V}} + \mathcal{S}$ *strongly normalises into the empty constellation* 0.

Since two kinds of tests are now performed, cancelling ordeals constitute a minor unknitting of blocks 2 and 3. In the companion paper (Gir16), non-determinism will allow us to sum an ordeal with several cancelling ones; the normal form being the sum of the possibilities, the usual correctness condition forces the cancellation condition for the additional parts.

A cancelling ordeal for literals consists in selecting between X and $\sim\!X$ for each propositional variable; the selection is done in the same way for all "occurrences" of each variable. Given such a switching, the ordeal consists of the stars:

$$[\![\frac{q_A(\mathtt{l}\cdot x), q_A(\mathtt{r}\cdot x)}{}]\!]$$

for any occurrence A of the literals $(X/\sim X, Y/\sim Y, \ldots)$ selected by the switching.

Being extremely sparse, these ordeals have a propensity to die, typically the switch "X" has nothing to match $\sim X$: in particular, identity links relating an "occurrence" of X with one of $\sim X$ will be cancelled. Any other combinations between literals (X/Y, X/X, …) can be shown to be incorrect by an adequate switching of both sides: X and Y since they are independent, X and X since they can be both put to "X": in the latter case, the constellation $[\![\frac{q_A(\mathtt{l}\cdot x), q_A(\mathtt{r}\cdot x)}{}]\!] + [\![\frac{p_A(x)}{q_A(x)}]\!] + [\![\frac{}{p_A(x), p_B(x)}]\!] + [\![\frac{p_B(x)}{q_B(x)}]\!] +$ $[\![\frac{q_B(\mathtt{l}\cdot x), q_B(\mathtt{r}\cdot x)}{}]\!]$ which yields a diagram actualising into $[\![\frac{q_B(\mathtt{r}\cdot x)}{q_B(\mathtt{r}\cdot x)}]\!]$: such a diagram is the typical vicious circle excluded by "condition 3" of strong normalisation.

Our cancelling ordeals only forbid "incorrect" links between literals. General links can be reduced — since the approach is insensitive to η-expansion — to links between a literal and a compound formula; a case that usual ordeals can handle, since the normal form, if it exists, could not be of the form $[\![p_\Gamma(x)]\!]$.

A last warning about these atoms: their treatment occurs *in the final stage* of the logical construction, when a universal quantification over propositions is performed, so that to keep the choice between X and $\sim X$ consistent, always X or always $\sim X$.

4.2. *Cut-elimination*

A *cut* is a conclusion $A \otimes \sim A$ put between square brackets:

$$\frac{\vdash \Gamma, A \otimes \sim A}{\vdash \Gamma, [A \otimes \sim A]}$$

The rule, so to speak, *predicts* the erasure of $A \otimes \sim A$, i.e., the conclusion $\vdash \Gamma$. This way of expressing the cut has the advantage to keep the cut rule in l'usine (see *infra*).

Cut-elimination is the actual plugging of p_A with $p_{\sim A}$ in order to produce a vehicle which eventually normalises into a proof of $\vdash \Gamma$. In order to activate this plugging, we shall use a pair of complementary colours, typically $p_A, p_{\sim A}$ and $p_A, p_{\sim A}$. To $\mathcal{V}$ with $p_A, p_{\sim A}$ painted `green`, i.e., replaced with $p_A, p_{\sim A}$, we add the *feedback* $[\![\frac{p_A(x), p_{\sim A}(x)}{}]\!]$.

It may be convenient to use different colours when normalising several cuts. Then a vehicle with two cuts may be painted `green` and `red`, with `magenta` and `cyan` feedbacks. As to normalisation, there are three options:

1 Normalise all cuts `green`/`magenta` and `red`/`cyan`.

2 Normalise the cuts `green`/`magenta`, then the residual cuts `red`/`cyan`.

3 Normalise the cuts `red`/`cyan`, then the residual cuts `green`/`magenta`.

The Church-Rosser property equates, *modulo* some precautions — deal with explicit forms rather than normal forms—, these three possibilities: hence one pair of colours is enough. From this, it will be possible to show the *compositionality* of cut, hence to develop a *synthetic* layer, e.g., a category-theoretic structure, a typical example of synthetic *a priori*, of usage. This point is a major knitting of transcendental syntax.

4.3. *Syntheticity* a priori*, a.k.a. l'usage*

Proof-nets were introduced to fix the drawbacks of syntax, in particular to provide a sort of symmetric natural deduction. A *sequentialisation* result reducing the new notion to sequent calculus was therefore needed. If this step was a historical necessity, it has no special transcendental status. We should thus try to understand this step under a different light and replace sequentialisation with something else, the adequation between two forms of syntheticity.

Correctness relates the locative vehicle to the spritual gabarit. This is one of the possible acceptions of the sense, the *sense-as-format* (i.e., as type), a.k.a. l'usine: the gabarit formats the vehicle. In Kantian terms, this should be styled *synthetic a posteriori*: experience (the testings according to given ordeals) tells us that the proof-net is correct.

The proof-net is d'usine when considered as a vehicle, its gabarit remaining implicit. But the combination "vehicle + gabarit", a finite set of constellations[7], is analytic, indeed performative (block 2).

There is, however, another acception of the sense, the *sense-as-use*: a proof yields consequences, typically corollaries. These consequences are handled by the *cut rule*, provided we anticipate the eventual erasure of the propositions between square brackets. But this cut-elimination relies on a heavy assumption, namely that the choices of gabarits for A and $\sim A$ are, in some sense, complementary. Since we cannot perform all possible cuts — there are infinitely many of them —, we must rely on mathematical reasoning, i.e., *predict*. This predictive part of the sense is the *synthetic a priori*, a.k.a. l'usage. Since it deals with the potential infinity of all uses, this *a priori* cannot be completely justified. Lingers a persistent, althought not quite reasonable, doubt of principle, the one best expressed by incompleteness.

The fact of using twice the same letter to denote a proposition — the so-called *occurrences* — is the ultimate logical prejudice. Bilocation lies at the very heart of syntheticity: how could we speak of the sense of a proposition that we can only see once? Now, in the identity link[8] $[\![p_A(x), p_{\sim A}(x)]\!]$, the coincidence between the two "occurrences" is justified by the "complementarity" between the ordeals for A and $\sim A$: due to the fact that we can actually *check* correctness, $A = A$ belongs in the *a posterori* of l'usine. The feedback $[\![\underline{p_A(x), p_{\sim A}(x)}]\!]$ is a sort of prediction, namely the convergence of cut-elimination — furthermore into a correct proof-net: this prediction, which is supposed to work for all correct proof-nets, cannot be justified by any checking, it belongs in the *a priori*. This *a priori* is the Capitol of thought; but beware of the Tarpeian Rock!

In terms of good old natural deduction, the distinction *a posteriori/a priori* is present in the two kinds of extremal propositions, minimal vs. maximal.

[7] Indeed a non deterministic constellation, see (Gir16)
[8] I prefer to avoid "axiom" which suggests some *a priori*.

4.4. *Adequation*

The adequation between *a priori* and *a posteriori* is known as cut-elimination. Incompleteness forbids any elementary cut-elimination theorem for full — second order — logic. This means that, in system **F**, the gap between *a priori* and *a posteriori* cannot be totally filled: a small foundational doubt forever lingers. This gap is however limited in the first order case, where we hardly feel the wind of the bullet of incompleteness.

Take a proof-net with conclusions $\Gamma, [X \otimes {\sim}X]$, and an ordeal $\llbracket \frac{p_{\sim X}(x)}{\boxed{q_{\sim X}(x)}} \rrbracket + \llbracket \frac{p_X(x)}{\boxed{q_X(x)}} \rrbracket +$ $\llbracket \frac{q_X(x), q_{\sim X}(x)}{\boxed{q_{X\otimes\sim X}(x)}} \rrbracket + \llbracket \frac{q_{X\otimes\sim X}(x)}{p_{X\otimes\sim X}(x)} \rrbracket + \mathcal{S}$, which normalises into $\llbracket \frac{p_X(x), p_{\sim X}(x)}{p_{X\otimes\sim X}(x)} \rrbracket + \mathcal{S}$. If $\boxed{\mathcal{V}} + \llbracket \frac{p_X(x), p_{\sim X}(x)}{p_{X\otimes\sim X}(x)} \rrbracket + \mathcal{S}$ strongly normalises, the same is true of $\mathcal{W} + \llbracket \frac{p_X(x), p_{\sim X}(x)}{} \rrbracket$, where $\mathcal{W}$ is $\mathcal{V}$ with $p_X, p_{\sim X}$ painted `green`, which normalises into some $\mathcal{W}'$ correct w.r.t. the ordeal $\mathcal{S}$: this handles the case of an atomic cut.

Take a cut on a complex proposition $A \otimes B$, i.e., a net $\mathcal{V}$ with conclusions $\Gamma, [C]$ with $C := (A \otimes B) \otimes ({\sim}A \parr {\sim}B)$. We *prove* cut-elimination by showing that *we can consider* $\mathcal{V}$ as a net with conclusions $\Gamma, [A \otimes {\sim}A], [B \otimes {\sim}B]$. In other terms, the analytics is unchanged (the same $\mathcal{V}$) only the synthetics is modified. We must therefore prove that correctness is preserved by this modification of ordeals.

An ordeal for $\Gamma, [C]$ is a union $\mathcal{T} + \mathcal{T}' + \mathcal{S}$, where $\mathcal{T}$ is either $\llbracket \frac{q_{\sim A}(x)}{\boxed{q_{\sim A \parr \sim B}(x)}} \rrbracket + \llbracket \frac{q_{\sim B}(x)}{} \rrbracket$ or $\llbracket \frac{q_{\sim B}(x)}{\boxed{q_{\sim A \parr \sim B}(x)}} \rrbracket + \llbracket \frac{q_{\sim A}(x)}{} \rrbracket$ and $\mathcal{T}' := \llbracket \frac{q_A(x), q_B(x)}{\boxed{q_{A\otimes B}(x)}} \rrbracket + \llbracket \frac{q_{A\otimes B}(x), q_{\sim A \parr \sim B}(x)}{\boxed{q_C(x)}} \rrbracket +$ $\llbracket \frac{q_C(x)}{p_C(x)} \rrbracket$ and $\mathcal{S} := \llbracket \frac{p_{A\otimes B}, p_{\sim A \parr \sim B}}{p_C(x)} \rrbracket$ $\mathcal{T} + \mathcal{T}' + \mathcal{S}$ normalises into $\mathcal{U}$ which is either $\mathcal{U}_1 := \llbracket \frac{q_A(x), q_B(x), q_{\sim A}(x)}{p_C(x)} \rrbracket + \llbracket \frac{q_{\sim B}(x)}{} \rrbracket$ or $\mathcal{U}_2 := \llbracket \frac{q_A(x), q_B(x), q_{\sim B}(x)}{p_C(x)} \rrbracket + \llbracket \frac{q_{\sim A}(x)}{} \rrbracket$.

Since $\boxed{\mathcal{V}} + \mathcal{S} + \mathcal{T} + \mathcal{T}'$ strongly normalises into $\llbracket p_\Gamma(x), p_C(x) \rrbracket$, $\boxed{\mathcal{V}} + \mathcal{S}$ strongly normalises into some $\mathcal{W}$ s.t. $\mathcal{W} + \mathcal{U}$ normalises into $\llbracket p_\Gamma(x), p_C(x) \rrbracket$. The part of $\mathcal{W}$ contributing to the normal form is made of three stars containing the various rays of $p_\Gamma(x)$ as well as $\boxed{q_A(x)}, \boxed{q_B(x)}, \boxed{q_{\sim A}(x)}, \boxed{q_{\sim B}(x)}$. $\boxed{q_A(x)}$ and $\boxed{q_B(x)}$ belong to two different stars $\sharp 1$ and $\sharp 2$; using $\mathcal{U}_1$, we see that $\boxed{q_{\sim A}(x)}$ can belong to neither of stars $\sharp 1, \sharp 2$; using $\mathcal{U}_2$ we see that $\boxed{q_{\sim B}(x)}$ can belong to neither of stars $\sharp 1, \sharp 2$. Hence they both belong to star $\sharp 3$.

Rewriting the cut amounts at replacing $\Gamma, [C]$ with $\Gamma, [D], [E]$, with $D = A \otimes {\sim}A$, $E = B \otimes {\sim}B$. An ordeal for these conclusions is $\mathcal{O} = \mathcal{S} + \llbracket \frac{q_A(x), q_{\sim A}(x)}{\boxed{q_D(x)}} \rrbracket + \llbracket \frac{q_D(x)}{p_D(x)} \rrbracket +$ $\llbracket \frac{q_B(x), q_{\sim B}(x)}{\boxed{q_E(x)}} \rrbracket + \llbracket \frac{q_E(x)}{p_E(x)} \rrbracket$ which normalises into $\mathcal{S} + \llbracket \frac{q_A(x), q_{\sim A}(x)}{p_D(x)} \rrbracket + \llbracket \frac{q_B(x), q_{\sim B}(x)}{p_E(x)} \rrbracket$. Due to the fact that the three stars of the normal form $\mathcal{W}$ of $\boxed{\mathcal{V}} + \mathcal{S}$ contain either $\boxed{q_A(x)}$ ($\sharp 1$) or $\boxed{q_B(x)}$ ($\sharp 2$) or $\llbracket \boxed{q_{\sim A}(x)}, \boxed{q_{\sim B}(x)} \rrbracket$ ($\sharp 3$), it is easily shown that $\boxed{\mathcal{V}} + \mathcal{O}$ strongly normalises into $\llbracket p_\Gamma(x), p_D(x), p_E(x) \rrbracket$.

Of course, we are still in need of an additional argument showing the convergence of the rewriting; but then we enter familiar waters.

5. Exponentials revisited

5.1. *A strategic retreat*

Transcendental syntax tells us that there can be nothing like the exponentials $!A$ and $?A$; one must thus operate a strategic retreat in direction of intuitionistic implication, in other terms, to restrict the use of $!A$ and $?A$ to combinations $A \olessthan B := \sim(A \Rightarrow \sim B)$ (i.e., $!A \otimes B$) and $A \ltimes B := \sim A \Rightarrow B$ (i.e., $?A \parr B$). Since $!A$ can be defined by means of $!A := A \olessthan \mathbf{1}$, this means that the multiplicative constants are rejected: they have no conditions of possibility.

Since intuitionistic implication $A \Rightarrow B$ is sometimes noted B^A, we keep the terminology "exponentials" for $\olessthan$ and $\ltimes$. As to notations, $\owedge$ and $\ovee$ are inadequate, since symmetric left/right; $\olessthan, \ltimes$ are asymetric, with the hint of an order, from left to right.

5.2. *Hidden conclusions*

Let us make an interesting experiment, namely, to underline some conclusions $\underline{A}$ (not all of them) in a proof-net. In terms of ordeals, this means replacing $[\![\, \frac{q_A(x)}{p_A(x)} \,]\!]$ with $[\![\, \frac{q_A(x)}{} \,]\!]$. The correctness criterion for a net with conclusions $\vdash \Gamma, \underline{\Delta}$ writes:

For any ordeal $\mathcal{S}$, $\boxed{\mathcal{V}} + \mathcal{S}$ *strongly normalises into* $[\![\, p_\Gamma(x) \,]\!]$.

In other terms, the $\underline{\Delta}$ are not visible in the output: they are, so to speak, *hidden.*

Hiding conclusions is indeed a relaxation of the usual correctness criterion: nobody forbids us from restoring the hidden conclusion A, i.e., from reverting to $[\![\, \frac{q_A(x)}{p_A(x)} \,]\!]$; let $\mathcal{T} := \mathcal{S} \setminus [\![\, \frac{q_A(x)}{} \,]\!] + [\![\, \frac{q_A(x)}{p_A(x)} \,]\!]$. If $\boxed{\mathcal{V}} + \mathcal{S}$ strongly normalises into $[\![\, p_\Gamma(x) \,]\!]$, so does $\boxed{\mathcal{V}} + \mathcal{T}$ whose normal form writes $[\![\, p_\Gamma(x), p_A(t_1), \ldots, p_A(t_n) \,]\!]$ for appropriate disjoint $t_1, \ldots, t_n$.

We see that underlining is a way to handle structural rules ($n = 0$ is weakening, the t_i handle the various copies in a contraction). There is no way to foresee neither the number n of instances nor the actual values of the t_i: they depend upon the ordeal $\mathcal{S}$. This is why the conclusion $\underline{A}$ had to be hidden. This also explains the impossibility of a connective like $?A$ which would, miraculously, guess the invisible!

As far as correctness (i.e., l'usine) is concerned, we thus grounded structural rules. But this is not enough: correctness should guarantee (*modulo* some reasoning) cut-elimination (i.e., the adequation usine/usage). Here our previous experience of GoI for exponentials (performative, block 2) is most precious: since (Gir89), exponentials are handled by means of tensorisations[9] $u \cdot v$, where the second component stands for the duplication instructions. It is thus legitimate to replace the $p_A(t_i)$ with $p_A(t_i \cdot u_i)$.

We can approximate this constraint by reverting to $[\![\, \frac{q_A(x)}{p_A(x)} \,]\!]$ in the hidden conclusions so as to get the slightly awkward reformulation:

For any ordeal $\mathcal{S}$, $\boxed{\mathcal{V}} + \mathcal{S}$ *strongly normalises into some* $[\![\, p_\Gamma(\kappa) + p_\Delta(x \cdot T) \,]\!]$.

[9] See annex A.2.

By $p_\Delta(x\cdot T)$, I mean that each A in Δ yields a certain number of "copies" $p_A(x\cdot t_i)$ of $p_A(x)$. Using techniques inspired from the switching $\ltimes_R$ (*infra*), this convenient formulation can be put back into the original form where A is really hidden. But this has only an interest of principle.

5.3. *Vehicles*

Just to show that we are dealing with familiar logic, let us see how proofs of sequent calculus (annex B) fit in this pattern. Observe that, whereas the formula A yields terms $p_A(t)$ when visible, it yields terms $p_A(t\cdot u)$ when hidden.

Dereliction: if the proof π of $\vdash \Gamma, \underline{\Delta}, \underline{A}$ comes from a proof ν of $\vdash \Gamma, \underline{\Delta}, A$, then $\pi^\bullet := \nu^\bullet$ in which the terms $p_A(t)$ have been replaced with $p_A(t\cdot \mathtt{d})$, where $\mathtt{d}$ is a constant.

Weakening: if the proof π of $\vdash \Gamma, \underline{\Delta}, \underline{A}$ comes from a proof ν of $\vdash \Gamma, \underline{\Delta}$, then $\pi^\bullet := \nu^\bullet$.

Contraction: if the proof π of $\vdash \Gamma, \underline{\Delta}, \underline{A}$ comes from a proof ν of $\vdash \Gamma, \underline{\Delta}, \underline{A'}, \underline{A"}$, then $\pi^\bullet := \nu^\bullet$ in which the terms $p_{A'}(t\cdot u)$ (resp. $p_{A"}(t\cdot u)$) have been replaced with $p_A(t\cdot(\mathtt{l}\cdot u))$ (resp. $p_A(t\cdot(\mathtt{r}\cdot u))$).

$\ltimes$-rule: if the proof π of $\vdash \Gamma, \underline{\Delta}, A\ltimes B$ comes from a proof ν of $\vdash \Gamma, \underline{\Delta}, \underline{A}, B$, then $\pi^\bullet := \nu^\bullet$, with p_A, p_B *defined* by $p_A(x) := p_{A\ltimes B}(\mathtt{l}\cdot x)$, $p_B(x) := p_{A\ltimes B}(\mathtt{r}\cdot x)$.

$\olessthan$-rule: if the proof π of $\vdash \Gamma', \underline{\Delta}, \underline{\Delta'}, A\olessthan B$ comes from proofs ν of $\vdash \underline{\Delta}, A$ and μ of $\vdash \Gamma', \underline{\Delta'}, B$, we define $p_A(x) := p_{A\ltimes B}(\mathtt{l}\cdot x), p_B(x) := p_{A\olessthan B}(\mathtt{r}\cdot x)$. We modify $\nu^\bullet$ into $\nu_1^\bullet$ by replacing all $p_A(t)$ with $p_A(t\cdot y)$, with y a fresh variable. Due to this variable, $\nu_1^\bullet$ is no longer a constellation; we homegeneise $\nu_1^\bullet$ into $\nu_2^\bullet$ by replacing all terms $p_C(t\cdot u)$ with $C\in\Delta$ with $p_C(t\cdot(u\cdot y))$. We define $\pi^\bullet := \nu_2^\bullet + \mu^\bullet$.

5.4. *Links*

The $\ltimes$ and $\olessthan$ links are written as follows:

$$\frac{A \quad B}{A\olessthan B} \qquad\qquad \frac{\underline{A} \quad B}{A\ltimes B}$$

$\underline{A}$ is short for an unspecified stock of "occurrences" of A, maybe none, like the $[A]$ of natural deduction was short for an unspecified stock of discharged hypotheses. This link thus includes weakening and contraction.

5.5. *Gabarits*

We now need to associate ordeals to exponentials; for this, we shall anticipate upon the future introduction (Gir16) of non-determinism, since non deterministic ordeals will be used to handle $\ltimes$. $\olessthan$ is also responsible for a complexification of the structure: a non linear term must be used in the switching $\olessthan_\delta$.

$\olessthan_\delta$**:** $[\![\dfrac{q_A(x\cdot x), q_B(x)}{\boxed{q_{A\olessthan B}(x)}}]\!]$.

$\varolessthan_{\mathtt{1}}$: $[\![\frac{q_A(x\cdot \mathtt{1}), q_B(x)}{q_{A\varolessthan B}(x)}]\!]$.

$\ltimes_R$: $[\![\frac{q_B(x)}{q_{A\ltimes B}(x)}]\!] + [\![\frac{q_A(x\cdot y)}{}]\!] + [\![\frac{q_A(x'\cdot y')}{}]\!]$, *except* if $x = x'$.

$\ltimes_{L'}$: $[\![\frac{q_A(x\cdot y)}{q_{A\ltimes B}(x\cdot y)}]\!]$.

First start with $\varolessthan$: the part of the ordeal leading to $q_A(x) := q_{A\varolessthan B}(\mathtt{1}\cdot x)$ must be "tensorised" with an extra variable: this means that any ray occuring in a star "above" A of the form T (resp. T, T), with $T = p_A(t)$ becomes $T\otimes y$ (resp. $T\otimes y$, $T\otimes y$), with $T\otimes y := p_A(t\cdot y)$. The two switchings circumvent the non-homegeneity of $[\![\frac{q_A(x\cdot y), q_B(x)}{q_{A\varolessthan B}(x)}]\!]$. If we remove the switch $\varolessthan_\delta$ in a correct proof-net, we get three stars corresponding to $A, B, A\varolessthan B$, say $\sharp 1, \sharp 2, \sharp 3$. The star $\sharp 3$ must contain $q_{A\varolessthan B}(x)$, hence $\sharp 1$ must contain $q_A(x\cdot t)$ with t *unifying* with both of $x, \mathtt{1}$. The only possibility is that of a fresh variable y: $\sharp 1$ contains $q_A(x\cdot y)$. The exotic possibility — which cannot be excluded in a non deterministic setting — of a sum $q_A(x\cdot x) + q_A(x\cdot \mathtt{1})$ would lead to an incorrect normal form, typically $p_\Gamma(x) + p_\Gamma(\mathtt{1})$ whatever switching we choose.

The switching $\ltimes_R := [\![\frac{q_B(x)}{q_{A\ltimes B}(x)}]\!] + [\![\frac{q_A(x\cdot y)}{}]\!] + [\![\frac{q_A(x'\cdot y')}{}]\!]$ emphasises the names of the variables, which usually hardly matter. The part $[\![\frac{q_B(x)}{q_{A\ltimes B}(x)}]\!] + [\![\frac{q_A(x\cdot y)}{}]\!]$ induces instanciations $[\![q_A(t_i\cdot u_i)]\!]$: we would like to ensure that the t_i are all equal to x. This is why we introduce a non deterministic alternative between $[\![\frac{q_A(x\cdot y)}{}]\!]$ and... itself (noted $[\![\frac{q_A(x'\cdot y')}{}]\!]$). A coherence relation governs the formation of diagrams: a $[\![\frac{q_A(x'\cdot y')}{}]\!]$ may coexist with some $[\![\frac{q_B(x)}{q_{A\ltimes B}(x)}]\!]$ or $[\![\frac{q_A(x\cdot y)}{}]\!]$ in the same diagram only if the actualisations of x' and x are distinct. Which is possible only in case of an actualisation $q_A(t_i\cdot u_i)$ with $t_i \neq x$, in case all the $q_A(t_i\cdot u_i)$ could be plugged as well to the variant $[\![q_A(x'\cdot y')]\!]$; this would produce a duplicate of the normal form, typically $p_\Gamma(x) + p_\Gamma(x)$.

The switching $\ltimes_{L'}$ is a sort of compensation for the impossibility of a plain left switching for $\ltimes$. The switching is however atypic: any ordeal using it is *cancelling*, i.e., $\mathcal{V} + \mathcal{S}$ must normalise into the empty constellation 0.

Let us explain how this cancellation works: the right switching already tells us that, on the side of A, we actually have a certain number of instances $q_A(x\cdot u_i)$. The normal form is obtained by plugging $\ltimes_R$ with a constellation made of two stars, $\sharp 1 = [\![q_{A\ltimes B}(x), \ldots]\!]$, $\sharp 2 := [\![q_B(x), \ldots]\!]$; a certain number of instances of $q_A(x\cdot u_i)$ are dispatched between $\sharp 1$ and $\sharp 2$. $\ltimes_{L'}$ plugs with the sole $\sharp 1$; if $\sharp 1 = [\![q_{A\ltimes B}(x), q_A(x\cdot u), \ldots]\!]$, then $\sharp 1 + \ltimes_{L'}$ contains the actualised diagram $[\![\frac{q_{A\ltimes B}(x)}{q_{A\ltimes B}(x\cdot u)}]\!]$: this contradicts "condition 3" of strong normalisation. The possibility of some $q_A(x\cdot u_i)$ in $\sharp 1$ being excluded, the constellation can only "die".

5.6. *Nested exponentials*

The replacement of $p_A(t)$ with $p_A(t) \cdot y$ corresponds to the building of an exponential box, the main — and by no means minor! — novelty being that the contents are no longer isolated. Now, what about nested boxes? The nesting number of a literal P of a conclusion A (or $\underline{A}$) being the number of ! and ? crossed when passing from P to A, we easily see that an original identity link $[\![p_X(x), p_{\sim X}(x)]\!]$ eventually becomes something like $[\![p_X((\ldots(x \cdot t_m)\ldots) \cdot t_1), p_{\sim X}(x)((\ldots(x \cdot u_n)\ldots) \cdot u_1)]\!]$, where m and n are the respective nesting numbers of X and $\sim X$.

This nesting, predictible from the logical formulas, must be taken into account when devising ordeals. Typically, the cancelling ordeal associated with the choice of the literal X (section 4.1) should include $[\![\frac{q_A((\ldots((\mathtt{l} \cdot x) \cdot y_1)\ldots) \cdot y_n), q_A((\ldots((\mathtt{r} \cdot x) \cdot y_1)\ldots) \cdot y_n)}{}]\!]$ when A is an occurrence of X of nesting number n.

5.7. *Cut-elimination*

Cut-elimination compels us into considering *hidden* cuts $[\underline{A \otimes \sim A}]$. We can thus replace a cut $\Gamma, \underline{\Delta}, [C]$, with $[C] = [(A \olessthan B) \otimes (\sim A \ltimes \sim B)]$ with $\Gamma, \underline{\Delta}, [\underline{A \otimes \sim A}], [B \otimes \sim B]$ while preserving correctness. In order not to complicate the discussion with irrelevant issues, we assume that $\Delta = \emptyset$.

Consider the sole switching $\olessthan_1, \ltimes_R$; using the same notations as in the multiplicative case, we get that $\mathcal{W}+\mathcal{U}$ normalises into $[\![p_\Gamma(x), p_C(x)]\!]$, with $C := (A \olessthan B) \otimes (\sim A \ltimes \sim B)$. The part of $\mathcal{W}$ contributing to the normal form is made of three stars containing the various rays of $p_\Gamma(x)$ as well as $\boxed{q_B(x)}$, $\boxed{q_{\sim B}(x)}$, various $\boxed{q_{\sim A}(x \cdot t_i)}$ and $\boxed{q_A(x \cdot y)}$. Star $\sharp 1$ contains $\boxed{q_A(x \cdot y)}$, star $\sharp 2$ contains $\boxed{q_B(x)}$, star $\sharp 3$ contains $\boxed{q_{\sim B}(x)}$. By using the switch $\ltimes_{L'}$, we conclude (see *supra* the explanation of the link) that the $\boxed{q_{\sim A}(x \cdot t_i)}$ cannot belong to stars $\sharp 1, 2$ that would be erased. From this, it is easy to conclude that the change of syntheticity to $\Gamma, \underline{\Delta}, [\underline{A \otimes \sim A}], [B \otimes \sim B]$ does preserve correctness. The hidden cut $[\underline{A \otimes \sim A}]$ consists in the star $[\![\frac{p_A(x \cdot y), p_{\sim A}(x \cdot y)}{}]\!]$, which matches the $\boxed{q_{\sim A}(x \cdot t_i)}$ and $\boxed{q_A(x \cdot y)}$; which amounts at matching, for each i, the star $[\![\frac{p_A(x \cdot t_i), p_{\sim A}(x \cdot t_i)}{}]\!]$ with $\boxed{q_{\sim A}(x \cdot t_i)}$ and $\boxed{q_A(x \cdot t_i)}$. The replacement of y with t_i is made possible because $\boxed{q_{\sim A}(x \cdot t_i)}$ and $\boxed{q_A(x \cdot y)}$ belong to distinct stars $\sharp 3$ and $\sharp 1$.

We just introduced *hidden* cuts; do they normalise? Replacing $[\![\frac{q_{A \otimes \sim A}(x \cdot y)}{}]\!]$ with $[\![\frac{q_{A \otimes \sim A}(x \cdot y)}{p_{A \otimes \sim A}(x \cdot y)}]\!]$ in an ordeal yields $[\![p_\Gamma(x), p_{A \otimes \sim A}(x \cdot t_1), \ldots, p_{A \otimes \sim A}(x \cdot t_n)]\!]$, a normal form which looks like n visible cuts, with two differences. First that n and the t_i may depend upon the ordeal, second that these cuts cannot be switched independently. The second problem, the only serious one, will be fixed in (Gir16) by the introduction of some non-determinism at the level of the switchings $\parr_L / \parr_R$.

Appendix A. Geometry of Interaction (GoI)

A few reminders about GoI and its relation to our present analytics.

A.1. *The unification algebra*

A.1.1. *Flows* A *flow* is an expression $t \leftharpoonup t'$ where t, t' are terms with quite the same variables. These common variables are internal to the flow, in other terms *bound*. Composition between $t \leftharpoonup t'$ and $u \leftharpoonup u'$ is obtained by *matching* t' and u. If θ is the principal unifier, we define composition by $(t \leftharpoonup t')(u \leftharpoonup u') := t\theta \leftharpoonup u'\theta$. Composition is thus a partial operation; if we formally add an empty flow 0 to take care of a possible failure of the matching: $(t \leftharpoonup t')(u \leftharpoonup u') := 0$, composition becomes associative, with neutral element $I := x \leftharpoonup x$.

If $\mathcal{T}$ is the set of closed terms, then any functional term t induces a subset $[t] \subset \mathcal{T}$, namely the set of all closed t_0 which unify with t; t, t' are *disjoint* when they don't match, i.e., when $[t] \cap [t'] = \emptyset$. Any flow $t \leftharpoonup t'$ induces a partial bijection $[t \leftharpoonup t']$ between the subsets $[t']$ and $[t]$ of $\mathcal{T}$. Let us fix a constant c; if t_0 is closed, then $[t \leftharpoonup t']t_0$ is defined when $(t \leftharpoonup t')(t_0 \leftharpoonup c) \neq 0$, in case it writes $[t \leftharpoonup t']t_0 \leftharpoonup c$. The condition "quite the same variables" ensures that $[t \leftharpoonup t']t_0$ is closed and that $[t \leftharpoonup t']$ is injective. Any flow $u \leftharpoonup u$ is idempotent; its associated function is the identity of the subset $[u] \subset \mathcal{T}$.

A.1.2. *The convolution algebra* One can introduce the *convolution algebra* of the monoid, i.e., the set of finite formal sums $\sum \lambda_i \phi_i$ where the ϕ_i are flows and the λ_i are complex coefficients, the improper flow 0 being identified with the empty sum. This algebra acts on the Hilbert space $\ell^2(\mathcal{T})$ by means of $(t \leftharpoonup t')(\sum_i \lambda_i t_i) := \sum_i \lambda_i [t \leftharpoonup t'] t_i$. The involution $(\sum \lambda_i (t_i \leftharpoonup t'_i))^* := \sum \bar{\lambda}_i (t'_i \leftharpoonup t_i)$ is implemented by the usual adjunction. The idempotents $t \leftharpoonup t$ correspond to the projections on the subspaces $\ell^2([t])$ and $t \leftharpoonup t'$ induces a partial isometry of *source* $\ell^2([t'])$ and *target* $\ell^2([t])$. The early versions of GoI, see (Gir89), did associate to proofs finite sums of flows. These sums were *partial isometries*; $u = \sum t_i \leftharpoonup t'_i$ is a partial isometry (i.e., $uu^*u = u$) if the targets t_i are pairwise *disjoint*, not unifiable, *idem* for the t'_i. The operators of GoI are indeed *partial symmetries* ($u = u^3 = u^*$): typically the identity axioms $(t \leftharpoonup t') + (t' \leftharpoonup t)$ $(t, t'$ disjoint).

A.2. *A few examples*

The unification algebra internalises the major algebraic constructions.

A.2.1. *Matrixes* If I is a finite set of closed terms, the $I \times I$ matrix (λ_{ij}) can be naturally represented by $\sum_{ij} \lambda_{ij} \, (i \leftharpoonup j)$.

A.2.2. *Direct sums* The flows $P := p(x) \leftharpoonup x, Q := q(x) \leftharpoonup x$ induce an isometric embedding of $\ell^2(\mathcal{T}) \oplus \ell^2(\mathcal{T})$ in $\ell^2(\mathcal{T})$: $x \oplus y \mapsto [P]x + [Q]y$. The isometricity comes from $P^*P = Q^*Q = I$, $P^*Q = Q^*P = 0$. The embedding is not surjective: this would require $PP^* + QQ^* = I$, in other terms that every term matches either $p(x)$ or $q(x)$.

P and Q have been heavily used in the early GoI, in particular for multiplicatives — and, modulo tensorisation with I, for contraction. They enable one to change the size of matrices in a flexible way. Usually, the only possibility is to *divide* the size, typically $\mathcal{M}_{mn}(\mathbb{C}) \simeq \mathcal{M}_m(\mathcal{M}_n(\mathbb{C}))$ replaces a $mn \times mn$ matrix with a $m \times m$ matrix whose entries are $n \times n$ matrices, i.e., blocks of size $n \times n$. Thanks to P, Q, one can replace a 3×3 matrix with a 2×2 one (with four "blocks" of sizes $2 \times 2, 2 \times 1, 1 \times 2, 1 \times 1$).

A.2.3. *Tensor products* The tensor product of two flows makes use of a binary function "$\cdot$" and is defined by $(t \leftharpoondown t') \otimes (u \leftharpoondown u') := t \cdot u \leftharpoondown t' \cdot u'$; the variables of the two flows must first be made distinct. This corresponds to an internalisation of the tensor product, which plays an essential role in the handling of exponentials, i.e., of repetition. The flow $T := (x \cdot y) \cdot z \leftharpoondown x \cdot (y \cdot z)$ compensates the want of associativity of the internal tensor: $T^*(((t \leftharpoondown t') \otimes (u \leftharpoondown u')) \otimes (v \leftharpoondown v'))T = (t \leftharpoondown t') \otimes ((u \leftharpoondown u') \otimes (v \leftharpoondown v'))$.

A.2.4. *Crown products* In the same style as T, the flow $\sigma := x_1 \cdot (x_2 \cdot (\ldots (x_{n-1} \cdot x_n) \ldots)) \leftharpoondown x_{\sigma(1)} \cdot (x_{\sigma(2)} \cdot (\ldots (x_{\sigma(n-1)} \cdot x_{\sigma(n)}) \ldots))$ induces a permutation of the constituents af a n-ary tensor. Crown products play an important role in GoI, typically in the interpretation of LOGSPACE computation (Gir12).

A.3. *Execution*

Geometry of Interaction represents proofs by pairs (u, σ), where u, σ are object of the convolution algebras. The *feedback* σ is a partial symmetry which represents the cuts. The normal form of (u, σ) is $(v, 0)$, with

$$v := (I - \sigma^2)u(I - \sigma u)^{-1}(I - \sigma^2)$$

Termination is expressed by the nilpotency of σu, hence the finite expansion:

$$v := (I - \sigma^2)(u + u\sigma u + u\sigma u\sigma u + \ldots + (u\sigma)^N u))(I - \sigma^2)$$

A.4. *From flows to stars*

The original proof-net criterion (Gir87) was formulated in terms of *trips*, translated as *permutations*, thus unitary operators in von Neumann algebras: this is Geometry of Interaction, see, e.g., (Gir11a). Taking into account the finiteness of the analytic layer, von Neumann algebras have been replaced with the unification algebra.

From the start, an alternative approach, graphlike, was proposed, namely the Danos-Regnier criterion (DR89). The graph approach is generalised by sorts of "thick graphs[10]" by Seiller (Sei13). My constellations are the common generalisation of thick graphs and unification algebra.

Since GoI is, in practice, restricted to hermitian operators, it does not harm to restrict to symmetric sums of flows, typically $(t \leftharpoondown t') + (t' \leftharpoondown t)$ which becomes $[\![\, t, t' \,]\!]$. However,

[10] Graphes épais.

since our objects are no longer operators, there are some difficulties in defining the analogue of composition, which only occurs in configurations like $u\sigma$. We can see σ as the swapping between two complementary colours, to the effect that $(u\sigma)^N u$ is made of all actualised diagrams of size $N + 1$: each sigma denotes an edge. Nilpotency is rendered by requiring that all diagrams of a certain size N are incorrect. The pre- and post-restrictions of v to $I - \sigma^2$ corresponds to the fact that we eventually keep the sole uncoloured actualised diagrams.

The graphlike approach — stars and constellations — still remains more user-friendly. However, flows are more flexible, since they are not committed to any kind of symmetry. Typically, the work of Bagnol (with Aubert, Pistone and Seiller) (ABPS14) which adapts my previous work on LOGSPACE — originally in terms of vN algebras — in terms of flows, exploits this additional flexibility. Another superiority of flows is that they have a more satisfactory structure w.r.t. the Church-Rosser property: the discrepancy observed in section 2.4 between normalisation in one and two steps is due to the presence of stars of arities > 2.

Appendix B. A sequent calculus

Sequents are of the form $\vdash \Gamma, \underline{\Delta}$, with $\Gamma \neq \emptyset$.

B.1. *Identity/Negation*

$$\frac{}{\vdash {\sim}A, A}\ (Id) \qquad \frac{\vdash \Gamma, \underline{\Delta}, A \otimes {\sim}A}{\vdash \Gamma, \underline{\Delta}, [A \otimes {\sim}A]}\ (Cut) \qquad \frac{\vdash \Gamma, \underline{\Delta}, \underline{A \otimes {\sim}A}}{\vdash \Gamma, \underline{\Delta}, [\underline{A \otimes {\sim}A}]}\ (\underline{Cut})$$

The two cut rules are nothing but some claim about the conclusion, namely that the configuration $[A \otimes {\sim}A]$ or $[\underline{A \otimes {\sim}A}]$ can be eliminated. If we believe — and we usually do — in deduction, we can thus ignore these bracketed propositions; if not, then we must see them as a prediction and/or a commitment: their elimination. This is the only place where the synthetic *a priori* occurs explicitly. This form of syntheticity has, of course, a deep influence on the choice of rules: the pairs ⊗/⅋, ⧀/⋉ must match both on *a priori* grounds, cut-elimination, but this remains implicit. They must also match on *a posteriori* grounds, typically when we decompose an identity link $\vdash A, {\sim}A$ into its constituents — what is known as η-expansion.

B.2. *Structural rules*

Underlined propositions are handled by means of structural rules: *dereliction, weakening, contraction* (and *exchange*, omitted).

$$\frac{\vdash \Gamma, \underline{\Delta}, A}{\vdash \Gamma, \underline{\Delta}, \underline{A}}\ (D) \qquad \frac{\vdash \Gamma, \underline{\Delta}}{\vdash \Gamma, \underline{\Delta}, \underline{A}}\ (W) \qquad \frac{\vdash \Gamma, \underline{\Delta}, \underline{A}, \underline{A}}{\vdash \Gamma, \underline{\Delta}, \underline{A}}\ (C)$$

B.3. *Logic*

$$\frac{\vdash \Gamma, \underline{\Delta}, A \quad \vdash \Gamma', \underline{\Delta}', B}{\vdash \Gamma, \Gamma', \underline{\Delta}, \underline{\Delta}', A \otimes B} \ (\otimes) \qquad \frac{\vdash \Gamma, \underline{\Delta}, A, B}{\vdash \Gamma, \underline{\Delta}, A \parr B} \ (\parr)$$

$$\frac{\vdash \underline{\Delta}, A \quad \vdash \Gamma', \underline{\Delta}', B}{\vdash \Gamma', \underline{\Delta}, \underline{\Delta}', A \owedge B} \ (\owedge) \qquad \frac{\vdash \Gamma, \underline{\Delta}, \underline{A}, B}{\vdash \Gamma, \underline{\Delta}, A \ltimes B} \ (\ltimes)$$

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